Division Algebras, Triality, and Exceptional Magic

A. Garrett Lisi

Pacific Science Institute, Makawao, HI, USA

E-mail: Gar@Li.si

ABSTRACT: We describe in explicit detail the rich relationship between division algebras, split composition algebras, triality, Clifford algebras, spinors, triality Lie algebras, generalized reflections, exceptional magic square Lie algebras, triality eigenspaces, 3-symmetric spaces, Vinberg theta algebras, and Exceptional Unification in particle physics.

KEYWORDS: Clifford algebras; spinors; division algebras; split composition algebras; triality; generalized reflections; exceptional Lie algebras; unification

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1 Introduction

A growing community of researchers has become interested in the application of division algebras and the corresponding split composition algebras to a structural description of the Standard Model of particle physics [1–18]. The use of triality in this context, relating three generations of fermions [19], has gathered increasing interest. The existence of the quaternion group, Q_8, within the CPT Group generated by charge, parity, and time conjugation symmetries, and its extension to the $CPTt$ Group — acting on three generations of fermions related by triality — strongly indicates division algebras are intricately woven into the fabric of reality. Despite this indication of usefulness, explicit mathematical descriptions of exactly how triality can be used in model building are sparse. It is the purpose of this work to remedy this deficit — to provide a detailed description of the mathematical scaffolding relating division algebras, triality, Clifford algebras, and Lie algebras related to the Standard Model and gravity. This paper largely follows and complements Baez's excellent paper, "The Octonions" [20], but in more painful detail, and with an intended application in physics.

We begin by introducing division algebras: the complex numbers, $\mathbb{C}$, quaternions, $\mathbb{H}$, and octonions, $\mathbb{O}$, and the related split-signature composition algebras, $\mathbb{C}'$, $\mathbb{H}'$, and $\mathbb{O}'$, and use these to construct Clifford algebras. The Clifford bivectors generate $Spin$ groups, which act on spinors with positive and negative-chiral halves. A structural isomorphism (or "confusion") then exists, between 2, 4, or 8-dimensional vectors, v, negative-chiral spinors, ψ, positive-chiral spinors, χ, and sets of three division algebra elements. (We casually use "division algebra", $\mathbb{D}$, to also encompass the corresponding split-signature composition algebras, and sometimes not the reals.)

A real, cyclic, trilinear triality function is defined by the division algebra product (or vice versa), and is invariant under the triality group of symmetries on its arguments. These symmetries relate to generalized reflections, producing duality automorphisms related to twistor incidence relations [21], as well as triality automorphisms, which transform and cycle the arguments. Each division algebra has a Lie algebra, its triality algebra, corresponding to this triality group. Each triality algebra is a subalgebra of the triality Lie algebra formed by the joining of a triality algebra with the three elements corresponding to a vector, negative-chiral spinor, and positive-chiral spinor. Using these elements, these triality Lie algebras can be expressed heuristically as $su(3, \mathbb{D})$ [5, 22]. Triality inner-automorphisms act within these Lie algebras, and can be displayed graphically in their root systems.

Two division algebras can be combined to construct a compound division algebra representation of a Clifford algebra. The corresponding two triality Lie algebras combine to give Lie algebras in the exceptional magic square [23–26]. These magic square Lie algebras, and their triality automorphisms, are formulated explicitly. Triality automorphisms partition these Lie algebras into three eigenspaces, with the triality invariant subalgebra acting on the positive eigenspace forming a Vinberg Θ-algebra [27]. It is proposed that three generations of fermions can mix within the positive and negative eigenspaces of the Θ-algebra decomposition. Explicit, detailed descriptions are provided for Lie algebras e_6, e_7, and e_8. The algebra of the $SO(10)$ Grand Unified Theory, including one generation of fermions, embeds in real compact e_6. Three generations of Dixon algebra fermions, related to $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, embed in complex e_7, or one real generation of fermions, with spin, embeds in compact real e_7. And three generations of fermions, with spin, related by triality, embed in real forms of e_8, along with Standard Model gauge, Higgs, and gravitational fields. The paper concludes with a brief description of division algebra automorphisms, which brings in the remaining exceptional Lie algebra, g_2.

2 Division Algebra Representation of Clifford Algebras

A n-dimensional division algebra, $\mathbb{D}$, or split-signature composition algebra, $\mathbb{D}'$, is spanned by its basis elements, e_a, which have a conjugation,

$$\tilde{e}_0 = e_{\tilde{0}} = e_0 = 1 \qquad \tilde{e}_1 = e_{\tilde{1}} = -e_1 \qquad \ldots \qquad \tilde{e}_{n-1} = e_{\widetilde{n-1}} = -e_{n-1}$$

and a multiplication table, $e_a e_b = {M_{ab}}^c e_c$, allowing the definition of its metric,

$$(e_a, e_b) = \tfrac{1}{2}(\tilde{e}_a e_b + \tilde{e}_b e_a) = n_{ab}$$

with $n_{ab} = \delta_{ab}$ for the usual division algebras, and n_{ab} having split signature, $\{+,-\}$, for the split-algebras. Under conjugation, division algebra multiplication satisfies

$$\widetilde{(e_a e_b)} = e_{\tilde{b}} e_{\tilde{a}} \qquad\qquad {M_{ab}}^{\tilde{c}} = {M_{\tilde{b}\tilde{a}}}^c$$

Standard multiplication tables, ${M_{ab}}^c e_c$, for the division algebras and their split-algebras are:

$$\mathbb{C}\ :\ \begin{bmatrix} e_0 & e_1 \\ e_1 & -e_0 \end{bmatrix}$$

$$\mathbb{H}\ :\ \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & -e_0 & e_1 \\ e_3 & e_2 & -e_1 & -e_0 \end{bmatrix}$$

$$\mathbb{O}\ :\ \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -e_0 & e_4 & e_7 & -e_2 & e_6 & -e_5 & -e_3 \\ e_2 & -e_4 & -e_0 & e_5 & e_1 & -e_3 & e_7 & -e_6 \\ e_3 & -e_7 & -e_5 & -e_0 & e_6 & e_2 & -e_4 & e_1 \\ e_4 & e_2 & -e_1 & -e_6 & -e_0 & e_7 & e_3 & -e_5 \\ e_5 & -e_6 & e_3 & -e_2 & -e_7 & -e_0 & e_1 & e_4 \\ e_6 & e_5 & -e_7 & e_4 & -e_3 & -e_1 & -e_0 & -e_2 \\ e_7 & e_3 & e_6 & -e_1 & e_5 & -e_4 & -e_2 & -e_0 \end{bmatrix}$$

$$\mathbb{O}'\ :\ \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -e_0 & e_3 & -e_2 & -e_5 & e_4 & -e_7 & e_6 \\ e_2 & -e_3 & -e_0 & e_1 & -e_6 & e_7 & e_4 & -e_5 \\ e_3 & e_2 & -e_1 & -e_0 & -e_7 & -e_6 & e_5 & e_4 \\ e_4 & e_5 & e_6 & e_7 & e_0 & e_1 & e_2 & e_3 \\ e_5 & -e_4 & -e_7 & e_6 & -e_1 & e_0 & e_3 & -e_2 \\ e_6 & e_7 & -e_4 & -e_5 & -e_2 & -e_3 & e_0 & e_1 \\ e_7 & -e_6 & e_5 & -e_4 & -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

$$\mathbb{H}'\ :\ \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_1 & e_0 & e_3 & e_2 \\ e_2 & -e_3 & -e_0 & e_1 \\ e_3 & -e_2 & -e_1 & e_0 \end{bmatrix} \tag{2.1}$$

$$\mathbb{C}'\ :\ \begin{bmatrix} e_0 & e_1 \\ e_1 & e_0 \end{bmatrix}$$

Division algebra multiplication allows the construction of chiral Clifford basis elements of $Cl(n)$, $Cl(0,n)$, or $Cl(\frac{n}{2},\frac{n}{2})$, which act on chiral *division algebra spinors*,

$$\begin{aligned}
\gamma_c &= \begin{bmatrix} 0 & \pm\tilde{e}_c \\ e_c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \pm(\bar{\Gamma}_c)^a{}_b \\ (\Gamma_c)^b{}_a & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pm M_{\tilde{c}\tilde{b}}{}^a \\ M_{ca}{}^{\tilde{b}} & 0 \end{bmatrix} \\
\Psi &= \begin{bmatrix} \psi \\ \tilde{\chi} \end{bmatrix} = \begin{bmatrix} \psi^a e_a \\ \chi^b \tilde{e}_b \end{bmatrix} \sim \begin{bmatrix} \psi^a Q_a^- \\ \chi^b Q_b^+ \end{bmatrix}
\end{aligned} \tag{2.2}$$

with the definition of a signature-adjusted matrix transpose, $(\bar{\Gamma}_c) = n_{cc}(\Gamma_c)^T$, and multiplication understood to be to the right by division algebra elements (accounting for non-associativity of octonions), or represented equivalently as $2n \times 2n$ real matrices built from the multiplication table coefficients,

$$(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}} \qquad\qquad (\bar{\Gamma}_c)^a{}_b = M_{\tilde{c}\tilde{b}}{}^a = M_{bc}{}^{\tilde{a}} = M_c{}^a{}_{\tilde{b}} = (\Gamma_c)_b{}^a = n_{cc}(\Gamma_c)^b{}_a$$

It is worth emphasizing the "trick" here, first brought to the author's attention by Mia Hughes, because it is at the heart of this paper. In our cases of interest a correct set of real, chiral, Clifford basis vector representative matrices, $(\Gamma_c)^b{}_a$, can be numerically identified with a division algebra multiplication table, $M_{ca}{}^{\tilde{b}}$, with a suitable shuffling of indices and a conjugation. We subsequently use M and Γ matrices interchangeably, depending on whether the division algebra or Clifford algebra context is more appropriate.

The representative Clifford basis vector elements, (2.2), satisfy the fundamental Clifford identity,

$$\gamma_a \cdot \gamma_b = \tfrac{1}{2}\left(\gamma_a\gamma_b + \gamma_b\gamma_a\right) = \pm\tfrac{1}{2}\begin{bmatrix} \tilde{e}_a e_b + \tilde{e}_b e_a & 0 \\ 0 & e_a\tilde{e}_b + e_b\tilde{e}_a \end{bmatrix} = \pm n_{ab} = \eta_{ab} \tag{2.3}$$

with the "$\pm$" signature in (2.2) and above usually chosen to be "$-$" for our purposes. Division algebra multiplication coefficients, and the corresponding Clifford matrix elements, satisfy a cyclic identity,

$$\bar{\Gamma}_{abc} = \bar{\Gamma}_{bca} = \Gamma_{acb} = \Gamma_{cba} = M_{ca\tilde{b}} = M_{ab\tilde{c}} = M_{\tilde{a}\tilde{c}b} = M_{\tilde{c}\tilde{b}a} \tag{2.4}$$

in which n_{ab} is used to lower indices. The Clifford pseudoscalar for each division algebra and split-algebra is $\gamma = \gamma_0 ... \gamma_{n-1} = \pm\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$.

The $\frac{1}{2}n(n-1)$ representative Clifford bivector basis elements, for $c < d$, are:

$$\begin{aligned}
\gamma_{cd} &= \begin{bmatrix} \pm\tilde{e}_c e_d & \\ & \pm e_c\tilde{e}_d \end{bmatrix} \sim \begin{bmatrix} \pm(\bar{\Gamma}_c)^a{}_b(\Gamma_d)^b{}_e & \\ & \pm(\Gamma_c)^b{}_a(\bar{\Gamma}_d)^a{}_f \end{bmatrix} \\
&= \begin{bmatrix} \pm M_{\tilde{c}\tilde{b}}{}^a M_{de}{}^{\tilde{b}} & \\ & \pm M_{ca}{}^{\tilde{b}} M_{\tilde{d}\tilde{f}}{}^a \end{bmatrix}
\end{aligned} \tag{2.5}$$

with it understood that, for example, e_d, multiplies to the right before $\tilde{e}_c$ multiplies the result. Since $e_c\tilde{e}_d = -e_d\tilde{e}_c$ for $c \neq d$, we also have the reverse-indexed bivectors, $\gamma_{dc} = -\gamma_{cd}$, and will sometimes account for these with a $\frac{1}{2}$ in sums, such as for $B = \frac{1}{2}B^{cd}\gamma_{cd}$. These bi-product division algebra operators are the chiral basis elements of the corresponding spin Lie algebra, which act on division algebra spinors. Spinors are the fundamental representation space of spin groups, which have spin Lie algebras spanned by Clifford algebra bivectors represented by matrices that act on the spinors. For the complex numbers and quaternions, multiplication is associative, so these bi-product basis elements are themselves purely imaginary complex numbers or quaternions, $\tilde{e}_c e_d \in \mathbb{B} = \mathrm{Im}(\mathbb{D})$, and the corresponding spin Lie algebras, $\mathbb{B}_2 = so(2) = u(1)$ and $\mathbb{B}_4 = so(4) = su(2) + su(2)$, are 1 and 6-dimensional. For the octonions, multiplication is not associative, and these octonionic bi-products span 28-dimensional $\mathbb{B}_8 = so(8) \neq \mathrm{Im}(\mathbb{O})$.

To add clarity for the reader, it is worth presenting the simplest example of this construction: $Cl(0,2)$ constructed from complex algebra. From the complex multiplication table, (2.1), and using $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$, we obtain a real chiral matrix representation of Clifford basis vectors, γ_c,

$$M_{00}{}^{\tilde{0}} = 1 \qquad M_{01}{}^{\tilde{1}} = M_{10}{}^{\tilde{1}} = -1 \qquad M_{11}{}^{\tilde{0}} = -1$$

$$\Gamma_0 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \qquad \Gamma_1 = \begin{bmatrix} & -1 \\ -1 & \end{bmatrix} \qquad \gamma_c = \begin{bmatrix} 0 & -\bar{\Gamma}_c \\ \Gamma_c & 0 \end{bmatrix}$$

To be painfully explicit, the resulting representative unit basis vectors and unit basis spinors of $Cl(0,2)$ are:

$$\gamma_0 = \begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix} \qquad \gamma_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

$$Q_0^- = \begin{bmatrix} 1 \\ \\ \\ \\ \end{bmatrix} \qquad Q_1^- = \begin{bmatrix} \\ 1 \\ \\ \\ \end{bmatrix} \qquad Q_0^+ = \begin{bmatrix} \\ \\ 1 \\ \\ \end{bmatrix} \qquad Q_1^+ = \begin{bmatrix} \\ \\ \\ 1 \end{bmatrix}$$

Physicists usually think of spinors, such as the unit spinors, $Q_c^\pm$, as matrix columns. Depending on context, they may also be thought of as ideal Clifford algebra elements.

From the division algebra representation of Clifford algebras, we have a natural *confusion* between sets of three n-dimensional division algebra elements related by division algebra multiplication, and corresponding sets of Clifford algebra vectors, negative real chiral spinors, and positive real chiral spinors, related by Clifford algebra multiplication,

$$\begin{aligned} v &= v^c e_c & &\sim & v &= v^c \gamma_c \\ \psi &= \psi^a e_a & &\sim & \psi &= \psi^a Q_a^- \\ \tilde{\chi} &= \chi^b \tilde{e}_b & &\sim & \chi &= \chi^b Q_b^+ \\ \tilde{\chi} &= v\,\psi & &\sim & \chi &= v\,\psi \end{aligned}$$

It is the chiral division algebra representative matrices of Clifford algebras, ${\Gamma_c}^b{}_a = M_{ca}{}^{\tilde{b}}$, that allow this direct identification between a set of vector, negative, and positive-chiral spinors, (v, ψ, χ), and a triplet of division algebra elements with the same coefficients, and their equivalent relationship under division algebra and Clifford multiplication,

$$\chi^b e_{\tilde{b}} = \tilde{\chi} = v\,\psi = v^c \psi^a M_{ca}{}^{\tilde{b}} e_{\tilde{b}} \qquad \sim \qquad \chi^b = v^c \psi^a (\Gamma_c)^b{}_a$$

This confusion of vectors and spinors with division algebra elements, and the division algebra construction of Clifford algebras, leads to the explicit construction and understanding of the structure of many Lie algebras and their automorphisms.

3 Generalized Reflections and Triality

Division algebras (and their split versions) have a cubic form — a real, cyclic, trilinear *triality function*, $T(v, \psi, \chi)$, of three elements, or, equivalently, of vectors and chiral spinors,

$$\chi^b v^c \psi^a \Gamma_{cba} = \bar{\chi} v\,\psi = T(v, \psi, \chi) = (\tilde{\chi}, v\,\psi) = \chi^b v^c \psi^a M_{ca\tilde{b}} \tag{3.1}$$

in which we again are using a signature-adjusted transpose, $\bar{Q}^+_b = n_{bb} Q^{+T}_b$, so the index in $Q^{+T}_b (\Gamma_c) Q^-_a = {\Gamma_c}^b{}_a$ will be lowered. The triality function is cyclic, $T(v, \psi, \chi) = T(\psi, \chi, v)$, by virtue of the cyclic nature of division algebra multiplication, (2.4). Although one usually considers the triality function as built from the division algebra product, it is possible, alternatively, to use the existence of a triality function, as a cyclic cubic form on a vector space, to define the division algebra product. The triality function is invariant under the *triality group*, $\mathrm{Tri}(\mathbb{D})$, with elements $r \in \mathrm{Tri}(\mathbb{D})$ satisfying:

$$r \;:\; (v, \psi, \chi) \;\mapsto\; (v', \psi', \chi') \qquad \ni \qquad T(v', \psi', \chi') = T(v, \psi, \chi)$$

The triality group acts linearly on (v, ψ, χ) as its representation space.

Consider reflections, R^u_v, through a unit-length Clifford vector or division algebra element,

$$-\,u \cdot u = u^a u^b n_{ab} = s_u = \tilde{u} u = \pm 1$$

in which the signature, s_u, is space-like, $+1$, for division algebra elements or $Cl(0, n)$ Clifford vectors, or can be time-like, -1, for some split-composition algebra elements or the corresponding time-like $Cl(\frac{n}{2}, \frac{n}{2})$ Clifford vectors. Note that we have chosen the "$-$" sign in (2.2, 2.3), to later match Lie algebra elements. Reflections, R^u_v, of a Clifford vector and spinor through u can be written as

$$v' = R^u_v v = -u v u^- \qquad\qquad \Psi' = R^u_v \Psi = \sqrt{s_u}\, u \gamma \Psi$$

with unit real or imaginary $\sqrt{s_u}$ introduced so that $R^u_v R^u_v = 1$. Although this expression matches a known description of reflections [28, 29], the use of the Clifford pseudoscalar allows the reflection of any Clifford algebra element to be described as a Clifford adjoint, $R^u A =$

$(u\gamma)A(u\gamma)^-$, with $(u\gamma)$ an element of the pin group [19].[1] These reflections can also be expressed using division algebra elements, and, since triality cycles vectors and spinors, we also have *generalized reflections*, R^u_m and R^u_p, acting as reflections through negative and positive chiral spinors, using division algebra multiplication,

$$\begin{array}{ccc} R^u_v & R^u_m & R^u_p \\ v' = R^u_v v = -s_u u\tilde{v}u & v' = R^u_m \chi = \sqrt{s_u}\tilde{\chi}\tilde{u} & v' = R^u_p \psi = \sqrt{s_u}\tilde{u}\tilde{\psi} \\ \psi' = R^u_v \chi = \sqrt{s_u}\tilde{u}\tilde{\chi} & \psi' = R^u_m \psi = -s_u u\tilde{\psi}u & \psi' = R^u_p v = \sqrt{s_u}\tilde{v}\tilde{u} \\ \chi' = R^u_v \psi = \sqrt{s_u}\tilde{\psi}\tilde{u} & \chi' = R^u_m v = \sqrt{s_u}\tilde{u}\tilde{v} & \chi' = R^u_p \chi = -s_u u\tilde{\chi}u \end{array} \tag{3.2}$$

These generalized reflections, through a space-like or time-like unit element, u, can be equivalently expressed as operations on Clifford basis vector and chiral spinor elements, and written out explicitly using indices,[1]

$$\begin{array}{ccc} R^u_v & R^u_m & R^u_p \\ \gamma'_c = (\delta^a_c - 2s_u u^a u_c)\gamma_a & \gamma'_c = \sqrt{s_u}u^a(\Gamma_a)^b{}_c Q^+_b & \gamma'_c = \sqrt{s_u}u^b(\bar{\Gamma}_b)^a{}_c Q^-_a \\ Q^-_a{}' = \sqrt{s_u}u^c(\bar{\Gamma}_c)^b{}_a Q^+_b & Q^-_a{}' = (\delta^b_a - 2s_u u^b u_a)Q^-_b & Q^-_a{}' = \sqrt{s_u}u^c(\Gamma_c)^b{}_a \gamma_b \\ Q^+_b{}' = \sqrt{s_u}u^c(\Gamma_c)^a{}_b Q^-_a & Q^+_b{}' = \sqrt{s_u}u^c(\bar{\Gamma}_c)^a{}_b \gamma_a & Q^+_b{}' = (\delta^a_b - 2s_u u^a u_b)Q^+_a \end{array} \tag{3.3}$$

In these expressions, γ_a are the basis vectors, Q^+_a are the basis positive chiral spinors, and Q^-_a are the basis negative chiral spinors. R^u_v is a reflection through the u unit vector, mapping vectors to vectors, positive chiral spinors to negative chiral spinors, and negative chiral spinors to positive chiral spinors. R^u_m is a generalized reflection through the u unit negative chiral spinor, mapping positive chiral spinors to vectors, negative chiral spinors to negative chiral spinors, and vectors to positive chiral spinors. R^u_p is a generalized reflection through the u unit positive chiral spinor, mapping negative chiral spinors to vectors, vectors to negative chiral spinors, and positive chiral spinors to positive chiral spinors. The triality function is anti-invariant under generalized reflections, such as

$$\begin{aligned} T(v', \psi', \chi') &= T(R^u_v v, R^u_v \chi, R^u_v \psi) = T(-s_u u\tilde{v}u, \sqrt{s_u}\tilde{u}\tilde{\chi}, \sqrt{s_u}\tilde{\psi}\tilde{u}) \\ &= (u\psi, (-u\tilde{v}u)\tilde{u}\tilde{\chi}) = -T(v, \chi, \psi) \end{aligned}$$

If we impose a constraint that $T(v, \psi, \chi) = 1$ for a matched set of $\{v, \psi, \chi\}$, then we can obtain expressions for either the vector, negative spinor, or positive spinor from the two others such that the constraint is satisfied,

$$v = \frac{1}{|\psi\chi|^2}\widetilde{\psi\chi} \qquad \psi = \frac{1}{|\chi v|^2}\widetilde{\chi v} \qquad \chi = \frac{1}{|v\psi|^2}\widetilde{v\psi}$$

This is called dualizing the triality function. More generally, generalized reflections through a non-unit-length division algebra element, v, give *duality functions*,[1] such as $\psi = \widetilde{\chi v}$. We

[1]This mathematical construction is likely original to the author.

can use division-Clifford algebra confusion to get the equivalent equation for a negative chiral spinor from the Clifford multiplication of a vector and a positive chiral spinor:

$$\psi = \tilde{v}\tilde{\chi} \quad \leftrightarrow \quad \psi^a = v^c \chi^b M_{\tilde{c}\tilde{b}}{}^a \quad \leftrightarrow \quad \psi^a = (v^c \bar{\Gamma}_c{}^a{}_b)\chi^b$$

For the quaternionic case, this is the "incidence relation" for a Euclidean twistor.[21] The main idea of twistor theory is that if you know a ψ and χ, the incidence relation lets you solve for v, so you can do fun things with a "twistor", (ψ, χ), that translates to things having to do with a vector, v. This relates triality to twistor theory, but is not the focus of this work.

Combining two generalized reflections of the same type gives a generalized rotation — an element of the triality group. Combining two generalized reflections of different types gives an element of the triality group that isn't a rotation. Combining four generalized reflections through two unit-length elements, u and w, gives a *triality automorphism*, such as[1]

$$t^{uw} = R_p^w R_v^u R_m^{\tilde{u}} R_p^{\tilde{u}}$$

an element of the triality group that takes vectors to negative spinors, negative spinors to positive spinors, and positive spinors to vectors,

$$t^{uw} : (v, \psi, \chi) \mapsto (v', \psi', \chi') = \left(\sqrt{s_u s_w}\tilde{w}(u\psi),\ \sqrt{s_u s_w}(\chi u)\tilde{w},\ s_u s_w w(\tilde{u}v\tilde{u})w\right)$$

with $T(v', \psi', \chi') = T(v, \psi, \chi)$. Via Clifford algebra confusion, pairs of unit elements, u and w, produce general triality automorphisms of sets of three division algebra elements or of the corresponding Clifford vector and spinors,

$$\begin{aligned}
v' &= \sqrt{s_u s_w}\tilde{w}(u\psi) = \sqrt{s_u s_w} w^d u^c \psi^a M_{\tilde{d}\tilde{b}}{}^f M_{ca}{}^{\tilde{b}} e_f \\
&\quad \sim \ v' = \sqrt{s_u s_w} w^d u^c \psi^a (\bar{\Gamma}_d)^f{}_b (\Gamma_c)^b{}_a Q_f^- = \sqrt{s_u s_w} w u \psi \\
\psi' &= \sqrt{s_u s_w}(\chi u)\tilde{w} = \sqrt{s_u s_w} \chi^b u^c w^d M_{bc}{}^{\tilde{a}} M_{\tilde{a}\tilde{d}}{}^f e_f \\
&\quad \sim \ \psi' = \sqrt{s_u s_w} \chi^b u^c w^d (\Gamma_d)^f{}_a (\bar{\Gamma}_c)^a{}_b Q_f^+ = \sqrt{s_u s_w} w u \chi \\
\chi' &= s_u s_w w(\tilde{u}v\tilde{u})w = s_u s_w w^a u^b v^c u^d w^e M_{af}{}^g M_{\tilde{b}c}{}^a M_{a\tilde{d}}{}^f M_{ge}{}^h e_h \\
&\quad \sim \ \chi' = v^b(\delta_b^a - 2 s_w w^a w_b)(\delta_a^c - 2 s_u u^c u_a)\gamma_c = s_u s_w w u v u w
\end{aligned} \tag{3.4}$$

Choosing $u = 1$ ($\sim\ u = \gamma_0$) and $w = 1$ ($\sim\ w = \gamma_0$) gives a *canonical triality automorphism*,

$$t \ : \ (v, \psi, \chi) \ \mapsto \ (v', \psi', \chi') = (\psi, \chi, v)$$

consistent with the invariance of triality under cyclic permutation of its arguments. Even numbers of generalized reflections — elements of the triality group — can be factored into canonical triality automorphisms and rotations, such as

$$t^{uw} = R_p^w R_v^u R_m^{\tilde{u}} R_p^{\tilde{u}} = R_p^w R_p^u\, t = R_p^{uw}\, t = t\, R_v^{uw}$$

A simple rotation, $R_v^{uw} = R_v^w R_v^u$, can be described as two reflections through vectors, u and w, spanning the plane of rotation, resulting in a transformation of vectors to vectors, negative chiral spinors to negative chiral spinors, and positive chiral spinors to positive chiral spinors. Alternatively, a rotation, R_v^B, can be described using a Clifford algebra rotor, $U = e^{B/2}$, acting on vectors and spinors,

$$v' = R_v^B v = UvU^- \simeq v + \tfrac{1}{2}Bv - \tfrac{1}{2}vB \qquad\qquad \Psi' = R_v^B \Psi = U\Psi \simeq \Psi + \tfrac{1}{2}B\Psi$$

in which

$$B = \tfrac{1}{2}B^{cd}\gamma_{cd} = \begin{bmatrix} B_m & \\ & B_p \end{bmatrix}$$

is a Clifford bivector, (2.5). Generalized rotations, $R_m^B = t^2 R_v^B t$ and $R_p^B = t R_v^B t^2$, can be described by combining rotations with a canonical triality automorphism.

The triality group of a division algebra, $\mathrm{Tri}(\mathbb{D})$, includes rotations and triality automorphisms. The Lie algebra of this triality group is the *triality algebra* of the division algebra, $\mathrm{tri}(\mathbb{D})$; the Lie algebra of generalized rotations. Its elements, $R \in \mathrm{tri}(\mathbb{D})$, satisfy:

$$R\,:\,(v,\psi,\chi) \mapsto (v',\psi',\chi') \qquad\qquad T(v',\psi,\chi) + T(v,\psi',\chi) + T(v,\psi,\chi') = 0$$

A bivector rotation generator, B, acts on chiral vectors and spinors as

$$v' = \begin{bmatrix} & v'_p \\ v'_m & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} & B_m v_p - v_p B_p \\ B_p v_m - v_m B_m & \end{bmatrix} \qquad\qquad \psi' = \tfrac{1}{2}B_m\psi \qquad\qquad \chi' = \tfrac{1}{2}B_p\chi$$

Which, using (3.1), we can see satisfies

$$\begin{aligned} 0 &= T(v',\psi,\chi) + T(v,\psi',\chi) + T(v,\psi,\chi') \\ &= \bar{\chi}\tfrac{1}{2}(B_p v_m - v_m B_m)\psi + \bar{\chi} v_m \tfrac{1}{2}B_m\psi + \bar{\chi}\tfrac{1}{2}\bar{B}_p v_m \psi \end{aligned}$$

using $\bar{B}_p = -B_p$. We can similarly calculate the action of generalized rotation generators on vectors and spinors by using a canonical triality automorphism. In this way, the rotation symmetry algebras of the complex numbers, $so(2) = u(1)$, quaternions, $so(4) = su(2) + su(2)$, octonions, $so(8)$, and split algebras are enlarged by triality. The resulting triality algebras [23, 24] are:

$$\begin{aligned} \mathrm{tri}(\mathbb{C}) &= u(1) + u(1) & \qquad \mathrm{tri}(\mathbb{C}') &= gl(1) + gl(1) \\ \mathrm{tri}(\mathbb{H}) &= su(2) + su(2) + su(2) & \qquad \mathrm{tri}(\mathbb{H}') &= sl(2) + sl(2) + sl(2) \\ \mathrm{tri}(\mathbb{O}) &= so(8) & \qquad \mathrm{tri}(\mathbb{O}') &= so(4,4) \end{aligned}$$

These triality algebras act on the triplets, (v,ψ,χ), as a vector space symmetrically under triality. If we combine them and close the Lie algebra brackets consistently, we obtain the *triality Lie algebras*.

4 Triality Lie Algebras

A good way to understand precisely how triality algebras act on the triplets, (v, ψ, χ), is via their embedding in the corresponding triality Lie algebras [23, 24],

$$
\begin{aligned}
su(3) &= \mathrm{tri}(\mathbb{C}) + \mathbb{C} + \mathbb{C} + \mathbb{C} = u(1) + u(1) + (1+\bar{1})_v + (1+\bar{1})_m + (1+\bar{1})_p \\
sp(3) &= \mathrm{tri}(\mathbb{H}) + \mathbb{H} + \mathbb{H} + \mathbb{H} = su(2) + su(2) + su(2) + (2,2,1)_v + (2,1,2)_m + (1,2,2)_p \\
f_{4(-52)} &= \mathrm{tri}(\mathbb{O}) + \mathbb{O} + \mathbb{O} + \mathbb{O} = so(8) + 8_v + 8_{s-} + 8_{s+} \\
sl(3) &= \mathrm{tri}(\mathbb{C}') + \mathbb{C}' + \mathbb{C}' + \mathbb{C}' = gl(1) + gl(1) + (1+\bar{1})_v + (1+\bar{1})_m + (1+\bar{1})_p \\
sp(6,\mathbb{R}) &= \mathrm{tri}(\mathbb{H}') + \mathbb{H}' + \mathbb{H}' + \mathbb{H}' = sl(2) + sl(2) + sl(2) + (2,2,1)_v + (2,1,2)_m + (1,2,2)_p \\
f_{4(4)} &= \mathrm{tri}(\mathbb{O}') + \mathbb{O}' + \mathbb{O}' + \mathbb{O}' = so(4,4) + 8_v + 8_{s-} + 8_{s+}
\end{aligned}
$$

Generalized reflections, generalized rotations, and triality automorphisms, are real automorphisms of triality Lie algebras. The structures of these Lie algebras fully elucidate these symmetries, and are worth examining in each case.

4.1 $su(3)$

Lie algebra elements, A, corresponding to the special unitary group, $SU(3)$, may be represented by 3×3 traceless, anti-Hermitian matrices of complex numbers, related to the eight basis Gell-Mann matrices,

$$
\begin{aligned}
A(B^1, B^2, v, \psi, \chi) &= \begin{bmatrix} i\,B^1 + \frac{i}{\sqrt{3}}B^2 & -v^0 + i\,v^1 & \psi^0 + i\,\psi^1 \\ v^0 + i\,v^1 & -i\,B^1 + \frac{i}{\sqrt{3}}B^2 & -\chi^0 + i\,\chi^1 \\ -\psi^0 + i\,\psi^1 & \chi^0 + i\,\chi^1 & -\frac{2i}{\sqrt{3}}B^2 \end{bmatrix} \\
&= \begin{bmatrix} V - M & -v^* & \psi \\ v & P - V & -\chi^* \\ -\psi^* & \chi & M - P \end{bmatrix} \\
&= B^1 T_1 + B^2 T_2 + v^a \gamma_a + \psi^a Q_a^- + \chi^a Q_a^+ \\
&= V\,H_v + M\,H_m + P\,H_p \\
&\quad + (v\,E_v^- - v^* E_v^+) + (\psi\,E_m^- - \psi^* E_m^+) + (\chi\,E_p^- - \chi^* E_p^+) \\
&\in su(3)
\end{aligned} \tag{4.1}
$$

with $\{v, \psi, \chi\}$ complex parameters, $v = v^0 + i\,v^1 = v^0 e_0 + v^1 e_1$, $\{B^1, B^2\}$ real parameters, related $\{V, M, P\}$ pure imaginary parameters, $\{T_1, T_2, \gamma_a, Q_a^-, Q_a^+\}$ Lie algebra basis generators related to the Gell-Mann matrices, $\{H_v, H_m, H_p\}$ Cartan generators, and $\{E_v^\pm, E_m^\pm, E_p^\pm\}$ root vector generators. Note that since $su(3)$ elements are traceless, V, M, and P correspond to only two degrees of freedom, B^1 and B^2 — the same $su(3)$ element is specified if V, M, and P are all shifted by a constant. These V, M, and P parameters are motivated by the existence of three overlapping $su(2)$ subalgebras, spanned by $\{H_v, \gamma_a\}$, $\{H_m, Q_a^-\}$, and $\{H_p, Q_a^+\}$. As a triality Lie algebra, $su(3)$ relates to the complex division algebra representation of $Cl(0,2)$, with basis vectors $\gamma_0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\gamma_1 = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]$. This $\mathbb{C}$ division algebra representation of $Cl(0,2)$ is not the representation from the complex multiplication table, (2.1), which is instead, from $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$,

$$M_{00}{}^{\tilde{0}} = 1 \qquad M_{01}{}^{\tilde{1}} = M_{10}{}^{\tilde{1}} = -1 \qquad M_{11}{}^{\tilde{0}} = -1$$

$$\Gamma_0 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \qquad \Gamma_1 = \begin{bmatrix} & -1 \\ -1 & \end{bmatrix} \qquad \gamma_c = \begin{bmatrix} 0 & -\bar{\Gamma}_c \\ \Gamma_c & 0 \end{bmatrix}$$

In general, a triality Lie algebra can either be described directly as $\sim su(3,\mathbb{D})$, or equivalently by constructing the corresponding Clifford algebra and its bivector and vector matrix representatives (identified with the upper-left 2×2 block in $su(3,\mathbb{D})$) which act on negative and positive-chiral spinors, then closing the algebra via the Lie brackets between spinors. The representation of the Clifford algebra may be by division algebra elements, by their matrix representatives, or from the equivalent division algebra multiplication table coefficients.

We can compute the $su(3)$ Lie brackets directly from the commutator of its representative matrices, (4.1),

$$\left[A\left(B_1^1, B_1^2, v_1, \psi_1, \chi_1\right), A\left(B_2^1, B_2^2, v_2, \psi_2, \chi_2\right)\right] = A\left(B_3^1, B_3^2, v_3, \psi_3, \chi_3\right)$$

$$\begin{aligned}
2i\, B_3^1 &= -2(v_1^* v_2 - v_2^* v_1) - (\psi_1 \psi_2^* - \psi_2 \psi_1^*) - (\chi_1 \chi_2^* - \chi_2 \chi_1^*) \\
\tfrac{2i}{\sqrt{3}} B_3^2 &= (\chi_1 \chi_2^* - \chi_2 \chi_1^*) + (\psi_1^* \psi_2 - \psi_2^* \psi_1) \\
v_3 &= -2i(B_1^1 v_2 - B_2^1 v_1) + (\chi_1^* \psi_2^* - \chi_2^* \psi_1^*) \\
\psi_3 &= i(B_1^1 + \sqrt{3} B_1^2)\psi_2 - i(B_2^1 + \sqrt{3} B_2^2)\psi_1 + (v_1^* \chi_2^* - v_2^* \chi_1^*) \\
\chi_3 &= i(B_1^1 - \sqrt{3} B_1^2)\chi_2 - i(B_2^1 - \sqrt{3} B_2^2)\chi_1 + (\psi_1^* v_2^* - \psi_2^* v_1^*)
\end{aligned}$$

Alternatively, using our basis elements, $\{T_1, T_2, \gamma_a, Q_a^-, Q_a^+\}$, the non-zero $su(3)$ brackets between them are, explicitly,[2]

$$\begin{array}{lll}
[T_1, \gamma_0] = -2\,\gamma_1 & \left[T_1, Q_0^-\right] = +Q_1^- & \left[T_1, Q_0^+\right] = +Q_1^+ \\
[T_1, \gamma_1] = +2\,\gamma_0 & \left[T_1, Q_1^-\right] = -Q_0^- & \left[T_1, Q_1^+\right] = -Q_0^+ \\
 & \left[T_2, Q_0^-\right] = +\sqrt{3}\,Q_1^- & \left[T_2, Q_0^+\right] = -\sqrt{3}\,Q_1^+ \\
 & \left[T_2, Q_1^-\right] = -\sqrt{3}\,Q_0^- & \left[T_2, Q_1^+\right] = +\sqrt{3}\,Q_0^+ \\
[\gamma_0, \gamma_1] = -2\,T_1 & \left[Q_0^-, Q_1^-\right] = T_1 + \sqrt{3}\,T_2 & \left[Q_0^+, Q_1^+\right] = T_1 - \sqrt{3}\,T_2 \\
\left[\gamma_a, Q_b^-\right] = -M_{\tilde{a}\tilde{b}}{}^c\, Q_c^+ & \left[\gamma_a, Q_b^+\right] = M_{\tilde{a}\tilde{b}}{}^c\, Q_c^- & \left[Q_a^-, Q_b^+\right] = -M_{\tilde{a}\tilde{b}}{}^c\, \gamma_c
\end{array} \tag{4.2}$$

[2]These formula were constructed and checked with the assistance of symbolic manipulation software, Sage and/or Mathematica.

and their anti-symmeterized partners.

With orthogonal Cartan subalgebra basis generators, $\{T_1, T_2\}$, or non-orthogonal Cartan basis generators, $\{H_v = T_1, H_m = (-\frac{1}{2}T_1 - \frac{\sqrt{3}}{2}T_2), H_p = (-\frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2)\}$, the root vectors and their Lie brackets are:[2]

$$
\begin{aligned}
E_v^+ &= \tfrac{1}{2}(-\gamma_0 - i\,\gamma_1) & [T_1, E_\alpha^\pm] &= \pm i\, g_\alpha^1\, E_\alpha^\pm & [E_v^\pm, E_m^\pm] &= \mp E_p^\mp \\
E_v^- &= \tfrac{1}{2}(+\gamma_0 - i\,\gamma_1) & [T_2, E_\alpha^\pm] &= \pm i\, g_\alpha^2\, E_\alpha^\pm & \left[E_m^\pm, E_p^\pm\right] &= \mp E_v^\mp \\
E_m^+ &= \tfrac{1}{2}(-Q_0^- - i\,Q_1^-) & & & \left[E_p^\pm, E_v^\pm\right] &= \mp E_m^\mp \\
E_m^- &= \tfrac{1}{2}(+Q_0^- - i\,Q_1^-) & [H_v, E_\alpha^\pm] &= \pm i\, v_\alpha\, E_\alpha^\pm & [E_v^+, E_v^-] &= -i\, H_v \\
E_p^+ &= \tfrac{1}{2}(-Q_0^+ - i\,Q_1^+) & [H_m, E_\alpha^\pm] &= \pm i\, m_\alpha\, E_\alpha^\pm & [E_m^+, E_m^-] &= -i\, H_m \\
E_p^- &= \tfrac{1}{2}(+Q_0^+ - i\,Q_1^+) & [H_p, E_\alpha^\pm] &= \pm i\, p_\alpha\, E_\alpha^\pm & \left[E_p^+, E_p^-\right] &= -i\, H_p
\end{aligned}
$$

with the $\{g_\alpha^1, g_\alpha^2, v_\alpha, m_\alpha, p_\alpha\}$ roots shown in Table 1. This structure of $su(3)$ is consistent with its triality decomposition, in which each of the three triples, $\{H_{v/m/p}, E^+_{v/m/p}, E^-_{v/m/p}\} \sim \{H, E^+, E^-\}$, corresponds to a different $su(2)$, related to each other by triality, with disjoint root vectors but overlapping Cartan generators. The relevant triality function is,

$$T(v, \psi, \chi) = v^c \psi^a \chi^b M_{ab\tilde{c}} = v^0\psi^0\chi^0 - v^0\psi^1\chi^1 - v^1\psi^1\chi^0 - v^1\psi^0\chi^1$$

and a canonical inner triality automorphism of $su(3)$ is:

$$t \;:\; A \;\mapsto\; A' = g_t\, A\, g_t^- \qquad\qquad g_t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \;\in\; SU(3)$$

in which g_t is an element of the 3×3 representation of the $SU(3)$ Lie group, and t transforms the generators, root vectors, and Cartan subalgebra elements as:

$$t \quad : \quad \gamma_a \mapsto Q_a^- \mapsto Q_a^+ \mapsto \gamma_a \qquad E_v^\pm \mapsto E_m^\pm \mapsto E_p^\pm \mapsto E_v^\pm \qquad H_v \mapsto H_p \mapsto H_m \mapsto H_v$$

This triality automorphism corresponds to a rotation on root space coordinates, (g_α^1, g_α^2), by t^-, and a transformation of the Cartan subalgebra basis elements, $\{T_1, T_2\}$, by the *triality matrix*, t,

$$\begin{bmatrix} g_\alpha^{1\prime} \\ g_\alpha^{2\prime} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} g_\alpha^1 \\ g_\alpha^2 \end{bmatrix} \qquad \begin{bmatrix} T_1{}' \\ T_2{}' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \qquad t = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Within the triality algebra of $su(3)$, which is also its Cartan subalgebra, the two basis generators, $\{T_1, T_2\}$, are each rotated between three directions by a canonical triality automorphism. Specifically, $T_1^I = H_v = T_1$, $T_1^{II} = H_p = -\frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2$, and $T_1^{III} = H_m = -\frac{1}{2}T_1 - \frac{\sqrt{3}}{2}T_2$.

Another way of looking at the decomposition of $su(3)$ is by grouping any $u(1)$ subalgebra of the $u(1)+u(1)$ triality subalgebra with the corresponding $E^\pm$ root vector pair to make a $su(2)$ subalgebra, resulting in the decomposition:

$$
\begin{aligned}
su(3) &= u(1) + (u(1) + (1 + \bar{1})_v) + (1 + \bar{1})_m + (1 + \bar{1})_p \\
&= u(1) + su(2) + 2_{+\sqrt{3}} + 2_{-\sqrt{3}}
\end{aligned}
$$

$su(3)$		g^1	g^2	v	m	p
▲	v^+	$+2$	0	$+2$	-1	-1
▼	v^-	-2	0	-2	$+1$	$+1$
▲	m^+	-1	$-\sqrt{3}$	-1	$+2$	-1
▼	m^-	$+1$	$+\sqrt{3}$	$+1$	-2	$+1$
▲	p^+	-1	$+\sqrt{3}$	-1	-1	$+2$
▼	p^-	$+1$	$-\sqrt{3}$	$+1$	$+1$	-2

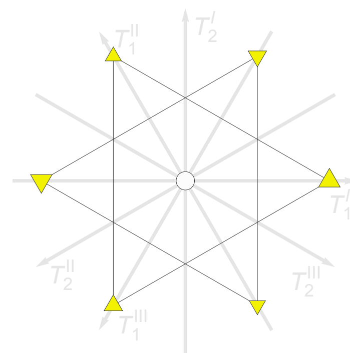


Table 1. Roots of $su(3)$ with respect to the orthogonal Cartan subalgebra basis generators, $\{T_1, T_2\}$, or non-orthogonal Cartan basis generators, $\{H_v, H_m, H_p\}$, and their canonical triality automorphism. The "particle" roots related by triality are shown as yellow triangles, ▲, of different sizes, while their "antiparticle" roots related by triality are inverted triangles, ▼.

in which the $\pm\sqrt{3}$ are the $u(1)$ charges, equal to the g^2 charges visible in the figure and Table 1. This $su(2)$ subalgebra of $su(3)$ corresponds to the upper left 2×2 block of the 3×3 matrix representation, (4.1), spanned by the T_1 and γ_a generators. A canonical triality automorphism of $su(3)$ with this decomposition produces three different $su(2)$ subalgebras.

The split-complex numbers, a composition algebra, are represented by $\{e'_0 = 1, e'_1 = I\}$, with $M'_{11}{}^0 = I^2 = 1$. Repeating our Lie algebra construction, using real Gell-Mann matrices, we get the Lie algebra $sl(3)$, with non-vanishing brackets:[2]

$$\begin{array}{lll}
[T'_1, \gamma'_0] = -2\,\gamma'_1 & [T'_1, Q'^-_0] = +Q'^-_1 & [T'_1, Q'^+_0] = +Q'^+_1 \\
[T'_1, \gamma'_1] = -2\,\gamma'_0 & [T'_1, Q'^-_1] = +Q'^-_0 & [T'_1, Q'^+_1] = +Q'^+_0 \\
 & [T'_2, Q'^-_0] = +\sqrt{3}\,Q'^-_1 & [T'_2, Q'^+_0] = -\sqrt{3}\,Q'^+_1 \\
 & [T'_2, Q'^-_1] = +\sqrt{3}\,Q'^-_0 & [T'_2, Q'^+_1] = -\sqrt{3}\,Q'^+_0 \\
[\gamma'_0, \gamma'_1] = -2\,T'_1 & [Q'^-_0, Q'^-_1] = T'_1 + \sqrt{3}\,T'_2 & [Q'^+_0, Q'^+_1] = T'_1 - \sqrt{3}\,T'_2 \\
[\gamma'_a, Q'^-_b] = -M'_{\tilde{a}\tilde{b}}{}^c\, Q'^+_c & [\gamma'_a, Q'^+_b] = M'_{\tilde{a}\tilde{b}}{}^c\, Q'^-_c & [Q'^-_a, Q'^+_b] = -M'_{\tilde{a}\tilde{b}}{}^c\, \gamma'_c
\end{array} \tag{4.3}$$

The triality automorphism structure for $sl(3)$ is the same as for $su(3)$.

4.2 $sp(3)$

The structure of the 21-dimensional symplectic Lie algebra, $sp(3)$ — the triality Lie algebra of the quaternions — is similar to that of $su(3)$. Instead of 3×3 traceless matrices of complex numbers, elements of $sp(3)$ can be represented by matrices of quaternions,

$$A(M, P, V, v, \psi, \chi) = \begin{bmatrix} M & -\tilde{v} & \psi \\ v & P & -\tilde{\chi} \\ -\tilde{\psi} & \chi & V \end{bmatrix} \in sp(3) = su(3, \mathbb{H}) \tag{4.4}$$

with $\{v,\psi,\chi\}$ quaternions and $\{M,P,V\}$ purely imaginary quaternions. The Lie brackets are thus:

$$[A(M_1,P_1,V_1,v_1,\psi_1,\chi_1),A(M_2,P_2,V_2,v_2,\psi_2,\chi_2)]=A(M_3,P_3,V_3,v_3,\psi_3,\chi_3)$$

$$\begin{aligned}
M_3 &= M_1M_2 - M_2M_1 - (\tilde{v}_1 v_2 - \tilde{v}_2 v_1) - (\psi_1\tilde{\psi}_2 - \psi_2\tilde{\psi}_1)\\
P_3 &= P_1P_2 - P_2P_1 - (\tilde{\chi}_1\chi_2 - \tilde{\chi}_2\chi_1) - (v_1\tilde{v}_2 - v_2\tilde{v}_1)\\
V_3 &= V_1V_2 - V_2V_1 - (\tilde{\psi}_1\psi_2 - \tilde{\psi}_2\psi_1) - (\chi_1\tilde{\chi}_2 - \chi_2\tilde{\chi}_1)\\
v_3 &= (P_1v_2 - P_2v_1) + (v_1M_2 - v_2M_1) + (\tilde{\chi}_1\tilde{\psi}_2 - \tilde{\chi}_2\tilde{\psi}_1)\\
\psi_3 &= (M_1\psi_2 - M_2\psi_1) + (\psi_1V_2 - \psi_2V_1) + (\tilde{v}_1\tilde{\chi}_2 - \tilde{v}_2\tilde{\chi}_1)\\
\chi_3 &= (V_1\chi_2 - V_2\chi_1) + (\chi_1P_2 - \chi_2P_1) + (\tilde{\psi}_1\tilde{v}_2 - \tilde{\psi}_2\tilde{v}_1)
\end{aligned}$$

Each of the three diagonal matrix elements is a $su(2)$ subalgebra, which act on two out of the three off-diagonal elements. The Cartan subalgebra basis consists of one element from each diagonal element, $\{T_3^M, T_3^P, T_3^V\}$.

The $sp(3)$ Lie algebra, and its brackets, relate to the Clifford algebra matrix representation of $Cl(0,4)$ from the quaternion multiplication table, as in (2.2), in which case (4.4) gives a 12×12 real representation. Or, using the usual Pauli matrix representation of quaternions,

$$e_0=\sigma_0 \qquad e_1=-i\,\sigma_1 \qquad e_2=-i\,\sigma_2 \qquad e_3=-i\,\sigma_3$$

(4.4) results in a 6×6 complex representation. Using the quaternion mulitplication table, (2.1), the $Cl(0,4)$ chiral representative matrices (2.2) are $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ and $-(\bar{\Gamma}_c)^a{}_b = -M_{\tilde{c}\tilde{b}}{}^a$, so

$$v \sim v^c\Gamma_c = \begin{bmatrix} v^0 & v^1 & v^2 & v^3\\ -v^1 & v^0 & -v^3 & v^2\\ -v^2 & v^3 & v^0 & -v^1\\ -v^3 & -v^2 & v^1 & v^0 \end{bmatrix} \qquad -\tilde{v} \sim -v^c\bar{\Gamma}_c = \begin{bmatrix} -v^0 & v^1 & v^2 & v^3\\ -v^1 & -v^0 & -v^3 & v^2\\ -v^2 & v^3 & -v^0 & -v^1\\ -v^3 & -v^2 & v^1 & -v^0 \end{bmatrix}$$

The negative and positive $Cl(0,4)$ chiral bivector matrices separate into independent degrees of freedom, corresponding to $so(4)=su(2)_M+su(2)_P$,

$$M \sim B_M = -\tfrac{1}{2}B^{ab}\bar{\Gamma}_a\Gamma_b = B_M^A\,\Gamma_A \qquad P \sim B_P = -\tfrac{1}{2}B^{ab}\Gamma_a\bar{\Gamma}_b = B_P^A\,\Gamma_A$$

in which $B^A_{M/P} = \mp B^{0A} + \frac{1}{2}\epsilon^{BCA}B^{BC}$, and the capital indices, $\{A,B,C\}$, are bivector indices, ranging over $\{1,2,3\}$.

The $sp(3)$ Lie algebra elements can be written in an orthoganal, Killing-normalized basis as

$$\begin{aligned}
A(M,P,V,v,\psi,\chi) &= M^AT_A^M + P^AT_A^P + V^AT_A^V + v^a\gamma_a + \psi^aQ_a^- + \chi^aQ_a^+\\
&= \begin{bmatrix} M^Ae_A & -v^a\frac{1}{\sqrt{2}}\tilde{e}_a & \psi^a\frac{1}{\sqrt{2}}e_a\\ v^a\frac{1}{\sqrt{2}}e_a & P^Ae_A & -\chi^a\frac{1}{\sqrt{2}}\tilde{e}_a\\ -\psi^a\frac{1}{\sqrt{2}}\tilde{e}_a & \chi^a\frac{1}{\sqrt{2}}e_a & V^Ae_A \end{bmatrix}
\end{aligned}$$

with the $sp(3)$ Lie brackets between these basis generators computed explicitly:[2]

$$
\begin{aligned}
\left[T_A^M, T_B^M\right] &= T_C^M (2M_{[AB]}{}^C) & \left[\gamma_a, Q_b^-\right] &= Q_c^+ \tfrac{1}{\sqrt{2}}(-M_{\tilde{b}\tilde{a}}{}^c) \\
\left[T_A^P, T_B^P\right] &= T_C^P (2M_{[AB]}{}^C) & \left[\gamma_a, Q_b^+\right] &= Q_c^- \tfrac{1}{\sqrt{2}}(M_{\tilde{a}\tilde{b}}{}^c) \\
\left[T_A^V, T_B^V\right] &= T_C^V (2M_{[AB]}{}^C) & \left[Q_a^-, Q_b^+\right] &= \gamma_c \tfrac{1}{\sqrt{2}}(-M_{\tilde{b}\tilde{a}}{}^c) \\
\left[T_A^M, \gamma_b\right] &= \gamma_c\,(-M_{bA}{}^c) & \left[T_A^M, Q_b^-\right] &= Q_c^-\,(M_{Ab}{}^c) \\
\left[T_A^P, \gamma_b\right] &= \gamma_c\,(M_{Ab}{}^c) & \left[T_A^P, Q_b^+\right] &= Q_c^+\,(-M_{bA}{}^c) \\
\left[T_A^V, Q_b^-\right] &= Q_c^-\,(-M_{bA}{}^c) & \left[T_A^V, Q_b^+\right] &= Q_c^+\,(M_{Ab}{}^c) \\
\left[\gamma_a, \gamma_b\right] &= T_C^M \tfrac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T_C^P \tfrac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C) \\
\left[Q_a^-, Q_b^-\right] &= T_C^V \tfrac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T_C^M \tfrac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C) \\
\left[Q_a^+, Q_b^+\right] &= T_C^P \tfrac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T_C^V \tfrac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C)
\end{aligned}
\tag{4.5}
$$

$sp(3)$		M	P	V
	$M^\pm$	± 2	0	0
	$P^\pm$	0	± 2	0
	$V^\pm$	0	0	± 2
	$\psi_e^\pm$	± 1	0	± 1
	$\psi_o^\pm$	± 1	0	∓ 1
	$\chi_e^\pm$	0	± 1	± 1
	$\chi_o^\pm$	0	∓ 1	± 1
	$v_e^\pm$	± 1	± 1	0
	$v_o^\pm$	∓ 1	± 1	0

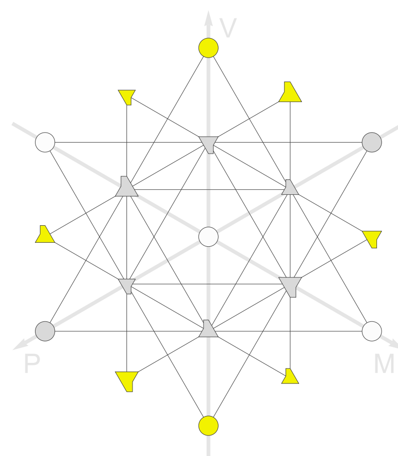


Table 2. Roots of $sp(3)$ with respect to the orthogonal Cartan subalgebra basis generators, $\{T_3^M, T_3^P, T_3^V\}$, and their canonical triality automorphism. Roots are labeled with circles, , of different colors for each triality-related $su(2)$, with upper-left triangles of different colors, , for even or odd charge combinations, and lower-right inverted triangles for their antiparticles, , with triangle glyphs of different sizes related by triality.

Generalized quaternionic reflections give automorphisms of $sp(3)$. For example, the generalized reflection along a unit positive spinor, R_p^0, gives

$$
R_p^0 \;:\; A(M,P,V,v,\psi,\chi) \;\mapsto\; A(M',P',V',v',\psi',\chi') \;=\; A(M,V,P,\tilde{\psi},\tilde{v},-\tilde{\chi}) \tag{4.6}
$$

which is an inner automorphism,

$$
R_p^0 \;\;:\;\; A \;\mapsto\; A' = g_R\, A\, g_R^- \qquad\qquad g_R = \begin{bmatrix} -1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \;\in\; SP(3) = SU(3,\mathbb{H})
$$

This corresponds to a reflection in root space coordinates, $(\alpha_M, \alpha_P, \alpha_V)$, by the R matrix, and a transformation of the Cartan subalgebra basis elements, $\{M, P, V\}$, by R,

$$\begin{bmatrix} \alpha'_M \\ \alpha'_P \\ \alpha'_V \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} \alpha_M \\ \alpha_P \\ \alpha_V \end{bmatrix} = \begin{bmatrix} \alpha_M \\ \alpha_V \\ \alpha_P \end{bmatrix} \qquad \begin{bmatrix} M' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} M \\ P \\ V \end{bmatrix} = \begin{bmatrix} M \\ V \\ P \end{bmatrix} \qquad R = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

This reflection visually reflects the $sp(3)$ root diagram in Table 2 across the M axis.

The collection of all possible generalized quaternion reflections, including their compositions, is the automorphism group, $SP(3)$, of the $sp(3)$ Lie algebra. A canonical triality automorphism of $sp(3)$ is:

$$t \quad : \quad \gamma_a \mapsto Q_a^- \mapsto Q_a^+ \mapsto \gamma_a \qquad T_A^M \mapsto T_A^V \mapsto T_A^P \mapsto T_A^M$$

which, as a composition of reflections, $t = R_p^0 R_v^0 R_m^0 R_p^0$, is an inner automorphism,

$$t \;:\; A \;\mapsto\; A' = g_t \, A \, g_t^- \qquad g_t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \;\in\; SP(3) = SU(3, \mathbb{H})$$

This corresponds to a rotation in root space coordinates, $(\alpha_M, \alpha_P, \alpha_V)$, by t^-, and a transformation of the Cartan subalgebra basis elements, $\{M, P, V\}$, by the triality matrix, t,

$$\begin{bmatrix} \alpha'_M \\ \alpha'_P \\ \alpha'_V \end{bmatrix} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} \alpha_M \\ \alpha_P \\ \alpha_V \end{bmatrix} = \begin{bmatrix} \alpha_P \\ \alpha_V \\ \alpha_M \end{bmatrix} \qquad \begin{bmatrix} M' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} M \\ P \\ V \end{bmatrix} = \begin{bmatrix} V \\ M \\ P \end{bmatrix} \qquad t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

Although one can describe these sorts of Lie algebra automorphisms as reflections or rotations of roots in root space, which correspond to reflections and rotations within the Cartan subalgebra and maps between root vectors, these descriptions do not specify the phases of the maps between root vectors. To obtain a consistent set of phases for such maps, it is usually easiest to describe the automorphisms directly, as transformations of the Lie algebra generators, and then transform to the Cartan-Weyl basis to get a complete description of the maps between root vectors, including phases.

Another way of looking at the decomposition of $sp(3)$ is by grouping any $su(2)+su(2)=so(4)$ subalgebra of the $su(3)+su(2)+su(2)$ triality subalgebra with the corresponding root vectors to make a $so(5)=sp(2)$ subalgebra, resulting in the decomposition:

$$\begin{aligned} sp(3) &= (su(2)_M + su(2)_P + (2,2,1)_v) + su(2)_V + (2,1,2)_m + (1,2,2)_p \\ &= sp(2) + su(2)_V + 4 \otimes 2 \end{aligned}$$

which can be seen in the figure and Table 2. This $sp(2)$ subalgebra of $sp(3)$ corresponds to the upper left 2×2 block of the 3×3 matrix representation, (4.4), spanned by the T_A^M, T_A^P, and γ_a generators. A canonical triality automorphism of $sp(3)$ with this decomposition produces three different $sp(2)$ subalgebras.

To describe the split real form, $sp(6,\mathbb{R})$, of the $sp(3)$ Lie algebra, we can use split-quaternions in (4.4) instead of quaternions, representing the basis elements as 2×2 real matrices,

$$e'_0 = \sigma_0 \qquad e'_1 = \sigma_1 \qquad e'_2 = -i\,\sigma_2 \qquad e'_3 = \sigma_3$$

Automorphisms of this split real Lie algebra are similar to those of compact $sp(3)$; however, an interesting difference can occur. For example, a reflection, R^3_v, around the $u = e'_3$ unit split-quaternion vector has $s_u = -1$, and the corresponding reflection element,

$$g_R = \begin{bmatrix} & -ie'_3 & \\ ie'_3 & & \\ & & -1 \end{bmatrix} \in SP(6,\mathbb{R}) = SU(3,\mathbb{H}')$$

produces a real $sp(6,\mathbb{R})$ inner automorphism,

$$M' = e'_3 P e'_3 \qquad P' = e'_3 M e'_3 \qquad V' = V \qquad v' = e'_3 \tilde{v} e'_3 \qquad \psi' = -ie'_3\tilde{x} \qquad \chi' = -i\tilde{\psi}e'_3$$

which doesn't look like a real automorphism, but is. This will be further discussed in the next section.

4.3 f_4

The triality Lie algebra of the octonions, f_4, has a similar structure to that of $sp(3)$:

$$\begin{aligned} A(B,v,\psi,\chi) &= \tfrac{1}{2}B^{ab}\gamma_{ab} + v^a\gamma_a + \psi^a Q^-_a + \chi^a Q^+_a \\ &= \begin{bmatrix} B_M & -\tilde{v} & \psi \\ v & B_P & -\tilde{\chi} \\ -\tilde{\psi} & \chi & B_V \end{bmatrix} \;\tilde{\in}\; su(3,\mathbb{O}) \sim f_4 \end{aligned} \tag{4.7}$$

but cannot be directly identified with $su(3,\mathbb{O})$ because of octonion non-associativity. To make the identification of elements of f_4 with matrices, (4.7), we need to define multiplication compositionally — with octonions multiplying to their right before they are then multiplied by octonions from their left. The f_4 representative matrix is then made of octonions and octonion bi-products. The matrix representation of $Cl(0,8)$ vectors, bivectors, and chiral spinors, from compositional octonionic multiplication, is

$$B = \tfrac{1}{2}B^{ab}\gamma_{ab} = \begin{bmatrix} -\frac{1}{2}B^{ab}\bar{\Gamma}_a\Gamma_b & \\ & -\frac{1}{2}B^{ab}\Gamma_a\bar{\Gamma}_b \end{bmatrix} \qquad v = v^c\gamma_c = \begin{bmatrix} & -v^c\bar{\Gamma}_c \\ v^c\Gamma_c & \end{bmatrix} \qquad \Psi = \begin{bmatrix} \psi^a Q^-_a \\ \chi^b Q^+_b \end{bmatrix}$$

using the octonion multiplication table, (2.1), and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$. The $su(3,\mathbb{O})$-inspired Lie brackets for the f_4 basis generators are:[2]

$$\begin{aligned}
[\gamma_{ab},\gamma_{cd}] &= 2\,\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} \\
[\gamma_{ab},\gamma_c] &= 2\,\{-n_{bc}\gamma_a + n_{ac}\gamma_b\} & [\gamma_a,\gamma_b] &= 2\,\gamma_{ab} \\
[\gamma_{ab},Q_c^-] &= Q_d^-(-\bar\Gamma_a\Gamma_b)^d{}_c & [\gamma_c,Q_a^-] &= Q_b^+(\Gamma_c)^b{}_a \\
[\gamma_{ab},Q_c^+] &= Q_d^+(-\Gamma_a\bar\Gamma_b)^d{}_c & \left[\gamma_c,Q_b^+\right] &= Q_a^-(-\bar\Gamma_c)^a{}_b \\
\left[Q_a^-,Q_b^-\right] &= \gamma_{cd}(\mp\bar\Gamma^c\Gamma^d)_{ab} & \left[Q_a^-,Q_b^+\right] &= \gamma_c(\mp\bar\Gamma^c)_{ab} \\
\left[Q_a^+,Q_b^+\right] &= \gamma_{cd}(\mp\Gamma^c\bar\Gamma^d)_{ab}
\end{aligned} \tag{4.8}$$

in which $n_{ab} = \delta_{ab}$ is used to raise or lower indices, and we have 28 bivector generators, γ_{ab}, identified with $-\gamma_{ba}$ for $a < b$. The choice of "$-$" signs above, in the "$\mp$", corresponds to the compact real form, $f_{4(-52)}$, while the choice of "+" signs gives $f_{4(-20)}$. For the split real form, $f_{4(4)}$, we use the split-octonionic representation of $Cl(4,4)$, with the split metric, $n'_{ab} = \mathrm{diag}(++++----)$. The positive and negative spinor metrics come from multiplying the positive signature Clifford vectors, $\gamma_0\gamma_1\gamma_2\gamma_3$, giving $n'^-_{ab} = n'^+_{ab} = n'_{ab}$. Using this Clifford representation and these metrics, the $f_{4(4)}$ brackets are the same as for $f_{4(-52)}$, above, except for some signs,

$$\left[Q_a^-,Q_b^+\right] = \gamma_c(-\bar\Gamma^c)_{ab} \qquad \left[Q_a^-,Q_b^-\right] = \gamma_{cd}(+\bar\Gamma^c\Gamma^d)_{ab} \qquad \left[Q_a^+,Q_b^+\right] = \gamma_{cd}(+\Gamma^c\bar\Gamma^d)_{ab}$$

For a canonical triality automorphism to exist, the metrics of the vectors, negative spinors, and positive spinors in $f_{4(-52)}$ or $f_{4(4)}$ must be equal and match the Killing form.

A canonical triality automorphism between vectors and spinors,

$$t \quad : \quad \gamma_a \mapsto Q_a^- \mapsto Q_a^+ \mapsto \gamma_a$$

is an automorphism of f_4, with corresponding automorphisms of its $so(8)$ subalgebra,

$$t \quad : \quad \gamma_{ab} \;\mapsto\; \gamma'_{ab} = \gamma_{cd}\,t^{cd}{}_{ab} = \tfrac{1}{2}\left[\gamma'_a,\gamma'_b\right] = \tfrac{1}{2}\left[Q_a^-,Q_b^-\right] = \gamma_{cd}\left(\mp\tfrac{1}{2}\bar\Gamma^c\Gamma^d\right)_{ab}$$

and

$$t^2 \quad : \quad \gamma_{ab} \;\mapsto\; \gamma''_{ab} = \gamma_{cd}\,t^{2\;cd}{}_{ab} = \tfrac{1}{2}\left[\gamma''_a,\gamma''_b\right] = \tfrac{1}{2}\left[Q_a^+,Q_b^+\right] = \gamma_{cd}\left(\mp\tfrac{1}{2}\Gamma^c\bar\Gamma^d\right)_{ab}$$

in which $a < b$ and the sum is over the 28 basis bivectors, $c < d$.

The description of f_4 can be made more explicitly octonionic by remembering that $\bar\Gamma_a\Gamma_b$ comes from multiplying to the right by e_b then multiplying to the right by $\tilde{e}_a$, so each $so(8)$ chiral bivector element corresponds to a sum of compositional octonion multiplications, $B_- \sim -\frac{1}{2}B^{ab}\tilde{e}_a e_b$. A canonical triality automorphism of $so(8)$ thus corresponds to a transformation of compositional multiplications of octonions,

$$t(\tilde{e}_a e_b) = t_{ab}{}^{cd}\,\tilde{e}_c e_d = e_a\tilde{e}_b$$

We can then describe the diagonal entries of the $\sim su(3,\mathbb{O})$ matrix explicitly as:

$$\begin{aligned} B_M &= -\tfrac{1}{2}B^{ab}\tilde{e}_a e_b \\ B_P &= t(B_M) = -\tfrac{1}{2}B^{ab}t_{ab}{}^{cd}\,\tilde{e}_c e_d = -\tfrac{1}{2}B^{ab}e_a\tilde{e}_b \\ B_V &= t(B_P) = t^2(B_M) = -\tfrac{1}{2}B^{ab}t_{ab}{}^{cd}\,e_c\tilde{e}_d = -\tfrac{1}{2}B^{ab}t^2_{ab}{}^{cd}\,\tilde{e}_c e_d \end{aligned}$$

This allows us to write the Lie brackets between $f_{4(-52)}$ elements,

$$[\,A(B_1,v_1,\psi_1,\chi_1)\,,A(B_2,v_2,\psi_2,\chi_2)\,] = A(B_3,v_3,\psi_3,\chi_3)$$

using compositional octonionic multiplication and triality [30],

$$\begin{aligned} B_3 &= B_1B_2 - B_2B_1 - (\tilde{v}_1v_2 - \tilde{v}_2v_1) - t^2(\tilde{\psi}_1\psi_2 - \tilde{\psi}_2\psi_1) - t(\tilde{\chi}_1\chi_2 - \tilde{\chi}_2\chi_1) \\ v_3 &= t^2(B_1)v_2 - t^2(B_2)v_1 + \tilde{\chi}_1\tilde{\psi}_2 - \tilde{\chi}_2\tilde{\psi}_1 \\ \psi_3 &= B_1\psi_2 - B_2\psi_1 + \tilde{v}_1\tilde{\chi}_2 - \tilde{v}_2\tilde{\chi}_1 \\ \chi_3 &= t(B_1)\chi_2 - t(B_2)\chi_1 + \tilde{\psi}_1\tilde{v}_2 - \tilde{\psi}_2\tilde{v}_1 \end{aligned} \tag{4.9}$$

in which the bivectors here are pairs of octonions multiplying to the right in order, such as $B_1\psi_2 = -\frac{1}{2}B_1^{ab}\psi_2^c\,\tilde{e}_a(e_be_c)$.

Generalized reflection symmetries of $f_{4(-52)}$, $f_{4(-20)}$, or $f_{4(4)}$, through a space-like or time-like unit vector, u, are real Lie algebra automorphisms, and their combinations comprise the corresponding F_4 Lie group. As F_4 group elements acting via their adjoint action on Lie algebra elements represented as elements of $su(3,\mathbb{O})$, (4.7), the three types of generalized reflections are

$$R^u_v : \begin{bmatrix} & \sqrt{s_u}\tilde{u} & \\ \sqrt{s_u}u & & \\ & & -1 \end{bmatrix} \qquad R^u_m : \begin{bmatrix} & & \sqrt{s_u}u \\ & -1 & \\ \sqrt{s_u}\tilde{u} & & \end{bmatrix} \qquad R^u_p : \begin{bmatrix} -1 & & \\ & & \sqrt{s_u}\tilde{u} \\ & \sqrt{s_u}u & \end{bmatrix}$$

Explicitly, in addition to the action of R^u_v, R^u_m, and R^u_p on f_4 vector and chiral spinor basis generators, (3.3), these each extend to corresponding actions on the $so(8)$ or $so(4,4)$ subalgebra basis generators,

$$\begin{aligned} R^u_v &: \gamma_{ab} \mapsto \gamma'_{ab} = \tfrac{1}{2}\,[R^u_v\gamma_a, R^u_v\gamma_b] = u\gamma_{ab}u = \tfrac{1}{2}(\delta^c_a - 2s_uu^cu_a)(\delta^d_b - 2s_uu^du_b)\gamma_{cd} \\ R^u_m &: \gamma_{ab} \mapsto \gamma'_{ab} = \tfrac{1}{2}\,[R^u_m\gamma_a, R^u_m\gamma_b] = \tfrac{1}{2}s_uu^e(\Gamma_e)^c{}_a u^f(\Gamma_f)^d{}_b(\mp\Gamma^g\bar{\Gamma}^h)_{cd}\gamma_{gh} \\ R^u_p &: \gamma_{ab} \mapsto \gamma'_{ab} = \tfrac{1}{2}\left[R^u_p\gamma_a, R^u_p\gamma_b\right] = \tfrac{1}{2}s_uu^e(\bar{\Gamma}_e)^c{}_a u^f(\bar{\Gamma}_f)^d{}_b(\mp\bar{\Gamma}^g\Gamma^h)_{cd}\gamma_{gh} \end{aligned}$$

producing automorphisms of the f_4 Lie algebra described via these maps of basis generators.

For triality automorphisms, from (3.4), using unit vectors u and w, we have t^{uw} : $A(\gamma_c, Q^-_a, Q^+_b, \gamma_{ab}) \mapsto A(\gamma'_c, {Q^-_a}', {Q^+_b}', \gamma'_{ab})$, with

$$\begin{aligned}
\gamma'_c &= \sqrt{s_u s_w} w^d (\bar\Gamma_d)^f{}_b u^a (\Gamma_a)^b{}_c Q^-_f \\
{Q^-_a}' &= \sqrt{s_u s_w} w^c (\Gamma_c)^f{}_d u^b (\bar\Gamma_b)^d{}_a Q^+_f \\
{Q^+_b}' &= (\delta^a_b - 2 s_w w^a w_b)(\delta^c_a - 2 s_u u^c u_a)\gamma_c \\
\gamma'_{ab} &= \tfrac{1}{2}\left[\gamma'_a, \gamma'_b\right] = \tfrac{1}{2} s_u s_w w^d (\bar\Gamma_d)^k{}_i u^c (\Gamma_c)^i{}_a w^f (\bar\Gamma_f)^m{}_j u^e (\Gamma_e)^j{}_b (\mp \bar\Gamma^g \Gamma^h)_{km} \gamma_{gh}
\end{aligned}$$

As an F_4 Lie group element acting via the adjoint, this typical generalized triality element is

$$t^{uw} = R^w_p R^u_v R^{\tilde u}_m R^{\tilde u}_p \quad : \quad \begin{bmatrix} & & 1 \\ \sqrt{s_u s_w} w \tilde u & & \\ & \sqrt{s_u s_w} \tilde w u & \end{bmatrix}$$

with it again understood that this compositional multiplication operation is ordered to act right-first — for example, $w\tilde u \psi = w(\tilde u \psi)$.

Note that real automorphisms of $f_{4(-20)}$ or the split real Lie algebra, $f_{4(4)}$, such as reflections, R^u_v, through a unit time-like vector, $s_u = -1$, or triality from a unit time-like u and space-like w, have explicit $\sqrt{s_u} = i$'s in them. This might appear to contradict the fact that these are real automorphisms, but this is not the case. For example, if we consider complex f_4 as $f_{4(4)}$ with complex coefficients, then the usual anti-linear complex conjugation operator, $\sigma = K$, acting on complex f_4 produces $f_{4(4)}$ as the invariant real subspace — the $f_{4(4)}$ real form. Reflections through a unit time-like vector, $\phi = R^u_v$, which have i's in them, are complex and not real automorphisms with respect to σ because $R^u_v K \neq K R^u_v$. However, if we consider the reflection, R^0_v, through the unit octonion or space-like Clifford vector, γ_0, which is a real involutive automorphism of $f_{4(4)}$, then an alternative anti-linear complex conjugation operator on complex f_4 exists, $\sigma' = R^0_v K$, which gives the same real form, $f_{4(4)}$, and determines that $\phi = R^u_v$ with a time-like u and $s_u = -1$ *is* a real automorphism of $f_{4(4)}$, because

$$\phi\,\sigma' = R^u_v R^0_v K = R^0_v K R^u_v = \sigma' \phi$$

The i's in R^u_v are precisely matched with the swapping of the conjugate Q^+ and Q^- generators by R^0_v in just such a way that this works. This same argument goes through for R^u_m, R^u_p, and triality automorphisms — which can all have explicit i's in them while being real automorphisms.

A canonical triality automorphism matrix, t, for $so(8)$ or $so(4,4)$ is a 28×28 rotation matrix, satisfying $tt^T = 1$, of real coefficients,

$$t^{cd}{}_{ab} = \left(\mp \tfrac{1}{2} \bar\Gamma^c \Gamma^d\right)_{ab} = \mp \tfrac{1}{2} M^{\tilde c}{}_{ea} M^d{}_b{}^e$$

derived from the octonionic multiplication table, with the sign "$-$" for compact $f_{4(-52)}$ and otherwise "$+$", and the 28 basis bivectors, $\gamma_\alpha = \gamma_{ab}$, indexed by $1 \leq \alpha = \frac{1}{2}(13a - a^2) + b \leq 28$, with $0 \le a < b \le 7$. (The larger, 64×64 triality automorphism matrix elements are also defined for $b \ge a$, and can be deduced from anti-symmetry.) The 28 basis bivectors can be separated into 7 disjoint sets of 4 intra-commuting basis bivectors, each set spanning a Cartan subalgebra intra-rotated by triality. The triality automorphism matrix, $t_\alpha{}^\beta$, can thus be re-ordered to be block diagonal, consisting of seven 4×4 Hadamard matrices. Typical blocks looks like:

$$t_1 = \begin{bmatrix} +1/2 & -1/2 & +1/2 & +1/2 \\ +1/2 & -1/2 & -1/2 & -1/2 \\ +1/2 & +1/2 & +1/2 & -1/2 \\ +1/2 & +1/2 & -1/2 & +1/2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} -1/2 & +1/2 & +1/2 & +1/2 \\ -1/2 & +1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & +1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & +1/2 \end{bmatrix}$$

but signs may vary, based on the octonionic multiplication table, or if we use a non-canonical triality automorphism.

For $so(4,4)$ triality automorphisms, the four bivectors, γ_{ab}, spanning each Cartan subalgebra rotated by triality can be constructed from Clifford basis vectors with space-space, time-time, or space-time signature. The triality-allowed signature sets for $so(4,4)$ Cartan bivectors are $\{ss, ss, tt, tt\}$ or $\{st, st, st, st\}$. If we choose one of these triality-adapted Cartan subalgebras for our f_4 Cartan-Weyl decomposition, then t is the triality matrix that rotates these Cartan basis generators, and $t^- = t^2 = t^T$ rotates the root coordinates.

There are two especially interesting f_4 Cartan subalgebra transformations we can do that emphasize the $sp(3)$ and $su(3)$ subalgebras of f_4, matching their previously described triality automorphisms:

$$c_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & & \\ 1/\sqrt{2} & 1/\sqrt{2} & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad c_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

$$t'_1 = c_1\, t_1\, c_1^- = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix} \qquad t'_2 = c_2\, t_2\, c_2^- = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & & \\ \sqrt{3}/2 & -1/2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

From t'_1 we see that triality cycles three $su(2)'s$ in f_4, while leaving a complimentary $su(2)$ invariant, matching $sp(3)$ triality; while from t'_2 we see that triality rotates in a single plane in 4-dimensional root space, matching $su(3)$ triality, while leaving a complimentary $su(3)$ invariant. The corresponding decompositions of $f_{4(-52)}$ are:

$$\begin{aligned} f_{4(-52)} &= so(8) + 8_v + 8_{s-} + 8_{s+} \\ &= su(2)_L + su(2)_R + su(2)_{W'} + su(2)_W + (2,2,2,2) \\ &\quad + ((2,2,1,1) + (1,1,2,2))_v + ((2,1,2,1) + (1,2,1,2))_{s-} \\ &\quad + ((1,2,2,1) + (2,1,1,2))_{s+} \end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
f_{4(-52)} &= so(8) + 8_v + 8_{s-} + 8_{s+} \\
&= u(1)_p + u(1)_B + su(3)_g + 3_I + \bar{3}_I + 3_{II} + \bar{3}_{II} + 3_{III} + \bar{3}_{III} \\
&\quad + (1 + \bar{1} + 3 + \bar{3})_v + (1 + \bar{1} + 3 + \bar{3})_{s-} + (1 + \bar{1} + 3 + \bar{3})_{s+}
\end{aligned} \tag{4.11}
$$

The f_4 roots matching decompositions (4.10) and (4.11) are shown in Table 3 and Table 4.

f_4		ω_T	ω_s	U	V
	$\omega_L^{\wedge/\vee}$	$\mp$	$\pm$	0	0
	$\omega_R^{\wedge/\vee}$	$\pm$	$\pm$	0	0
	$W'^{\pm}$	0	0	$\pm$	$\pm$
	$W^{\pm}$	0	0	$\mp$	$\pm$
	$e_T^{\wedge/\vee}\ \phi_I^{/*}$	$\pm$	0	$\mp$	0
	$e_T^{\wedge/\vee}\ \phi_{II}^{/*}$	0	$\pm$	0	$\pm$
	$e_T^{\wedge/\vee}\ \phi_{III}^{/*}$	0	$\mp$	0	$\pm$
	$e_s^{\wedge/\vee}\ \phi_I^{*/}$	0	$\pm$	$\pm$	0
	$e_s^{\wedge/\vee}\ \phi_{II}^{*/}$	0	$\mp$	$\pm$	0
	$e_s^{\wedge/\vee}\ \phi_{III}^{*/}$	$\pm$	0	0	$\mp$
	$e_T^{\wedge/\vee}\ \phi^{*/}$	$\pm$	0	$\pm$	0
	$e_T^{\wedge/\vee}\ \phi_{\pm}$	$\pm$	0	0	$\pm$
	$\nu_{eL}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$
	$\nu_{\mu L}^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$
	$\nu_{\mu R}^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$
	$\mu_L^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$
	$\mu_R^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$
	$\nu_{\tau L}^{\wedge}\ \tau_L^{\vee}$	0	0	$\mp$	0
	$\nu_{\tau L}^{\vee}\ \tau_L^{\wedge}$	0	0	0	$\pm$
	$\nu_{\tau R}^{\wedge}\ \tau_R^{\vee}$	$\pm$	0	0	0
	$\nu_{\tau R}^{\vee}\ \tau_R^{\wedge}$	0	$\pm$	0	0

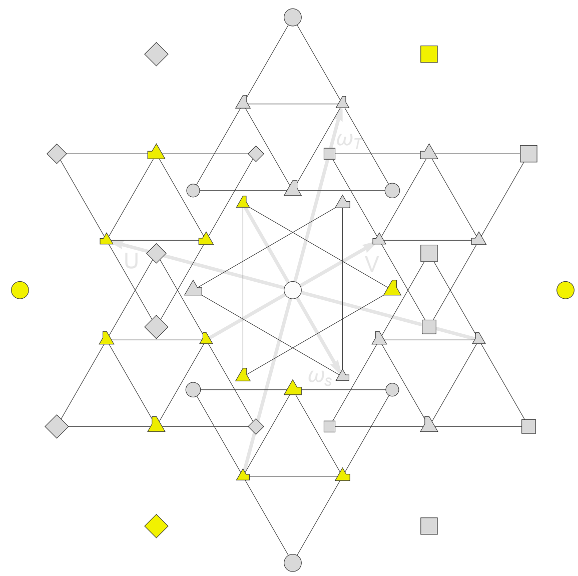


Table 3. The 48 roots of f_4, labeled suggestively as elementary particles, matching the branching decomposition of (4.10). In this table ω_T and ω_s are Euclidean boost and spin quantum numbers, while U and V combine to give weak and weaker charge, $W = \frac{1}{2}(-U + V)$ and $W' = \frac{1}{2}(U + V)$ — see (7.2). The gravitational spin connection and weaker boson roots, $\omega_{L/R}^{\wedge/\vee}$ and $W'^{\pm}$, correspond to $su(2)$'s related by triality, while the weak boson roots, $W^{\pm}$, correspond to a triality invariant $su(2)_W$. The $e_{s/T}^{\wedge/\vee}\phi$ are gravitational frame-Higgs boson roots, related by triality or triality invariant. The first generation lepton roots, $e_{L/R}^{\wedge/\vee}$ and $\nu_{eL/R}^{\wedge/\vee}$, with spin up or down and left or right chirality, correspond to a 8_{s+} of $so(8)$ and relate to the second and third generation leptons by triality. The spin and charges of these second and third generation leptons are not correct when considered independently of their triality relationship to the first.

f_4		p	x	y	z
●	g	0	$(+1$	-1	$0\,)$
● ● ●	$X_I^{rgb}\ \bar{X}_I^{rgb}$	0	$(\pm 1$	± 1	$0\,)$
● ● ●	$X_{II}^{rgb}\ \bar{X}_{II}^{rgb}$	± 1	$(\mp 1$	0	$0\,)$
● ● ●	$X_{III}^{rgb}\ \bar{X}_{III}^{rgb}$	∓ 1	$(\mp 1$	0	$0\,)$
▲	l_I	$-1/2$	$-1/2$	$-1/2$	$-1/2$
▼	$\bar{l}_I$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
▲ ▲ ▲	$q_I^{(rgb)}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
▼ ▼ ▼	$\bar{q}_I^{(rgb)}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
▲	l_{II}	$-1/2$	$+1/2$	$+1/2$	$+1/2$
▼	$\bar{l}_{II}$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
▲ ▲ ▲	$q_{II}^{(rgb)}$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
▼ ▼ ▼	$\bar{q}_{II}^{(rgb)}$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
▲	l_{III}	$+1$	0	0	0
▼	$\bar{l}_{III}$	-1	0	0	0
▲ ▲ ▲	$q_{III}^{(rgb)}$	0	$(-1$	0	$0\,)$
▼ ▼ ▼	$\bar{q}_{III}^{(rgb)}$	0	$(+1$	0	$0\,)$

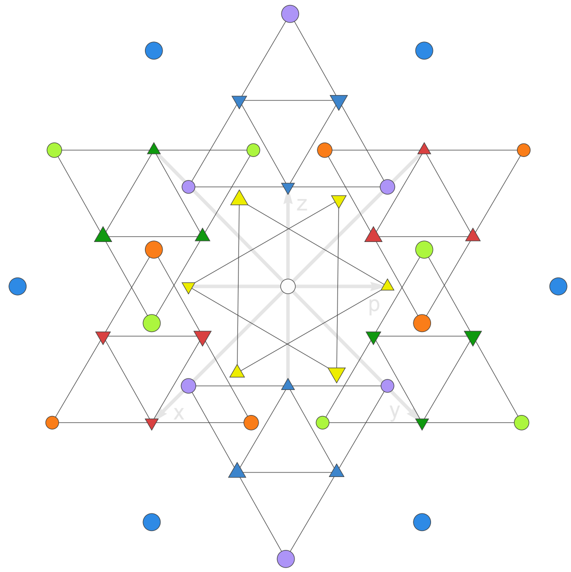


Table 4. The 48 roots of f_4, labeled suggestively as elementary particles, showing the triality invariant $su(3)_g$ and triality-related roots, as per (4.11). Coordinates in parenthesis are permuted over specified columns. The g roots are triality invariant gluons and the X are strongly charged (colored) GUT X-bosons related by triality. The leptons and anti-leptons, l_I and $\bar{l}_I$, and colored quarks and anti-quarks, q_I and $\bar{q}_I$, are in an 8_{s+} and relate to two other generations by triality. The (x, y, z) charges correspond to color and B charge, (7.2), while p indicates particles vs antiparticles.

It is interesting to perform an eigenspace decomposition of $f_{4(-52)}$ with respect to its canonical triality automorphism. This can be done computationally, or via sufficient staring at the figure in Table 4. The f_4 decomposition in (4.11) recombines into three triality eigenspaces,

$$
\begin{aligned}
f_{4(-52)} &= so(8) + 8_v + 8_{s-} + 8_{s+} \\
&= u(1)_p + u(1)_B + su(3)_g + 3_I + \bar{3}_I + 3_{II} + \bar{3}_{II} + 3_{III} + \bar{3}_{III} \\
&\quad + (1 + \bar{1} + 3 + \bar{3})_v + (1 + \bar{1} + 3 + \bar{3})_{s-} + (1 + \bar{1} + 3 + \bar{3})_{s+} \\
&= [su(3)_g + 3 + \bar{3} + 1 + \bar{1} + 3 + \bar{3}]_0 + [u(1) + 3 + \bar{3} + 1 + \bar{1} + 3 + \bar{3}]_{\pm 1} \\
&= \Big[so(7) + u(1)_w\Big]_0 + \Big[7^v_{\pm 1} + 8^s_{\mp 1/2}\Big]_{\pm 1} \\
&= so(9) + 16^s
\end{aligned}
\tag{4.12}
$$

with eigenvalues $e^{2\pi i k/3}$, in which $k \in \{0, +1, -1\}$. Algebraically, the $so(8)$ contains a triality invariant $su(3)$ and the triality invariant average of three $(3 + \bar{3})$'s, which combine into an invariant g_2 subalgebra of the $so(8)$. Similarly, the three triality-related 8's produce a triality invariant average 8, which is a $7 + 1$ under the invariant g_2. These combine to give $[so(7) + u(1)_w]$ as the triality invariant subalgebra of f_4. The remaining representation spaces split into $k = \pm 1$ triality eigenspaces, each including a vector, 7^v, and spinor, 8^s, of the $so(7)$, with w charges of ± 1 and $\mp {}^1\!/_2$. The invariant $[so(7) + u(1)_w]$ lives inside a $so(9)$ along with the $7^v_{\pm 1}$, which are intra-rotated by triality. Triality acts on the complex conjugate representation spaces, 7^v_{+1} and 7^v_{-1}, as multiplication by $e^{\pm 2\pi i/3}$. This corresponds to a rotation of real Lie algebra generators by ${}^{\pm 2\pi}\!/_3$ in 7 independent planes spanned by the orthogonal "real", $7^{\mathbb{R}}_v = \frac{1}{2}(7^v_{+1} + 7^v_{-1})$, and "imaginary", $7^{\mathbb{I}}_v = \frac{1}{2i}(7^v_{+1} - 7^v_{-1})$, generators within $so(9)$ — mapping between three sets of non-orthogonal 7^v vectors. Similarly, triality acts on the complex $8^s_{\mp 1/2}$ representation spaces as multiplication by $e^{\pm 2\pi i/3}$, corresponding to a rotation of ${}^{\pm 2\pi}\!/_3$ in 8 independent planes spanned by the orthogonal real, $8^{\mathbb{R}}_s = \frac{1}{2}(8^s_{-1/2} + 8^s_{+1/2})$, and imaginary, $8^{\mathbb{I}}_s = \frac{1}{2i}(8^s_{-1/2} - 8^s_{+1/2})$, generators — mapping between three sets of non-orthogonal 8^s spinor generators within 16^s.

For a compact Lie algebra, such as $f_{4(-52)}$, the complex conjugate pairs of root vectors, and conjugate pairs of triality eigenvectors, correspond to sign-conjugate pairs of roots. The $f_{4(-52)}$ roots matching the triality eigenspace decomposition, (4.12), are shown in Table 5. For a non-compact Lie algebra, such as split real $f_{4(4)}$, triality-automorphisms among root vectors can get more complicated.

f_4		w	x	y	z
●	g	0	$(+1$	-1	$0)$
● ● ●	X^{PS}	0	$(\pm 1$	± 1	$0)$
● ● ●	X^{GG}	0	$(\pm 1$	0	$0)$
■	Φ	$+1$	0	0	0
■ ■ ■	Φ_+	$+1$	$(+1$	0	$0)$
◆ ◆ ◆	Φ_-	$+1$	$(-1$	0	$0)$
■	Φ^*	-1	0	0	0
◆ ◆ ◆	Φ^*_+	-1	$(-1$	0	$0)$
■ ■ ■	Φ^*_-	-1	$(+1$	0	$0)$
▲	l	$-1/2$	$-1/2$	$-1/2$	$-1/2$
▼	$\bar{l}$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
▲ ▲ ▲	q	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
▼ ▼ ▼	$\bar{q}$	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
▼	l^*	$+1/2$	$+1/2$	$+1/2$	$+1/2$
▲	$\bar{l}^*$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
▼ ▼ ▼	q^*	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
▲ ▲ ▲	$\bar{q}^*$	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$

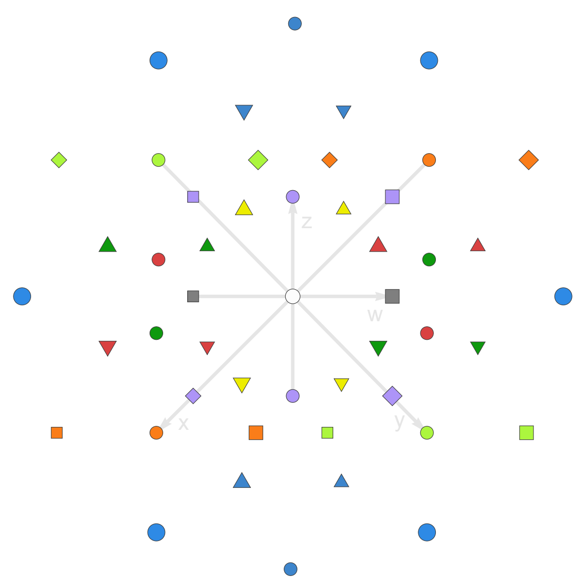


Table 5. The 48 roots of f_4, labeled suggestively as elementary particles, sorted into triality eigenspaces, as per (4.12). Roots with $w \in \{+1, -1/2\}$ and $w \in \{-1, +1/2\}$ are triality rotated by $e^{+2\pi i/3}$ and $e^{-2\pi i/3}$. The g are gluons, X^{PS} Pati-Salam X bosons, X^{GG} Georgi-Glashow X bosons, all invariant under triality. The Φ are Higgs bosons, related to their conjugates, Φ^*, by triality. The l, $\bar{l}$, q, $\bar{q}$ are leptons, anti-leptons, quarks, and anti-quarks, related to their conjugates, l^*, $\bar{l}^*$, q^*, $\bar{q}^*$, by triality.

5 Exceptional Magic

To obtain larger exceptional Lie algebras, we can consider the tensor product of division algebras or their split versions, such as the 32-dimensional vector space $\mathbb{H}\otimes\mathbb{O}$. From any two division algebras, $\mathbb{D}'$ and $\mathbb{D}$, of dimension n' and n and signature (p',q') and (p,q), we can construct chiral representations of Clifford algebras, $Cl(p'+p,q'+q)$ or $Cl(q'+p,p'+q)$, in a similar manner to the construction of Clifford division algebra representations. In a *Clifford compound division algebra representation*, $(n'+n)$-dimensional Clifford basis vectors are expressed as:

$$v = v^\alpha\gamma_\alpha = \begin{bmatrix} 0 & v_- \\ v_+ & 0 \end{bmatrix} \qquad v_- = \begin{bmatrix} 1\otimes\tilde{x} & \pm\tilde{y}'\otimes 1 \\ y'\otimes 1 & -1\otimes x \end{bmatrix} \qquad v_+ = \begin{bmatrix} 1\otimes x & \pm\tilde{y}'\otimes 1 \\ y'\otimes 1 & -1\otimes\tilde{x} \end{bmatrix}$$

which may be understood as matrices of inter-commuting division algebra elements, $x\in\mathbb{D}$ and $y'\in\mathbb{D}'$, or as $\mathbb{R}(4(n'\times n))$ matrices via their multiplication coefficients. The result of multiplying two vectors is

$$u\,v = \begin{bmatrix} u_-v_+ & 0 \\ 0 & u_+v_- \end{bmatrix}$$

$$u_-v_+ = \begin{bmatrix} \tilde{w} & \pm\tilde{z}' \\ z' & -w \end{bmatrix}\begin{bmatrix} x & \pm\tilde{y}' \\ y' & -\tilde{x} \end{bmatrix} = \begin{bmatrix} 1\otimes\tilde{w}x \pm \tilde{z}'y'\otimes 1 & \pm\tilde{y}'\otimes\tilde{w} \mp \tilde{z}'\otimes\tilde{x} \\ z'\otimes x - y'\otimes w & \pm z'\tilde{y}'\otimes 1 + 1\otimes w\tilde{x} \end{bmatrix}$$

from which we see that the result of squaring a Clifford vector is

$$v\,v = \begin{bmatrix} 1\otimes\tilde{x}x \pm \tilde{y}'y'\otimes 1 & 0 \\ 0 & \pm y'\tilde{y}'\otimes 1 + 1\otimes x\tilde{x} \end{bmatrix}$$

and so the represented Clifford algebra has signature $(p'+p,q'+q)$ or $(q'+p,p'+q)$, depending on the choice of $\pm$. The chiral bivector part of uv is a representative element of a spin Lie algebra, $so(p'+p,q'+q)$ or $so(q'+p,p'+q)$, which is

$$u_-v_+ \;\in\; \begin{bmatrix} (\mathbb{B}'_M\otimes 1)\oplus(1\otimes\mathbb{B}_M) & \mathbb{D}'\otimes\mathbb{D} \\ \mathbb{D}'\otimes\mathbb{D} & (\mathbb{B}'_P\otimes 1)\oplus(1\otimes\mathbb{B}_P) \end{bmatrix}$$

with the direct sum of bi-products on the diagonal, and the tensor product of the division algebras on the off diagonal. Expanding upon the previous description of Lie algebras using $su(3,\mathbb{D})$, we have a family of Lie algebras described heuristically as:

$$\begin{bmatrix} \mathbb{B}'_M\oplus\mathbb{B}_M & \mathbb{D}'_{\tilde{v}}\otimes\mathbb{D}_{\tilde{v}} & \mathbb{D}'_\psi\otimes\mathbb{D}_\psi \\ \mathbb{D}'_v\otimes\mathbb{D}_v & \mathbb{B}'_P\oplus\mathbb{B}_P & \mathbb{D}'_{\tilde{\chi}}\otimes\mathbb{D}_{\tilde{\chi}} \\ \mathbb{D}'_{\tilde{\psi}}\otimes\mathbb{D}_{\tilde{\psi}} & \mathbb{D}'_\chi\oplus\mathbb{D}_\chi & \mathbb{B}'_V\otimes\mathbb{B}_V \end{bmatrix} \;\sim\; su(3,\mathbb{D}'\otimes\mathbb{D})$$

This family is the *magic square of Lie algebras* [24], shown in Table 6. Each member of the magic square has a canonical triality automorphism, t, constructed from the triality automorphisms of its constituent parts. Every member also has three $so(p'+p,q'+q)\sim su(2,\mathbb{D}'\otimes\mathbb{D})$ subalgebras, related by a canonical triality automorphism.

$\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{C}'$	$\mathbb{H}'$	$\mathbb{O}'$
$\mathbb{R}$	$su(2)$	$su(3)$	$sp(3)$	f_4	$sl(3)$	$sp(6,\mathbb{R})$	$f_{4(4)}$
$\mathbb{C}$	$su(3)$	$2\,su(3)$	$su(6)$	e_6	$sl(3,\mathbb{C})$	$su(3,3)$	$e_{6(2)}$
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	e_7	$sl(3,\mathbb{H})$	$sp(6,\mathbb{H})$	$e_{7(-5)}$
$\mathbb{O}$	f_4	e_6	e_7	e_8	$e_{6(-26)}$	$e_{7(-25)}$	$e_{8(-24)}$
$\mathbb{C}'$	$sl(3)$	$sl(3,\mathbb{C})$	$sl(3,\mathbb{H})$	$e_{6(-26)}$	$2sl(3)$	$sl(6,\mathbb{R})$	$e_{6(6)}$
$\mathbb{H}'$	$sp(6,\mathbb{R})$	$su(3,3)$	$sp(6,\mathbb{H})$	$e_{7(-25)}$	$sl(6,\mathbb{R})$	$so(6,6)$	$e_{7(7)}$
$\mathbb{O}'$	$f_{4(4)}$	$e_{6(2)}$	$e_{7(-5)}$	$e_{8(-24)}$	$e_{6(6)}$	$e_{7(7)}$	$e_{8(8)}$

Table 6. The magic square Lie algebras, constructed from pairs of division algebras or their split versions.

An easy way to construct the Lie brackets of any member, $\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}}$, of the magic square is by suitably joining the Lie brackets of its constituent pairing of $su(2)$, $su(3)$, $sp(3)$, f_4, $sl(3)$, $sp(6,\mathbb{R})$, or $f_{4(4)}$ subalgebras, corresponding to its compound triality decomposition,

$$\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}} = \mathrm{Tri}(\mathbb{D}') \,+\, \mathrm{Tri}(\mathbb{D}) \,+\, \mathbb{D}'_v\otimes\mathbb{D}_v \,+\, \mathbb{D}'_m\otimes\mathbb{D}_m \,+\, \mathbb{D}'_p\otimes\mathbb{D}_p$$

For the compact real e_6, e_7, and e_8 Lie algebras, their compound triality decompositions are:

$$e_6 = u(1)+u(1)+so(8)+(1{+}\bar{1})_v\otimes 8_v+(1{+}\bar{1})_m\otimes 8_{s-}+(1{+}\bar{1})_p\otimes 8_{s+}$$

$$e_7 = su(2)_M+su(2)_P+su(2)_V+so(8)+\,(2,2,1)_v\otimes 8_v+(2,1,2)_m\otimes 8_{s-}+(1,2,2)_p\otimes 8_{s+}$$

$$e_8 = so(8)' \,+\, so(8) \,+\, 8'_v\otimes 8_v \,+\, 8'_{s-}\otimes 8_{s-} \,+\, 8'_{s+}\otimes 8_{s+}$$

The root system of any magic square Lie algebra may be constructed by suitably joining the roots of the constituent pairing. The triality matrix for its root system is a block diagonal matrix, t, constructed from the triality matrices of its constituents, such as

$$t = \begin{bmatrix} +1/2 & -1/2 & +1/2 & +1/2 & 0 & 0 & 0 & 0 \\ +1/2 & -1/2 & -1/2 & -1/2 & 0 & 0 & 0 & 0 \\ +1/2 & +1/2 & +1/2 & -1/2 & 0 & 0 & 0 & 0 \\ +1/2 & +1/2 & -1/2 & +1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & +1/2 & +1/2 & +1/2 \\ 0 & 0 & 0 & 0 & -1/2 & +1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & -1/2 & +1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & -1/2 & -1/2 & +1/2 \end{bmatrix} \tag{5.1}$$

for a real form of e_8. Although the roots of $sl(3)$, $sp(6,\mathbb{R})$, and $f_{4(4)}$ may have a mixture of imaginary and real components, a canonical triality automorphism only rotates between all compact or between all noncompact Cartan generators, and the triality matrices for these Lie algebras can only inter-mix all real or all imaginary root components.

The main advantage of having explicit expressions for the structure of a Lie algebra and its triality automorphisms, over the description via roots and a triality matrix, is that the triality matrix alone doesn't determine the phases of the triality maps between root vectors or generators. We could employ tricks to find these phases, but it's usually easier to find them from a direct division algebra description and the corresponding explicit triality automorphism.

6 Triality Eigenspaces, Vinberg Θ-Groups, and Gradings

For any Lie algebra automorphism, Θ, of order n, the Lie algebra separates into n different eigenspaces, with eigenvalues ω^k, with $\omega = e^{2\pi i/n}$ and integral $0 \le k \le n-1$. If n is odd, then $\omega^k = \omega^{-(k-n)}$ and we can index these spaces with ${}^{-(n-1)}/_2 \le k \le {}^{(n-1)}/_2$. For a triality automorphism, $\Theta = t$, the Lie algebra separates into three complex $\mathfrak{g}_k$ eigenspaces,

$$\mathfrak{g} \;=\; \mathfrak{g}_{-1} \;+\; \mathfrak{g}_0 \;+\; \mathfrak{g}_{+1}$$

with $\mathfrak{g}_0$ the real triality invariant subalgebra of $\mathfrak{g}$, and the Lie brackets satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The $\mathfrak{g}_{\pm 1}$ eigenspaces are spanned by complex conjugate sets of eigenvectors, $V_i^{+1} \in \mathfrak{g}_{+1}$ and $V_i^{+1\,*} = V_i^{-1} \in \mathfrak{g}_{-1}$. These conjugate pairs of coset basis generators combine into "real" and "imaginary" orthogonal pairs of real vectors,

$$\begin{aligned} V_i^{\mathbb{R}} &= \tfrac{1}{2}\left(V_i^{+1} + V_i^{-1}\right) \\ V_i^{\mathbb{I}} &= \tfrac{1}{2i}\left(V_i^{+1} - V_i^{-1}\right) \end{aligned} \qquad\qquad \Theta : \begin{bmatrix} V_i^{\mathbb{R}} \\ V_i^{\mathbb{I}} \end{bmatrix} \mapsto \begin{bmatrix} V'^{\mathbb{R}}_i \\ V'^{\mathbb{I}}_i \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} V_i^{\mathbb{R}} \\ V_i^{\mathbb{I}} \end{bmatrix}$$

which are rotated 120° in their plane by triality. The real Lie algebra, spanned by $\{V_a^0, V_i^{\mathbb{R}}, V_i^{\mathbb{I}}\}$, is invariant under these triality rotations[1] — the triality automorphism. These orthogonal pairs of basis vectors, $\{V_i^{\mathbb{R}}, V_i^{\mathbb{I}}\}$, can also be replaced by non-orthogonal triples,

$$V_i^{I} = V_i^{\mathbb{R}} \qquad\quad V_i^{II} = -\tfrac{1}{2}V_i^{\mathbb{R}} - \tfrac{\sqrt{3}}{2}V_i^{\mathbb{I}} \qquad\quad V_i^{III} = -\tfrac{1}{2}V_i^{\mathbb{R}} + \tfrac{\sqrt{3}}{2}V_i^{\mathbb{I}}$$

which are permuted by triality, $\Theta : V_i^{I} \mapsto V_i^{II} \mapsto V_i^{III} \mapsto V_i^{I}$. It is an intriguing possibility that three such sets of triality-related generators may correspond to the three generations of fermions in the Standard Model of particle physics.

The complement of $\mathfrak{g}_0$ in $\mathfrak{g}$ is spanned by the coset generators of a 3-symmetric space, G/G_0 [31],

$$\mathfrak{g}/\mathfrak{g}_0 \;=\; \mathfrak{g}_{+1} \;+\; \mathfrak{g}_{-1}$$

Just as a symmetric space is invariant under inversion of the coset generators, this complex 3-symmetric space is invariant under $V_i^k \to e^{2\pi i k/3}\, V_i^k$, and the corresponding real 3-symmetric space is invariant under 120° rotations in the spanning $\{V_i^{\mathbb{R}}, V_i^{\mathbb{I}}\}$ planes.

The Lie algebra $\mathfrak{g}_\Theta = \mathfrak{g}_0 + \mathfrak{g}_{+1}$, with $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_{+1}] \subset \mathfrak{g}_{+1}$ brackets from $\mathfrak{g}$, and $[\mathfrak{g}_{+1}, \mathfrak{g}_{+1}] = 0$ brackets vanishing, is not necessarily a subalgebra of the real Lie algebra, $\mathfrak{g}$. As a Lie algebra, $\mathfrak{g}_\Theta$ is known as the Vinberg Θ-algebra of the corresponding Vinberg Θ-group [27]. For some of the magic square Lie algebras, under triality automorphisms, their Vinberg Θ-algebras, including their triality invariant subalgebras, $\mathfrak{g}_0$, and $\mathfrak{g}_{+1}$ eigenspaces, are:[2]

$$\begin{aligned} su(3)_\Theta &= \Big[u(1) + u(1)\Big]_0 + \Big[1_{(-\sqrt{3},1)} + 1_{(0,-2)} + 1_{(\sqrt{3},1)}\Big]_{+1} \\ sp(3)_\Theta &= \Big[so(4) + u(1)_w\Big]_0 + \Big[3^{su(2)}_{+1} + 4^{v}_{-1/2}\Big]_{+1} \\ f_{4(-52)\Theta} &= \Big[so(7) + u(1)_w\Big]_0 + \Big[7^{v}_{+1} + 8^{s}_{-1/2}\Big]_{+1} \end{aligned}$$

$$
\begin{aligned}
f_{4(4)\Theta} &= \Big[so(4,3)+u(1)_w\Big]_0 + \Big[7^v_{+1}+8^s_{-1/2}\Big]_{+1} \\
e_{6(-78)\Theta} &= \Big[so(8)+u(1)+u(1)\Big]_0 + \Big[(1+1+1)\otimes 8^{s-}\Big]_{+1} \\
e_{7(-133)\Theta} &= \Big[so(10)+su(2)+u(1)_w\Big]_0 + \Big[10^v_{+1}+2\otimes 16^{s+}_{-1/2}\Big]_{+1} \\
e_{8(-248)\Theta} &= \Big[so(14)+u(1)_w\Big]_0 + \Big[14^v_{+1}+64^{s-}_{-1/2}\Big]_{+1} \\
e_{8(8)\Theta} &= \Big[so(8,6)+u(1)_w\Big]_0 + \Big[14^v_{+1}+64^{s-}_{-1/2}\Big]_{+1} \\
e_{8(-24)\Theta} &= \Big[so(4,10)+u(1)_w\Big]_0 + \Big[14^v_{+1}+64^{s-}_{-1/2}\Big]_{+1}
\end{aligned}
\tag{6.1}
$$

in which the subscripts are the $u(1)_w$ charge and triality eigenvalue, k. A good way to think of Vinberg Θ-algebras and Θ-groups is as the corresponding Lie algebras and Lie groups modded by an automorphism, $\mathfrak{g}_\Theta = \mathfrak{g}/\Theta$. The dimension of a Vinberg Θ-algebra is $\dim \mathfrak{g}_\Theta = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{+1}$. The complex triality eigenspace decomposition of a real Lie algebra might or might not correspond to a graded decomposition of the Lie algebra into a real subalgebra and real representation spaces.

The non-compact real forms of e_8 do have triality-related five-gradings and two-gradings,

$$
\begin{aligned}
e_{8(8)} &= 14^v_{-1}+64^{s+}_{+1/2}+\Big[so(7,7)+so(1,1)_{w^{\mathbb{R}}}\Big]+64^{s-}_{-1/2}+14^v_{+1} = so(8,8)+128^{s+} \\
e_{8(-24)} &= 14^v_{-1}+64^{s+}_{+1/2}+\Big[so(3,11)+so(1,1)_{w^{\mathbb{R}}}\Big]+64^{s-}_{-1/2}+14^v_{+1} = so(4,12)+128^{s+}
\end{aligned}
\tag{6.2}
$$

which include subalgebras of the same dimension but different signatures from the corresponding triality invariant subalgebra. Interestingly, the extended graviGUT algebra [32],

$$
so(3,11)+64^{s-}_{-1/2}+14^v_{-1}
$$

is embedded within $e_{8(-24)}$, and can describe the gravitational, gauge, and scalar fields of the Standard Model acting on one generation of fermions, with spin [13]. Note that the intersection of the graviGUT gauge algebra, $so(3,11)$, and the triality invariant gauge algebra, $[so(4,10)+u(1)]$, within the $so(4,12)$ subalgebra of $e_{8(-24)}$ is $so(3,10)$, which includes precisely the spin and GUT quantum numbers needed to identify a 64-real-dimensional generation of fermions. The triality-map of $64^{s-}_{-1/2}$ spinors exists as rotations within the 128^{s+} of the two-grading. Explicitly, the $\psi_i \in 64^{s-}_{-1/2}$ can be identified with a Standard Model generation while the $\psi^m_i \in 64^{s+}_{+1/2}$ are non-physical "mirror" fermions [33]. Since these $w^{\mathbb{R}} = \pm 1/2$ states are not $w = \pm 1/2$ triality eigenstates, triality does not act on them as $e^{\pm 2\pi i/3}$. Rather, a canonical triality automorphism in $e_{8(8)}$ or $e_{8(-24)}$ acts on the $64^{s-}_{-1/2}$ fermions as a $2\pi/3$ rotation in the planes spanned by $\{\psi^{\mathbb{R}}_i, \psi^{m\mathbb{I}}_i\}$ and $\{\psi^{m\mathbb{R}_i}, \psi^{\mathbb{I}}_i\}$. This suggests the triality rotation of these fermion degrees of freedom within the 128^{s+} may relate to the existence of three generations of Standard Model fermions.

7 e_6

The compact e_6 Lie algebra has a compound triality decomposition from combining $su(3)$ and f_4,

$$\begin{aligned} e_{6(-78)} &= u(1) + u(1) + so(8) + (1{+}\bar{1})_v \otimes 8_v + (1{+}\bar{1})_m \otimes 8_{s-} + (1{+}\bar{1})_p \otimes 8_{s+} \\ &\sim su(3, \mathbb{C} \otimes \mathbb{O}) \end{aligned} \tag{7.1}$$

and a triality automorphism that maps between the triplet of complex octonions, $\mathbb{C}\otimes\mathbb{O}$. The corresponding set of e_6 basis elements is

$$\{\, T'_1,\, T'_2,\, \gamma_{ab},\, \gamma_{a'a},\, Q^-_{a'a},\, Q^+_{a'a} \,\}$$

with primed index, a', ranging over complex indices, $\{0, 1\}$, un-primed index, a, ranging over octonion indices, $\{0, ..., 7\}$, and the bivector index, ab, ranging over the 28 $so(8)$ basis generator permutations with $a < b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of $su(3)$ and $f_{4(-52)}$, (4.2) and (4.8):[2]

$$\begin{aligned}
[\gamma_{ab}, \gamma_{cd}] &= 2\,\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} \\
[\gamma_{ab}, \gamma_{a'c}] &= 2\,\{-n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b}\} \\
\left[\gamma_{ab}, Q^-_{a'c}\right] &= Q^-_{a'd}(-\bar{\Gamma}_a\Gamma_b)^d{}_c \\
\left[\gamma_{ab}, Q^+_{a'c}\right] &= Q^+_{a'd}(-\Gamma_a\bar{\Gamma}_b)^d{}_c \\
[T'_1, \gamma_{0'a}] &= -2\,\gamma_{1'a} \\
\left[T'_1, Q^-_{0'a}\right] &= +Q^-_{1'a} \\
\left[T'_1, Q^+_{0'a}\right] &= +Q^+_{1'a} \\
[T'_1, \gamma_{1'a}] &= +2\,\gamma_{0'a} \\
\left[T'_1, Q^-_{1'a}\right] &= -Q^-_{0'a} \\
\left[T'_1, Q^+_{1'a}\right] &= -Q^+_{0'a} \\
\left[\gamma_{a'a}, Q^-_{b'b}\right] &= -M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\Gamma_a)^c{}_b Q^+_{c'c} \\
\left[Q^-_{a'a}, Q^+_{b'b}\right] &= M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\bar{\Gamma}^c)_{ab}\gamma_{c'c}
\end{aligned}
\qquad
\begin{aligned}
\left[T'_2, Q^-_{0'a}\right] &= +\sqrt{3}\,Q^-_{1'a} \\
\left[T'_2, Q^+_{0'a}\right] &= -\sqrt{3}\,Q^+_{1'a} \\
\left[T'_2, Q^-_{1'a}\right] &= -\sqrt{3}\,Q^-_{0'a} \\
\left[T'_2, Q^+_{1'a}\right] &= +\sqrt{3}\,Q^+_{0'a} \\
[\gamma_{0'a}, \gamma_{1'b}] &= 2\,T'_1 n_{ab} \\
\left[Q^-_{0'a}, Q^-_{1'b}\right] &= -(T_1 + \sqrt{3}\,T_2)n^-_{ab} \\
\left[Q^+_{0'a}, Q^+_{1'b}\right] &= -(T_1 - \sqrt{3}\,T_2)n^+_{ab} \\
[\gamma_{a'a}, \gamma_{b'b}] &= -2\,n'_{a'b'}\gamma_{ab} \\
\left[Q^-_{a'a}, Q^-_{b'b}\right] &= n'^-_{a'b'}\gamma_{cd}(\bar{\Gamma}^c\Gamma^d)_{ab} \\
\left[Q^+_{a'a}, Q^+_{b'b}\right] &= n'^+_{a'b'}\gamma_{cd}(\Gamma^c\bar{\Gamma}^d)_{ab} \\
\left[\gamma_{a'a}, Q^+_{b'b}\right] &= M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\bar{\Gamma}_a)^c{}_b Q^-_{c'c}
\end{aligned}$$

in which $M'_{\tilde{a}'\tilde{b}'}{}^{c'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ are complex and octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are complex and octonion metrics, also used to raise or lower indices. Different real forms of e_6 come from combining different real forms of $su(3)$ and f_4, using correspondingly different multiplication tables and metrics. A canonical triality automorphism of $e_{6(-78)}$ is

$$\begin{aligned}
t \quad : \quad T'_1 &\mapsto T''_1 = -\tfrac{1}{2}T'_1 - \tfrac{\sqrt{3}}{2}T'_2 \\
T'_2 &\mapsto T''_2 = \tfrac{\sqrt{3}}{2}T'_1 - \tfrac{1}{2}T'_2 \\
\gamma_{ab} &\mapsto \gamma'_{ab} = \gamma_{cd}t^{cd}{}_{ab} = \gamma_{cd}\left(-\tfrac{1}{2}\bar{\Gamma}^c\Gamma^d\right)_{ab}
\end{aligned}
\qquad
\begin{aligned}
\gamma_{a'a} &\mapsto \gamma'_{a'a} = Q^-_{a'a} \\
Q^-_{a'a} &\mapsto Q'^-_{a'a} = Q^+_{a'a} \\
Q^+_{a'a} &\mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{aligned}$$

from combining the triality automorphisms of the constituent $su(3)$ and $f_{4(-52)}$.

The triality decomposition of e_6, (7.1), may be concatenated, using $so(10) = u(1) + so(8) + (1{+}\bar{1})_v \otimes 8_v$ to get

$$e_{6(-78)} = so(10) + u(1) + 16_{s+} + 16_{s-}$$

which includes the algebra of the $SO(10)$ Grand Unified Theory of particle physics, including fermion states, without spin. Decomposing this into Standard Model representation spaces, we have:

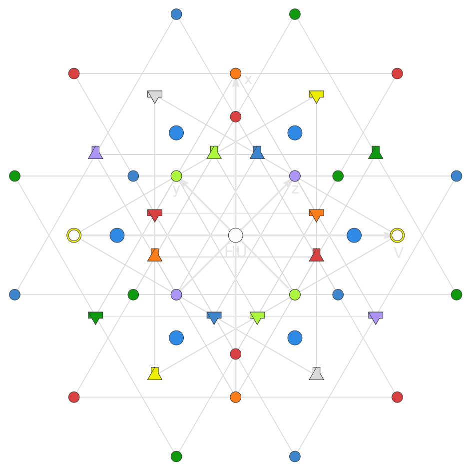
$$\begin{aligned} e_{6(-78)} &= su(2)_{W'} + su(2)_W + su(3)_g + u(1)_B + u(1)_H \\ &+ \big((2,2;3_{-2/3} + \bar{3}_{+2/3}) + (1,1;3_{4/3} + \bar{3}_{-4/3})\big)_0 \\ &+ \big((1,2;1_{-1} + 3_{+1/3}) + (2,1;1_{+1} + \bar{3}_{-1/3})\big)_{+\sqrt{3}/2} \\ &+ \big((2,1;1_{-1} + 3_{+1/3}) + (1,2;1_{+1} + \bar{3}_{-1/3})\big)_{-\sqrt{3}/2} \end{aligned}$$

The Cartan subalgebra generators, $\{H, U, V, x, y, z\}$, of $u(1)_H + so(10)$ or of $u(1)_H + u(1)_U + so(8)$, correspond to root coordinates, or charges, which combine to give the familiar strong, g, weak, W, hypercharge, Y, and electric charge, Q, of the Standard Model, as well as chirality (or helicity), H, the weaker charge, W', baryon minus lepton number, B, and other GUT charges:

$$\begin{aligned} W &= \tfrac{1}{2}(-U + V) \\ W' &= \tfrac{1}{2}(U + V) \\ B &= \tfrac{2}{3}(x + y + z) \\ g^3 &= \tfrac{1}{2}(-x + y) \\ g^8 &= \tfrac{1}{2\sqrt{3}}(-x - y + 2z) \\ Y &= 2W' + B = \tfrac{1}{2}(U + V) + \tfrac{2}{3}(x + y + z) \\ X &= 2W' - \tfrac{3}{2}B = \tfrac{1}{2}(U + V) - (x + y + z) \\ Q &= W + \tfrac{1}{2}Y = V + \tfrac{1}{2}B = V + \tfrac{1}{3}(x + y + z) \\ Z &= W - \tfrac{3}{10}Y = -\tfrac{4}{5}U + \tfrac{1}{5}(V - x - y - z) \end{aligned} \tag{7.2}$$

The e_6 roots, labeled as elementary particle states, are shown in Table 7. The $su(3)_g$ gluons, g, and Pati-Salam bosons, $X^{PS}_{\pm 2/3}$, are in a $so(6) = su(4)$ subalgebra of $so(10)$, which has $so(4) = su(2)_W + su(2)_{W'}$ in its complement, with $su(2)_{W'}$ the W' bosons. The Georgi-Glashow bosons, $X^{GG}_{\pm 2/3}$ and $X^{GG}_{\pm 1/3}$, are in an $su(5)$ subalgebra of $so(10)$, and the GUT bosons, $X_{\pm 4/3}$ and $X_{\pm 1/3}$, are in the rest of $so(10)$. The left-handed fermions and antifermions, with scaled helicity, $H = +\sqrt{3}/2$, are in a 16_{s+} spinor of $so(10)$, while their conjugates, the right-handed fermions and antifermions, with $H = -\sqrt{3}/2$, are in a 16_{s-}.

With respect to the compound division algebra decomposition, (7.1), we have $\{W^\pm, W'^\pm, X^{GG}_{\pm 1/3}, X_{\pm 1/3}\}$ in $(1{+}\bar{1})_v \otimes 8_v$, the up-type fermions and antifermions in $(1{+}\bar{1})_m \otimes 8_{s-}$, and the down-type fermions and antifermions in $(1{+}\bar{1})_p \otimes 8_{s+}$. A canonical triality automorphism of e_6, although pretty, does not appear to correspond to anything especially interesting.



e_6		H	U	V	x	y	z
	g	0	0	0	$(+1$	-1	$0)$
	$X^{PS}_{\pm 2/3}$	0	0	0	$(\pm 1$	± 1	$0)$
	$X^{GG}_{\pm 2/3}$	0	0	± 1	$(\mp 1$	0	$0)$
	$X_{\pm 4/3}$	0	0	± 1	$(\pm 1$	0	$0)$
	$W^{\pm}$	0	∓ 1	± 1	0	0	0
	$W'^{\pm}$	0	± 1	± 1	0	0	0
	$X^{GG}_{\pm 1/3}$	0	∓ 1	0	$(\pm 1$	0	$0)$
	$X_{\pm 1/3}$	0	± 1	0	$(\pm 1$	0	$0)$
	ν_L	$+\sqrt{3}/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	ν_R	$-\sqrt{3}/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_L$	$+\sqrt{3}/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_R$	$-\sqrt{3}/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$u_L^{(rgb)}$	$+\sqrt{3}/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$u_R^{(rgb)}$	$-\sqrt{3}/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{u}_L^{(rgb)}$	$+\sqrt{3}/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$\bar{u}_R^{(rgb)}$	$-\sqrt{3}/2$	$+1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
	e_L	$+\sqrt{3}/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	e_R	$-\sqrt{3}/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}_L$	$+\sqrt{3}/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{e}_R$	$-\sqrt{3}/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$d_L^{(rgb)}$	$+\sqrt{3}/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$d_R^{(rgb)}$	$-\sqrt{3}/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{d}_L^{(rgb)}$	$+\sqrt{3}/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$\bar{d}_R^{(rgb)}$	$-\sqrt{3}/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$

Table 7. The 72 roots of e_6 labeled as elementary particles. There are gluons, g, Pati-Salam X bosons, X^{PS}, Georgi-Glashow X bosons, X^{GG}, and remaining SO(10) GUT X bosons. The W and W' are the weak and weaker (GUT) bosons. One generation of fermions is included: left and right-chiral neutrinos and anti-neutrinos, $\nu_{L/R}$ and $\bar{\nu}_{L/R}$, electrons and positrons, $e_{L/R}$ and $\bar{e}_{L/R}$, and up and down quarks and anti-quarks of different colors, $u_{L/R}$, $\bar{u}_{L/R}$, $d_{L/R}$, and $\bar{d}_{L/R}$. For the charges, see (7.2).

8 e_7

The e_7 Lie algebra has a compound triality decomposition from combining $sp(3)$ and f_4,

$$\begin{aligned} e_7 =\, & su(2)_M + su(2)_P + su(2)_V + so(8) \\ & + (2,2,1)_v \otimes 8_v + (2,1,2)_m \otimes 8_{s-} + (1,2,2)_p \otimes 8_{s+} \\ \sim & su(3, \mathbb{H}\otimes\mathbb{O}) \end{aligned} \tag{8.1}$$

and a triality automorphism that maps between the triplet of quaterni-octonions, $\mathbb{H}\otimes\mathbb{O}$. The corresponding set of e_7 basis elements is

$$\{ T^M_{A'},\, T^P_{A'},\, T^V_{A'},\, \gamma_{ab},\, \gamma_{a'a},\, Q^-_{a'a},\, Q^+_{a'a} \}$$

with primed index, a', ranging over quaternion indices, $\{0,...,3\}$, primed capitals, A', ranging over imaginary quaternion indices, $\{1,...,3\}$, un-primed index, a, ranging over octonion indices, $\{0,...,7\}$, and the bivector index, ab, ranging over the 28 $so(8)$ basis generator permutations with $a<b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of $sp(3)$ and $f_{4(-52)}$, (4.5) and (4.8):[2]

$$\begin{aligned}
[\gamma_{ab},\gamma_{cd}] &= 2\,\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} \\
[\gamma_{ab},\gamma_{a'c}] &= 2\,\{-n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b}\} \\
\left[\gamma_{ab},Q^-_{a'c}\right] &= Q^-_{a'd}(-\bar\Gamma_a\Gamma_b)^d{}_c & \left[\gamma_{ab},Q^+_{a'c}\right] &= Q^+_{a'd}(-\Gamma_a\bar\Gamma_b)^d{}_c \\
\left[T^M_{A'},T^M_{B'}\right] &= T^M_{C'}(2M'_{[A'B']}{}^{C'}) & \left[T^M_{A'},\gamma_{b'a}\right] &= \gamma_{c'a}\,(-M'_{b'A'}{}^{c'}) \\
\left[T^P_{A'},T^P_{B'}\right] &= T^P_{C'}(2M'_{[A'B']}{}^{C'}) & \left[T^P_{A'},\gamma_{b'a}\right] &= \gamma_{c'a}\,(M'_{A'b'}{}^{c'}) \\
\left[T^V_{A'},T^V_{B'}\right] &= T^V_{C'}(2M'_{[A'B']}{}^{C'}) & \left[T^V_{A'},Q^-_{b'a}\right] &= Q^-_{c'a}\,(-M'_{b'A'}{}^{c'}) \\
\left[T^M_{A'},Q^-_{b'a}\right] &= Q^-_{c'a}\,(M'_{A'b'}{}^{c'}) & \left[\gamma_{a'a},Q^-_{b'b}\right] &= -\tfrac{1}{\sqrt2}M'_{\tilde a'\tilde b'}{}^{c'}(\Gamma_a)^c{}_b Q^+_{c'c} \\
\left[T^P_{A'},Q^+_{b'a}\right] &= Q^+_{c'a}\,(-M'_{b'A'}{}^{c'}) & \left[\gamma_{a'a},Q^+_{b'b}\right] &= \tfrac{1}{\sqrt2}M'_{\tilde a'\tilde b'}{}^{c'}(\bar\Gamma_a)^c{}_b Q^-_{c'c} \\
\left[T^V_{A'},Q^+_{b'a}\right] &= Q^+_{c'a}\,(M'_{A'b'}{}^{c'}) & \left[Q^-_{a'a},Q^+_{b'b}\right] &= \tfrac{1}{\sqrt2}M'_{\tilde a'\tilde b'}{}^{c'}(\bar\Gamma^c)_{ab}\gamma_{c'c}
\end{aligned}$$

$$\begin{aligned}
[\gamma_{a'a},\gamma_{b'b}] &= -\left(T^M_{C'}\tfrac12(M'_{\tilde b'a'}{}^{C'} - M'_{\tilde a'b'}{}^{C'}) + T^P_{C'}\tfrac12(M'_{b'\tilde a'}{}^{C'} - M'_{a'\tilde b'}{}^{C'})\right) n_{ab} - 2\,n'_{a'b'}\gamma_{ab} \\
\left[Q^-_{a'a},Q^-_{b'b}\right] &= -\left(T^V_{C'}\tfrac12(M'_{\tilde b'a'}{}^{C'} - M'_{\tilde a'b'}{}^{C'}) + T^M_{C'}\tfrac12(M'_{b'\tilde a'}{}^{C'} - M'_{a'\tilde b'}{}^{C'})\right) n^-_{ab} + n'^-_{a'b'}\gamma_{cd}(\bar\Gamma^c\Gamma^d)_{ab} \\
\left[Q^+_{a'a},Q^+_{b'b}\right] &= -\left(T^P_{C'}\tfrac12(M'_{\tilde b'a'}{}^{C'} - M'_{\tilde a'b'}{}^{C'}) + T^V_{C'}\tfrac12(M'_{b'\tilde a'}{}^{C'} - M'_{a'\tilde b'}{}^{C'})\right) n^+_{ab} + n'^+_{a'b'}\gamma_{cd}(\Gamma^c\bar\Gamma^d)_{ab}
\end{aligned}$$

in which $M'_{\tilde a'\tilde b'}{}^{c'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde b}$ are quaternion and octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are quaternion and octonion metrics, also used to raise or lower indices. Different real forms of e_7 come from combining different real forms of $sp(3)$ and f_4, using correspondingly different multiplication tables and metrics.

A canonical triality automorphism of $e_{7(-133)}$ is

$$\begin{aligned}
t \quad : \quad & \gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{cd}t^{cd}{}_{ab} = \gamma_{cd}\left(-\tfrac12\bar\Gamma^c\Gamma^d\right)_{ab} \\
& T^M_{A'} \mapsto T'^M_{A'} = T^V_{A'} \qquad T^V_{A'} \mapsto T'^V_{A'} = T^P_{A'} \qquad T^P_{A'} \mapsto T'^P_{A'} = T^M_{A'} \\
& \gamma_{a'a} \mapsto \gamma'_{a'a} = Q^-_{a'a} \qquad Q^-_{a'a} \mapsto Q'^-_{a'a} = Q^+_{a'a} \qquad Q^+_{a'a} \mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{aligned}$$

from combining the triality automorphisms of $sp(3)$ and $f_{4(-52)}$.

The triality decomposition of e_7, (8.1), may be concatenated using

$$so(12) = su(2)_M + su(2)_P + so(8) + (2,2)_v \otimes 8_v$$

to get

$$e_{7(-133)} = so(12) + su(2) + 2 \otimes 32_{s+}$$

which includes the algebra of the $SO(10)$ Grand Unified Theory of particle physics, including fermion states with spin. The Cartan subalgebra generators, $\{\omega, U, V, h, x, y, z\}$, of $su(2)_\omega + so(12)$ or of $su(2)_\omega + su(2)_W + su(2)_{W'} + so(8)$, correspond to charges which combine to give the strong, g, weak, W, hypercharge, Y, and electric charge, Q, of the Standard Model, as well as the weaker charge, W', and reversed chirality (or helicity), h. The triality invariant subalgebra of $e_{7(-133)}$ is

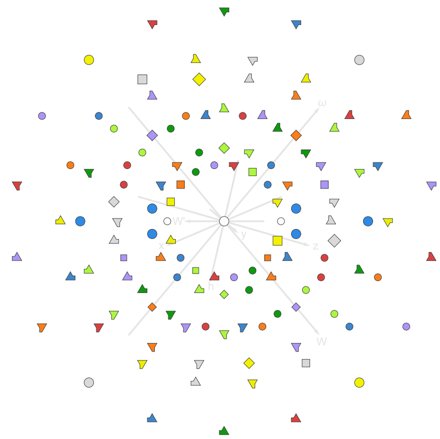
$$\mathfrak{g}_0 = so(10) + su(2)_\omega + u(1)_h$$

which is a subalgebra of $so(12) + su(2)$. With respect to this subalgebra, the -1 triality eigenspace is

$$\mathfrak{g}_1 = 10^v_{-1} + 2 \otimes 16^{s+}_{+1/2}$$

which includes vector and spinor representation spaces with $h \in \{-1, +1/2\}$. The $2 \otimes 16^{s+}_{+1/2}$ representation space matches a set of left-chiral Standard Model fermions and antifermions, while the 10^v_{-1} matches a set of GUT Higgs fields, Φ. For $e_{7(-133)}$ and triality, the Vingberg Θ-algebra matches the $SO(10)$ GUT with a Higgs and one left-chiral generation of fermions with spin acted on by a $su(2)_\omega$. Recall that the triality automorphism rotates this representation space within the $2 \otimes 32_{s+}$ representation space of $so(12) + su(2)$, permuting $3 \otimes 32$ non-linearly-independent sets of vectors. It is compelling to consider these spaces as three different left-chiral generations of fermions. The $2 \otimes 32_{s+}$ representation space is spanned by the $[2 \otimes 16^{s+}_{+1/2}]_{-1}$ eigenvectors and the $[2 \otimes 16^{s-}_{-1/2}]_{+1}$ conjugate eigenvectors. These conjugate fermions, such as $(e^{\wedge}_L)^* = \bar{e}^{\vee}_R$, can be interpreted in e_7 as either the right-chiral fermions and antifermions, or as the conjugate partner directions to the Standard Model fermions, allowing three triality-related generations to inhabit the 64-dimensional space. The geometry of the Higgs states, with 10^v_{-1} rotated by triality in a 20-dimensional space spanned by 10^v_{-1} and 10^v_{+1}, can be treated similarly.

The e_7 roots, labeled suggestively as elementary particles, with conjugate states related by triality, are shown in Table 8. Note that, to incorporate three generations, the right-chiral fermions and antifermions here need to be interpreted as the conjugate root vectors of the left-chiral fermions and antifermions, with the triality automorphism intra-rotating their real and imaginary parts. Alternatively, e_7 could have just a single generation of fermions, including the right-chiral fermions and antifermions.



e_7		ω	W	W'	h	x	y	z
	$\omega^{\wedge/\vee}$	$\pm\sqrt{2}$	0	0	0	0	0	0
	$W^{\pm}$	0	$\pm\sqrt{2}$	0	0	0	0	0
	$W'^{\pm}$	0	0	$\pm\sqrt{2}$	0	0	0	0
	g	0	0	0	0	$(+1$	-1	$0)$
	$X^{PS}_{\pm 2/3}$	0	0	0	0	$(\pm 1$	± 1	$0)$
	$X^{GG}_{\pm 2/3}$	0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(\mp 1$	0	$0)$
	$X_{\pm 4/3}$	0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(\pm 1$	0	$0)$
	$X^{GG}_{\pm 1/3}$	0	$\pm 1/\sqrt{2}$	$\mp 1/\sqrt{2}$	0	$(\pm 1$	0	$0)$
	$X_{\pm 1/3}$	0	$\mp 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(\pm 1$	0	$0)$
	$\Phi_0 \quad \Phi^m_0$	0	$\mp 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	∓ 1	0	0	0
	$\Phi^*_0 \quad \Phi^{*m}_0$	0	$\pm 1/\sqrt{2}$	$\mp 1/\sqrt{2}$	∓ 1	0	0	0
	$\Phi_{+1}\ \Phi^m_{-1}$	0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	∓ 1	0	0	0
	$\Phi_{-1}\ \Phi^m_{+1}$	0	$\mp 1/\sqrt{2}$	$\mp 1/\sqrt{2}$	∓ 1	0	0	0
	$\Phi_{+1/3}\ \Phi^m_{-1/3}$	0	0	0	∓ 1	$(\pm 1$	0	$0)$
	$\Phi_{-1/3}\ \Phi^m_{+1/3}$	0	0	0	∓ 1	$(\mp 1$	0	$0)$
	$\nu^{\wedge/\vee}_{eL}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}^{\wedge/\vee}_{eL}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$e^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$u^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{u}^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$d^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{d}^{\wedge/\vee}_{L}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$\nu^{\wedge/\vee}_{eR}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}^{\wedge/\vee}_{eR}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$e^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$u^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{u}^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$d^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{d}^{\wedge/\vee}_{R}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$

Table 8. The 126 roots of e_7, suggestively labeled as elementary particles, with triality eigenspaces corresponding to $h = 0$, $h \in \{-1, +1/2\}$, and $h \in \{+1, -1/2\}$. The bosons are the same as for e_6 plus a spatial spin connection, $\omega^{\wedge/\vee}$, and some more Higgs fields, Φ. There is a full set of left-chiral fermions and antifermions with spin, $f^{\wedge/\vee}_L$ and $\bar{f}^{\wedge/\vee}_L$. The fermions labeled $f^{\wedge/\vee}_R$ and $\bar{f}^{\wedge/\vee}_R$ can be considered either the right-chiral partners, or as conjugates related to the left-chiral fermions by triality.

9 e_8

The e_8 Lie algebra has a compound triality decomposition from combining two f_4's,

$$\begin{aligned} e_8 &= so(8)' + so(8) + 8'_v \otimes 8_v + 8'_{s-} \otimes 8_{s-} + 8'_{s+} \otimes 8_{s+} \\ &\sim su(3, \mathbb{O} \otimes \mathbb{O}) \end{aligned}$$

and a triality automorphism that maps between the triplet of octo-octonions, $\mathbb{O} \otimes \mathbb{O}$. The corresponding set of e_8 basis elements is

$$\{\, \gamma_{a'b'},\, \gamma_{ab},\, \gamma_{a'a},\, Q^-_{a'a},\, Q^+_{a'a} \,\} \tag{9.1}$$

with primed and unprimed indices, a' and a, ranging over octonion indices, $\{0, ..., 7\}$, and the bivector indices, $a'b'$ and ab, each ranging over the 28 $so(8)$ basis generator permutations with $a < b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of two $f_{4(-52)}$'s, (4.8):[2]

$$\begin{aligned}
[\gamma_{a'b'}, \gamma_{c'd'}] &= 2\left\{ n'_{a'c'}\gamma_{b'd'} - n'_{a'd'}\gamma_{b'c'} - n'_{b'c'}\gamma_{a'd'} + n'_{b'd'}\gamma_{a'c'} \right\} \\
[\gamma_{ab}, \gamma_{cd}] &= 2\left\{ n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac} \right\} \\
[\gamma_{a'b'}, \gamma_{c'a}] &= 2\left\{ -n'_{b'c'}\gamma_{a'a} + n'_{a'c'}\gamma_{b'a} \right\} \\
[\gamma_{ab}, \gamma_{a'c}] &= 2\left\{ -n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b} \right\} \\
[\gamma_{a'a}, \gamma_{b'b}] &= -2\, n_{ab}\gamma_{a'b'} - 2\, n'_{a'b'}\gamma_{ab} \\
\left[\gamma_{a'a}, Q^-_{b'b}\right] &= (\Gamma_{a'})^{c'}{}_{b'}(\Gamma_a)^c{}_b Q^+_{c'c} \\
\left[\gamma_{a'a}, Q^+_{b'b}\right] &= (\bar\Gamma_{a'})^{c'}{}_{b'}(\bar\Gamma_a)^c{}_b Q^-_{c'c} \\
\left[Q^-_{a'a}, Q^-_{b'b}\right] &= n^-_{ab}\gamma_{c'd'}(\bar\Gamma^{c'}\Gamma^{d'})_{a'b'} + n'^-_{a'b'}\gamma_{cd}(\bar\Gamma^c\Gamma^d)_{ab} \\
\left[Q^+_{a'a}, Q^+_{b'b}\right] &= n^+_{ab}\gamma_{c'd'}(\Gamma^{c'}\bar\Gamma^{d'})_{a'b'} + n'^+_{a'b'}\gamma_{cd}(\Gamma^c\bar\Gamma^d)_{ab} \\
\left[Q^-_{a'a}, Q^+_{b'b}\right] &= (\bar\Gamma^{c'})_{a'b'}(\bar\Gamma^c)_{ab}\gamma_{c'c}
\end{aligned}$$

$$\begin{aligned}
\left[\gamma_{a'b'}, Q^-_{c'a}\right] &= Q^-_{d'a}(-\bar\Gamma_{a'}\Gamma_{b'})^{d'}{}_{c'} \\
\left[\gamma_{a'b'}, Q^+_{c'a}\right] &= Q^+_{d'a}(-\Gamma_{a'}\bar\Gamma_{b'})^{d'}{}_{c'} \\
\left[\gamma_{ab}, Q^-_{a'c}\right] &= Q^-_{a'd}(-\bar\Gamma_a\Gamma_b)^d{}_c \\
\left[\gamma_{ab}, Q^+_{a'c}\right] &= Q^+_{a'd}(-\Gamma_a\bar\Gamma_b)^d{}_c
\end{aligned}$$

in which $(\Gamma_{c'})^{b'}{}_{a'} = M'_{c'a'}{}^{\tilde b'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde b}$ are octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are octonion metrics, also used to raise or lower indices. Different real forms of e_8 come from using one or two copies of $f_{4(4)}$, incorporating split-octonion multiplication tables and metrics, or using two $f_{4(-52)}$'s and flipping some signs to get split real $e_{8(8)}$.

A canonical triality automorphism of compact real $e_{8(-248)}$ is

$$\begin{aligned}
t \quad : \quad \gamma_{a'b'} &\mapsto \gamma'_{a'b'} = \gamma_{c'd'} t^{c'd'}{}_{a'b'} = \gamma_{c'd'}\left(-\tfrac{1}{2}\bar\Gamma^{c'}\Gamma^{d'}\right)_{a'b'} \\
\gamma_{ab} &\mapsto \gamma'_{ab} = \gamma_{cd} t^{cd}{}_{ab} = \gamma_{cd}\left(-\tfrac{1}{2}\bar\Gamma^c\Gamma^d\right)_{ab} \\
\gamma_{a'a} &\mapsto \gamma'_{a'a} = Q^-_{a'a} \qquad Q^-_{a'a} \mapsto Q'^-_{a'a} = Q^+_{a'a} \qquad Q^+_{a'a} \mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{aligned}$$

from combining the triality automorphisms of $f_{4(-52)}$'s. For quaternionic $e_{8(-24)}$ or split real $e_{8(8)}$, some real triality automorphisms — obtained by mixing triality automorphisms of f_4's — may contain explicit i's. The e_8 roots and canonical triality automorphism, with a triality matrix (5.1) from combining triality matrices of two f_4's, are shown in Table 9 and Figure 1.

e_8		ω_T	ω_s	U	V	p	x	y	z
	$\omega_I^{\wedge/\vee}$	∓1	±1	0	0	0	0	0	0
	$\omega_{II}^{\wedge/\vee}$	∓1	∓1	0	0	0	0	0	0
	$\omega_{III}^{\wedge/\vee}$	0	0	∓1	∓1	0	0	0	0
	$W^\pm$	0	0	∓1	±1	0	0	0	0
	$e_T^{\wedge/\vee}\,\phi_I^{/*}$	±1	0	∓1	0	0	0	0	0
	$e_T^{\wedge/\vee}\,\phi_{II}^{/*}$	0	±1	0	±1	0	0	0	0
	$e_T^{\wedge/\vee}\,\phi_{III}^{/*}$	0	∓1	0	±1	0	0	0	0
	$e_s^{\wedge/\vee}\,\phi_I^{*/}$	0	±1	±1	0	0	0	0	0
	$e_s^{\wedge/\vee}\,\phi_{II}^{*/}$	0	∓1	±1	0	0	0	0	0
	$e_s^{\wedge/\vee}\,\phi_{III}^{*/}$	±1	0	0	∓1	0	0	0	0
	$e_T^{\wedge/\vee}\,\phi^{*/}$	±1	0	±1	0	0	0	0	0
	$e_T^{\wedge/\vee}\,\phi_\pm$	±1	0	0	±1	0	0	0	0
	g	0	0	0	0	0	$(+1$	-1	$0\,)$
	$X_I^{rgb}\ \bar{X}_I^{rgb}$	0	0	0	0	0	$(\pm1$	±1	$0\,)$
	$X_{II}^{rgb}\ \bar{X}_{II}^{rgb}$	0	0	0	0	±1	$(\mp1$	0	$0\,)$
	$X_{III}^{rgb}\ \bar{X}_{III}^{rgb}$	0	0	0	0	∓1	$(\mp1$	0	$0\,)$
	$\nu_{eL}^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_{eL}^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_{eR}^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}_L^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{e}_R^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$u_L^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$-1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$u_R^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$+1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{u}_L^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$-1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{u}_R^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$+1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$d_L^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$+1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$d_R^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$-1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{d}_L^{\wedge/\vee}$	$\pm1/2$	$\pm1/2$	$+1/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{d}_R^{\wedge/\vee}$	$\mp1/2$	$\pm1/2$	$-1/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\nu_{\mu L}^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\nu_{\mu R}^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm1/2$	$\pm1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_{\mu L}^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm1/2$	$\pm1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_{\mu R}^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\mu_L^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$+1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\mu_R^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm1/2$	$\pm1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\mu}_L^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm1/2$	$\pm1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\mu}_R^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$-1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$c_L^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$-1/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$c_R^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm1/2$	$\pm1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{c}_L^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm1/2$	$\pm1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{c}_R^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$+1/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$s_L^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$+1/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$s_R^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm1/2$	$\pm1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{s}_L^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm1/2$	$\pm1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{s}_R^{\wedge/\vee}$	$\mp1/2$	$\mp1/2$	$-1/2$	$+1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\nu_{\tau L}^{\wedge}\ \bar{\nu}_{\tau R}^{\vee}$	0	0	∓1	0	±1	0	0	0
	$\nu_{\tau L}^{\vee}\ \bar{\nu}_{\tau R}^{\wedge}$	0	0	0	±1	±1	0	0	0
	$\nu_{\tau R}^{\wedge}\ \bar{\nu}_{\tau L}^{\vee}$	±1	0	0	0	±1	0	0	0
	$\nu_{\tau R}^{\vee}\ \bar{\nu}_{\tau L}^{\wedge}$	0	±1	0	0	±1	0	0	0
	$\tau_L^{\wedge}\ \bar{\tau}_R^{\vee}$	0	0	0	∓1	±1	0	0	0
	$\tau_L^{\vee}\ \bar{\tau}_R^{\wedge}$	0	0	±1	0	±1	0	0	0
	$\tau_R^{\wedge}\ \bar{\tau}_L^{\vee}$	0	∓1	0	0	±1	0	0	0
	$\tau_R^{\vee}\ \bar{\tau}_L^{\wedge}$	∓1	0	0	0	±1	0	0	0
	$t_L^{\wedge}\ \bar{t}_R^{\vee}$	0	0	∓1	0	0	$(\mp1$	0	$0\,)$
	$t_L^{\vee}\ \bar{t}_R^{\wedge}$	0	0	0	±1	0	$(\mp1$	0	$0\,)$
	$t_R^{\wedge}\ \bar{t}_L^{\vee}$	±1	0	0	0	0	$(\mp1$	0	$0\,)$
	$t_R^{\vee}\ \bar{t}_L^{\wedge}$	0	±1	0	0	0	$(\mp1$	0	$0\,)$
	$b_L^{\wedge}\ \bar{b}_R^{\vee}$	0	0	0	∓1	0	$(\mp1$	0	$0\,)$
	$b_L^{\vee}\ \bar{b}_R^{\wedge}$	0	0	±1	0	0	$(\mp1$	0	$0\,)$
	$b_R^{\wedge}\ \bar{b}_L^{\vee}$	0	∓1	0	0	0	$(\mp1$	0	$0\,)$
	$b_R^{\vee}\ \bar{b}_L^{\wedge}$	∓1	0	0	0	0	$(\mp1$	0	$0\,)$

Table 9. The 240 roots of the compact real form, $e_{8(-248)}$, suggestively labeled as elementary particles. $\omega_{I,II,III}^{\wedge/\vee}$ are gravitational spin connection roots related by triality, $W^\pm$ are weak bosons, $e_{s/T}^{\wedge/\vee}\phi$ are gravitational frame-Higgs roots, g are gluons, and X are colored X bosons. There is a complete Standard Model generation of fermions with spin, related to two other "generations" by triality.

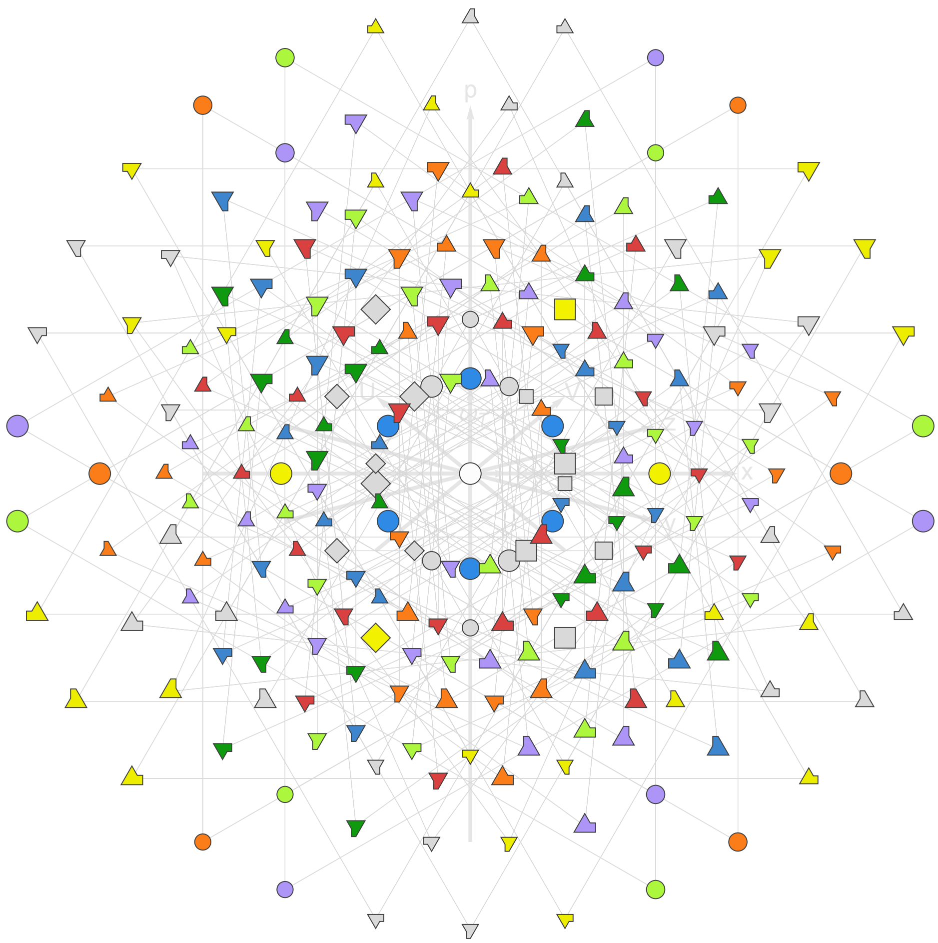


Figure 1. The 240 roots of e_8, suggestively labeled as elementary particles, with "generations" related by triality.

The particle assignment and canonical triality automorphism, (5.1), in Table 9 and Figure 1, is consistent with split-real $e_{8(8)}$, quaternionic $e_{8(-24)}$, or compact $e_{8(-248)}$. Using the split-octonions and/or octonions, corresponding to Clifford algebras $Cl(4,4)$ or $Cl(0,8)$, the Cartan subalgebras correspond to Clifford basis bivectors with signatures:

$$
\begin{aligned}
\{\omega_T, \omega_s, U, V; p, x, y, z\} &= \{\gamma_{3'\,4'}, \gamma_{1'\,2'}, \gamma_{5'\,6'}, \gamma_{7'\,8'}; \gamma_{7\,8}, \gamma_{1\,2}, \gamma_{3\,4}, \gamma_{5\,6}\} \\
e_{8(8)} &: \{\gamma_{++}, \gamma_{++}, \gamma_{++}, \gamma_{++}; \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}\} \\
e_{8(-24)} &: \{\gamma_{++}, \gamma_{++}, \gamma_{--}, \gamma_{--}; \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}\} \\
e_{8(-248)} &: \{\gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}; \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}\}
\end{aligned}
$$

The only way to accommodate three triality-related linearly-independent fermion generations in e_8 is by having Euclidean rather than Lorentzian spacetime and spinors.

More realistic models, with three triality-related generations, including Lorentzian spacetime and spinors, are achieved by considering the triality eigenspaces, (6.1), of split-real $e_{8(8)}$ or quaternionic $e_{8(-24)}$, and their triality-related five-gradings, (6.2). The relevant Cartan subalgebra generators compatible with e_8 triality eigenspaces are:

$$
\begin{aligned}
\{\omega_t, \omega_s, U, V, w, x, y, z\} &= \{\gamma_{3\,15}, \gamma_{1\,2}, \gamma_{5\,6}, \gamma_{7\,8}, \gamma_{4\,16}, \gamma_{9\,10}, \gamma_{11\,12}, \gamma_{13\,14}\} \\
e_{8(8)} &: \{\gamma_{++}, \gamma_{++}, \gamma_{++}, \gamma_{++}, \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}\} \\
e_{8(-24)} &: \{\gamma_{++}, \gamma_{++}, \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}, \gamma_{--}\}
\end{aligned} \tag{9.2}
$$

Within these Cartans, for $e_{8(8)}$ or $e_{8(-24)}$, $w = \gamma_{4\,16}$ is the $u(1)$ generator in a triality invariant eigenspace, either $[so(8,6)+u(1)_w]$ or $[so(4,10)+u(1)_w]$. The corresponding triality automorphism acts as multiplication by $e^{+2\pi i/3}$ on the $[14^v_{+1} + 64^{s-}_{-1/2}]$ and $e^{-2\pi i/3}$ on the complex conjugate, $[14^v_{-1} + 64^{s+}_{+1/2}]$. These same triality automorphisms, Θ, act in interesting ways within the five-graded decompositions, (6.2). The Cartans for the matched five-graded decompositions differ from the Cartans for the triality eigenspaces by a kind of Wick rotation:

$$
\begin{aligned}
\{\omega_t^{\mathbb{R}}, \omega_s, U, V, w^{\mathbb{R}}, x, y, z\} &= \{\gamma_{3\,4}, \gamma_{1\,2}, \gamma_{5\,6}, \gamma_{7\,8}, \gamma_{15\,16}, \gamma_{9\,10}, \gamma_{11\,12}, \gamma_{13\,14}\} \\
e_{8(8)} &: \{\gamma_{+-}, \gamma_{++}, \gamma_{++}, \gamma_{++}, \gamma_{+-}, \gamma_{--}, \gamma_{--}, \gamma_{--}\} \\
e_{8(-24)} &: \{\gamma_{+-}, \gamma_{++}, \gamma_{--}, \gamma_{--}, \gamma_{+-}, \gamma_{--}, \gamma_{--}, \gamma_{--}\}
\end{aligned} \tag{9.3}
$$

This Wick rotation exchanges the temporal γ_4 and a spatial γ_{15}. The resulting $\omega_t^{\mathbb{R}} = \gamma_{3\,4}$ and $w^{\mathbb{R}} = \gamma_{15\,16}$ have real eigenvalues, while $\omega_t = \gamma_{3\,15}$ and $w = \gamma_{4\,16}$ have imaginary eigenvalues.

An explicit description of a continuous Wick rotation within e_8 is given by defining a psuedo-similarity transformation acting on e_8 Clifford vector, bivector, and spinor generators,[1]

$$
S_\theta = e^{i\frac{\theta}{2}\gamma_4\gamma_{15}} \qquad \gamma_a \mapsto \gamma_a^\theta = S_\theta \gamma_a S_\theta^- \qquad \gamma_{ab} \mapsto \gamma_{ab}^\theta = S_\theta \gamma_{ab} S_\theta^- \qquad \Psi \mapsto \Psi_\theta = S_\theta \Psi
$$

with $0 \le \theta \le \frac{\pi}{2}$. This corresponds to a Wick rotation of time by $t \mapsto t_\theta = e^{i\theta} t$. For $\theta = \frac{\pi}{2}$ this gives Euclidean time, $t_E = i\,t$, and

$$
S_E = \frac{1}{\sqrt{2}}(1 + i\gamma_4\gamma_{15}) \qquad \gamma_4 \mapsto \gamma_4^E = i\gamma_{15} \qquad \gamma_{15} \mapsto \gamma_{15}^E = i\gamma_4 \qquad \Psi \mapsto \Psi_E = S_E \Psi
$$

Some Cartan generators change under Wick rotation,

$$S_E\{\gamma_{3\,4},\gamma_{15\,16}\}S_E^- = \{i\gamma_{3\,15}, i\gamma_{4\,16}\}$$

so root coordinates, for α^L and α^E, differ by these i's, and the Lorentzian and Euclidean spinorial root vectors differ by $\Psi^E_{\alpha^E} = S_E \Psi^L_{\alpha^L}$. This Wick rotation allows us to transform between the Euclidean e_8 triality eigenspaces, (6.1),

$$\begin{aligned} e_{8(8)} &\sim \Big[14^v_{-1} + 64^{s+}_{+1/2}\Big]_{-1} + \Big[so(8,6)+u(1)_w\Big]_0 + \Big[14^v_{+1} + 64^{s-}_{-1/2}\Big]_{+1} \\ e_{8(-24)} &\sim \Big[14^v_{-1} + 64^{s+}_{+1/2}\Big]_{-1} + \Big[so(4,10)+u(1)_w\Big]_0 + \Big[14^v_{+1} + 64^{s-}_{-1/2}\Big]_{+1} \end{aligned} \tag{9.4}$$

and the Lorentzian spaces of the e_8 five-gradings, (6.2).

In the spinor representation space of a compact or non-compact spin group, root vectors corresponding to fermions have complex conjugate root vectors called "conjugate fermions", which have complex conjugate roots — the conjugate root coordinates having opposite signs for the compact Cartan generators, such as $\gamma_{1\,2}$ spin, and the same signs for non-compact Cartan generators, such as $\gamma_{3\,4}$ boost. For spinors of non-compact spin groups there can also be "mirror fermions", for which the mirror root coordinates have the same signs for the compact Cartan generators, and opposite signs for non-compact Cartan generators. In the five-graded decompositions, (6.2),

$$\begin{aligned} e_{8(8)} &= 14^v_{-1} + 64^{s+}_{+1/2} + \Big[so(7,7)+so(1,1)_{w^{\mathbb{R}}}\Big] + 64^{s-}_{-1/2} + 14^v_{+1} \;=\; so(8,8) + 128^{s+} \\ e_{8(-24)} &= 14^v_{-1} + 64^{s+}_{+1/2} + \Big[so(3,11)+so(1,1)_{w^{\mathbb{R}}}\Big] + 64^{s-}_{-1/2} + 14^v_{+1} \;=\; so(4,12) + 128^{s+} \end{aligned}$$

the $64^{s-}_{-1/2}$ spinors of $so(7,7)$ or $so(3,11)$ are one generation of Standard Model fermions, and the $64^{s+}_{+1/2}$ are their mirrors. The triality automorphisms, Θ, for which $[so(8,6)+u(1)_w]$ and $[so(4,10)+u(1)_w]$ are invariant subalgebras, mix the Standard Model fermions, $64^{s-}_{-1/2}$, and mirror fermions, $64^{s+}_{+1/2}$, in an interesting way.

For the mixed Cartan, (9.3), the Standard Model fermion root vectors, such as $e_L^\wedge = \Psi_j$ (for some j) and their conjugate antifermions, $\bar{e}_R^\vee = \Psi_j^*$, are in $64^{s-}_{-1/2}$, while their mirrors, $e_L^{\wedge m} = \Psi_j^m$ and $\bar{e}_R^{\vee m} = \Psi_j^{*m} = \Psi_j^{m*}$, are in $64^{s+}_{+1/2}$, and all are in the 128^{s+} of $so(8,8)$ or $so(4,12)$, mixed by Θ. If we carry out the Wick rotation, $\gamma_4 \leftrightarrow i\gamma_{15}$, to go from the Euclidean triality-eigenspace decomposition to the Lorentzian five-grading decomposition, then we see that the triality automorphism mixes between real and imaginary fermions and their mirrors,

$$\Theta \quad : \quad \begin{bmatrix} \Psi_j \\ \Psi_j^m \end{bmatrix} \mapsto \begin{bmatrix} \Psi'_j \\ \Psi'^m_j \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Psi_j \\ \Psi_j^m \end{bmatrix}$$

The $64^{s-}_{-1/2}$ is inhabited by the real, $\Psi_j^{\mathbb{R}}$, and imaginary, $\Psi_j^{\mathbb{I}}$, parts of the complex fermion root vectors, Ψ_j. And the $64^{s-}_{-1/2}$ is inhabited by the real and imaginary parts of the complex mirror fermion root vectors. The triality automorphism mixes these real and imaginary parts of fermions and their mirrors,

$$\begin{aligned} \Psi_j^{\mathbb{R}} &= \tfrac{1}{2}\left(\Psi_j + \Psi_j^*\right) \\ \Psi_j^{\mathbb{I}} &= \tfrac{1}{2i}\left(\Psi_j - \Psi_j^*\right) \\ \Psi_j^{m\mathbb{R}} &= \tfrac{1}{2}\left(\Psi_j^m + \Psi_j^{m*}\right) \\ \Psi_j^{m\mathbb{I}} &= \tfrac{1}{2i}\left(\Psi_j^m - \Psi_j^{m*}\right) \end{aligned} \qquad \Theta \;:\; \begin{bmatrix} \Psi_j^{\mathbb{R}} \\ \Psi_j^{\mathbb{I}} \\ \Psi_j^{m\mathbb{I}} \\ \Psi_j^{m\mathbb{R}} \end{bmatrix} \mapsto \begin{bmatrix} \Psi_j'^{\mathbb{R}} \\ \Psi_j'^{\mathbb{I}} \\ \Psi_j'^{m\mathbb{I}} \\ \Psi_j'^{m\mathbb{R}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & & & -\frac{\sqrt{3}}{2} \\ & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \\ & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \\ \frac{\sqrt{3}}{2} & & & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Psi_j^{\mathbb{R}} \\ \Psi_j^{\mathbb{I}} \\ \Psi_j^{m\mathbb{I}} \\ \Psi_j^{m\mathbb{R}} \end{bmatrix}$$

a rotation by $120°$ in 64 independent planes in the 128^{s+} of $so(8,8)$ or $so(4,12)$. This triality automorphism acts between three sets of complex state vectors,

$$\begin{aligned} \Psi_j^I &= \Psi_j^{\mathbb{R}} + i\Psi_j^{\mathbb{I}} = \Psi_j \\ \Psi_j^{II} &= -\tfrac{1}{2}\Psi_j^{\mathbb{R}} - \tfrac{1}{2}i\Psi_j^{\mathbb{I}} + \tfrac{\sqrt{3}}{2}i\Psi_j^{m\mathbb{R}} - \tfrac{\sqrt{3}}{2}\Psi_j^{m\mathbb{I}} = -\tfrac{1}{2}\Psi_j + \tfrac{\sqrt{3}}{2}i\Psi_j^m \\ \Psi_j^{III} &= -\tfrac{1}{2}\Psi_j^{\mathbb{R}} - \tfrac{1}{2}i\Psi_j^{\mathbb{I}} - \tfrac{\sqrt{3}}{2}i\Psi_j^{m\mathbb{R}} + \tfrac{\sqrt{3}}{2}\Psi_j^{m\mathbb{I}} = -\tfrac{1}{2}\Psi_j - \tfrac{\sqrt{3}}{2}i\Psi_j^m \end{aligned}$$

as a cyclic permutation, $\Theta : \Psi_j^I \mapsto \Psi_j^{II} \mapsto \Psi_j^{III} \mapsto \Psi_j^I$. Using this description, mirror fermion root vectors can be expressed as

$$\Psi_j^m = \tfrac{1}{i\sqrt{3}}\left(\Psi_j^{II} - \Psi_j^{III}\right) \tag{9.5}$$

The triality automorphism mixing is similar for the vectorial root vectors, $w \in 14^v_{\pm 1}$, which are spanned by $\gamma_{x\,15}$ and $\gamma_{x\,16}$ bivector generators in $so(8,8)$ or $so(4,12)$. The vectorial root vectors mix as

$$\Theta \;:\; \begin{bmatrix} w \\ w^m \end{bmatrix} \mapsto \begin{bmatrix} w' \\ w'^m \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} w \\ w^m \end{bmatrix}$$

Note that we have chosen the simplest possible triality automorphism, Θ, compatible with a Wick rotation from a five-grading, (6.2), to triality eigenspaces. The corresponding elementary particle assignments and the triality relationship to their mirrors are shown in Table 10 and Figure 2. Different choices of $u(1)_w$ within $so(8,8)$ or $so(4,12)$ correspond to different triality automorphisms that can give more complicated mixing between the $64^{s-}_{-1/2}$ fermions and mirrors, and between the $14^v_{\pm 1}$ vectors. This suggests the three sets of 64 triality-related generators may correspond to the three generations of fermions in the Standard Model of particle physics.

We have identified a canonical triality automorphism of $e_{8(8)}$ or $e_{8(-24)}$ with the adjoint action of a $u(1)$ generator, $w = \gamma_{4\,16}$, in a $so(8,8)$ or $so(4,12)$ subalgebra. It is reasonable to ask what this Lie algebra generator is in our triality decomposition, 9.1. The answer is

$$w = \tfrac{1}{\sqrt{3}}\left(\gamma_1 + Q_1^- + Q_1^+\right)$$

the normalized Lie algebra element corresponding to the sum of three unit division algebra elements, vector, negative chiral spinor, and positive chiral spinor. The canonical triality automorphism is the exponentiation of the adjoint action of this generator, $t = e^{-2\pi/3\,\mathrm{Ad_w}}$.

e_8		$\omega_t^{\mathbb{R}}$	ω_s	U	V	$w^{\mathbb{R}}$	x	y	z
	$\omega_L^{\wedge/\vee}$	∓ 1	± 1	0	0	0	0	0	0
	$\omega_R^{\wedge/\vee}$	± 1	± 1	0	0	0	0	0	0
	$W^{\pm}$	0	0	∓ 1	± 1	0	0	0	0
	$W'^{\pm}$	0	0	± 1	± 1	0	0	0	0
	g	0	0	0	0	0	$(+1$	-1	$0\,)$
	$X^{PS}_{\pm 2/3}$	0	0	0	0	0	$(\pm 1$	± 1	$0\,)$
	$X^{GG}_{\pm 1/3}$	0	0	∓ 1	0	0	$(\pm 1$	0	$0\,)$
	$X^{GG}_{\mp 2/3}$	0	0	0	∓ 1	0	$(\pm 1$	0	$0\,)$
	$X_{\pm 1/3}$	0	0	± 1	0	0	$(\pm 1$	0	$0\,)$
	$X_{\pm 4/3}$	0	0	0	± 1	0	$(\pm 1$	0	$0\,)$
	$e_t^{\wedge/\vee}\phi_0$	± 1	0	$+1$	0	0	0	0	0
	$e_s^{\wedge/\vee}\phi_0$	0	± 1	$+1$	0	0	0	0	0
	$e_t^{\wedge/\vee}\phi_0^*$	± 1	0	-1	0	0	0	0	0
	$e_s^{\wedge/\vee}\phi_0^*$	0	± 1	-1	0	0	0	0	0
	$e_t^{\wedge/\vee}\phi_+$	± 1	0	0	$+1$	0	0	0	0
	$e_s^{\wedge/\vee}\phi_+$	0	± 1	0	$+1$	0	0	0	0
	$e_t^{\wedge/\vee}\phi_-$	± 1	0	0	-1	0	0	0	0
	$e_s^{\wedge/\vee}\phi_-$	0	± 1	0	-1	0	0	0	0
	$e_t^{\wedge/\vee}\phi_{+1/3}$	± 1	0	0	0	0	$(+1$	0	$0\,)$
	$e_s^{\wedge/\vee}\phi_{+1/3}$	0	± 1	0	0	0	$(+1$	0	$0\,)$
	$e_t^{\wedge/\vee}\phi_{-1/3}$	± 1	0	0	0	0	$(-1$	0	$0\,)$
	$e_s^{\wedge/\vee}\phi_{-1/3}$	0	± 1	0	0	0	$(-1$	0	$0\,)$
	$e_t^{\wedge/\vee}w$	± 1	0	0	0	$+1$	0	0	0
	$e_s^{\wedge/\vee}w$	0	± 1	0	0	$+1$	0	0	0
	$w\phi_0\ w\phi_0^*$	0	0	± 1	0	$+1$	0	0	0
	$w\phi_\pm$	0	0	0	± 1	$+1$	0	0	0
	$w\phi_{\pm 1/3}$	0	0	0	0	$+1$	$(\pm 1$	0	$0\,)$
	$\nu_{eL}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar\nu_{eL}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar\nu_{eR}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar e_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar e_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$u_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$u_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar u_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar u_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$d_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$d_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar d_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar d_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$e_t^{\wedge/\vee}w^m$	± 1	0	0	0	-1	0	0	0
	$e_s^{\wedge/\vee}w^m$	0	± 1	0	0	-1	0	0	0
	$w^*\phi_0\ w^m\phi_0^*$	0	0	± 1	0	-1	0	0	0
	$w^m\phi_\pm$	0	0	0	± 1	-1	0	0	0
	$w^*\phi_{\pm 1/3}$	0	0	0	0	-1	$(\pm 1$	0	$0\,)$
	$\nu_{eL}^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\nu_{eR}^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar\nu_{eL}^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar\nu_{eR}^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar e_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar e_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$u_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$u_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar u_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar u_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$d_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$d_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar d_L^{\wedge/\vee m}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar d_R^{\wedge/\vee m}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$

Table 10. The 240 roots of quaternionic and split real e_8, suggestively labeled as elementary particles, matching triality eigenspaces. The triality invariant eigenspace includes the spin connection, $\omega_{L/R}^{\wedge/\vee}$, weak and weaker bosons, W and W', gluons, g, various X bosons, and frame-Higgs, $e_{s/t}^{\wedge/\vee}\phi$. The $+1$ triality eigenspace includes frame-w roots, $e_{s/t}^{\wedge/\vee}w$ and $w\phi$, and a full Standard Model generation of fermions with spin. The -1 triality eigenspace includes their mirrors.

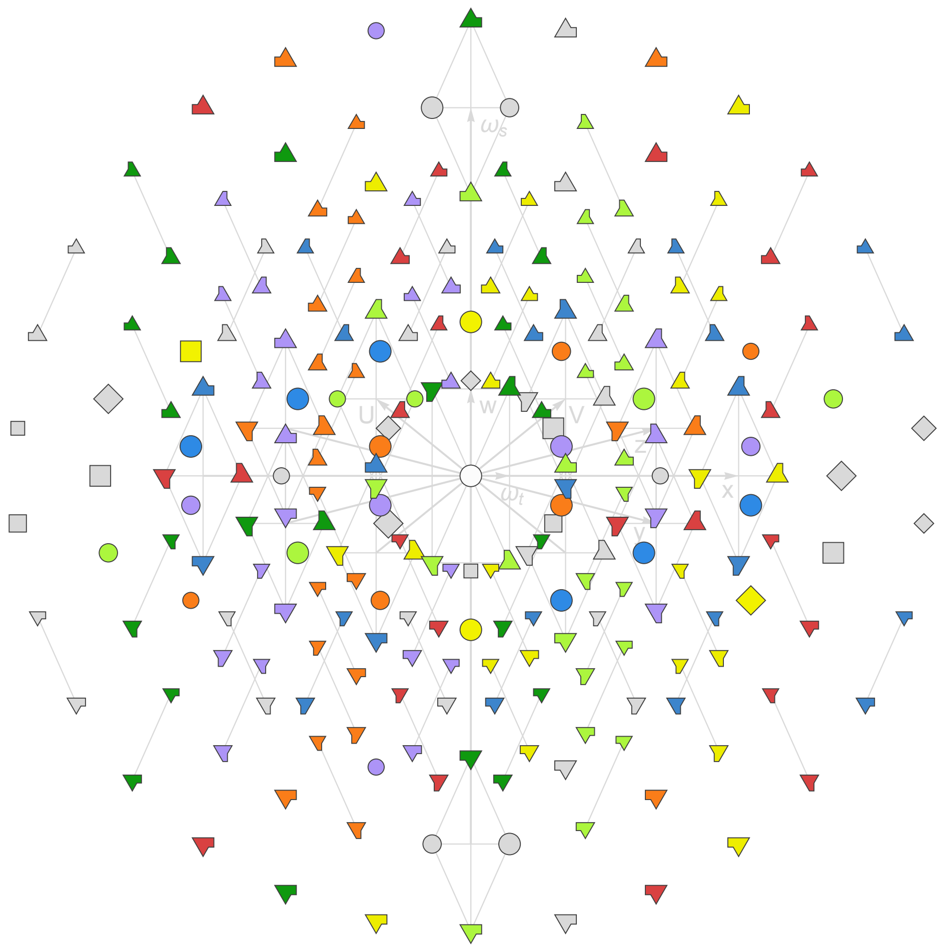


Figure 2. The 240 roots of e_8, suggestively labeled as elementary particles, with mirrors related by triality.

10 Exceptional Unification

The suggested assignments of elementary particle labels to roots of e_6, e_7, and e_8 correspond to deeper underlying theories of Exceptional Unification. Each deserves a long description; we only discuss them briefly here. One thing all have in common is a unification of bosons (gauge field 1-forms) and fermions (spinor Grassmann number fields) in a superconnection valued in an exceptional Lie algebra, usually corresponding to a two-grading. This is sensible within the context of Lie Group Cosmology [34], with Grassmann number fields understood as 1-forms along spinorial directions within the Lie group. The supercurvature gives the usual curvature 2-form of the connection 1-form plus an exterior-Dirac derivative of the fermions, which can be used with a generalized-Hodge star to build a generalized Yang-Mills action that includes fermions. The exceptional Lie brackets produce the correct interactions between gauge fields and spinorial fermions, which is encouraging. Beyond these commonalities, Exceptional Unification models include the gauge fields of the $SO(10)$ Grand Unified Theory: the electroweak W and Z and γ, and gluons, g, as well as the "weaker" W' and Z' gauge bosons, Pati-Salam $so(6)$ gauge bosons, X^{PS}, Georgi-Glashow $su(5)$ gauge bosons, X^{GG}, and other GUT gauge bosons, X. The most minimal model is based on e_6, a subalgebra of e_7 and of e_8.

The particle assignment within e_6, shown in Table 7, largely ignores triality and corresponds to the $SO(10)$ Grand Unified Theory with one generation of fermions and antifermions. The five-dimensional Cartan subalgebra of $so(10)$ corresponds to charges (U, V, x, y, z), with U and V combining to produce $su(2)$ weak charge, W, and $su(2)$ weaker charge, W', while (x, y, z) combine to give strong $su(3)$ color charges and $u(1)$ baryon minus lepton number charge, B, (7.2). These further combine to give hypercharge, Y, and electric charge, Q. The sixth fundamental Cartan subalgebra element within e_6 is scaled helicity, H, corresponding to a $u(1)$ gauge field. One generation of fermions, without spin, is embedded in e_6, with left-chiral fermions and antifermions in a 16_{s+} spinor of $so(10)$, and right-chiral fermions and antifermions in a 16_{s-}.

The particle assignment within e_7, shown in Table 8, relates to the work of Dixon, Furey, and Hughes, with $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ fermions [4, 8, 9]. While it is possible to fit three linearly-independent generations of Standard Model fermions into e_7, related by triality, one cannot do it with both the left and right-chiral partners and the antiparticle partners – one must choose either, and is thus forced to use a complex form of e_7 to fit three generations, which is problematic for the gauge fields. Alternatively, as shown in Table 8, one can include one generation of left-chiral fermions and antifermions, with spin, all in the same triality eigenspace, $[2 \otimes 16^{s+}_{+1/2}]$. There are then two choices. One could consider the right-chiral fermions and antifermions to be in $[2 \otimes 16^{s-}_{-1/2}]$, and call it a day, with a one-generation e_7 model. Or, using the triality-rotation of the left-chiral generation within $[2 \otimes 32_{s+}]$, one could consider this a three-generation model, but again only with left-chiral fermions and antifermions. In either case the gravitational frame is missing from e_7, and it is not clear what to make of the $\Phi \in 10^v_{\pm 1}$ — though it includes a needed Higgs field. The Φ appears to be a scalar or possibly Grassmann field, useful for $SO(10)$ and Standard Model symmetry breaking, but understanding its nature and a consistent E7 Theory awaits future work.

The particle assignment within e_8, shown in Table 9 and Figure 1, matches the particle assignment in "An Exceptionally Simple Theory of Everything" [12]. Each generation of particles matches to an $\mathbb{O} \otimes \mathbb{O}$ or to an $\mathbb{O}' \otimes \mathbb{O}$, depending on whether the $(\omega_T, \omega_s, U, V)$ corresponds to an $so(8)$ or $so(4,4)$, and each linearly-independent generation is related to the others by triality. The (p, x, y, z) $so(8)$ charges include a particle or antiparticle charge, p, which mixes with B under triality. This $so(8)$ can act on 8's of the same signature as those of the first $so(8)$ for the compact real form, or 8's of opposite signature for the split real form. The main downside of this embedding of the three generation Standard Model in e_8 is that it is necessarily Euclidean, and must be part of some larger theory to describe our Lorentzian reality. Also, the natural two-grading of e_8 does not match this assignment of fermions. Several variations of this embedding of three linearly-independent generations of fermions within different real forms of e_8 are possible, but all have these drawbacks.

A different particle assignment within quaternionic or split real e_8, matching its natural two-grading and its triality eigenspaces, is shown in Table 10 and Figure 2. Technically, the blocks of triality eigenspaces are correct for this table of roots only when the compact ω_t and w Cartan generators are used, and the triality eigenspaces are inhabited by mixtures of particles and their mirrors when using the non-compact $\omega_t^{\mathbb{R}}$ and $w^{\mathbb{R}}$ Cartan generators. This produces a nice result: triality automorphisms of non-compact e_8 can rotate three full generations of fermions between standard fermions and their mirrors. Early criticism of E8 Theory held that these mirror fermions (also called an "anti-generation") imply the theory cannot work [33], but this is not our view. In our model, the mirror fermions are replaced by the second and third generations, related to the first generation by triality. Of course, this implies the three generations of fermions are not linearly-independent, and we do expect there to be generational mixing. Triality also mixes the frame-Higgs root vectors that have a temporal component, such as $\{e_t\phi_0, e_t\phi_0^*, e_t w, e_t w^m\}$. This may introduce multi-fingered time, with these fields interacting differently with the three generations of fermions. This new description of E8 Theory, using triality eigenspaces, has many promising features. All three generations of fermions live in the odd part of e_8's natural two-grading. The gravitational spin connection, frame-Higgs, and $SO(10)$ GUT gauge fields largely live in the triality invariant eigenspace. And the spin and gauge charges of the three generations of fermions, from $\{\omega_s, U, V, x, y, z\}$, are the same, producing identical Standard Model quantum numbers for all three generations of fermions. Further development of this model seems warranted.

11 Division Algebra Automorphisms and g_2

The division algebras, and their split algebras, are each invariant under transformation by elements of their *automorphism group*, $\Phi \in G_{\mathbb{D}}$ [4]. These group transformations leave multiplication invariant, $\Phi(a)\Phi(b) = \Phi(ab)$. The complex numbers are invariant under complex conjugation. The quaternions are invariant under $SO(3)$ rotations of their imaginary elements. The octonions are not invariant under $SO(7)$ rotations of their imaginary elements, but under a subgroup, G_2, that preserves octonionic non-associativity. The automorphism groups of the split algebras are similar,

$$\begin{aligned} G_{\mathbb{C}} &= \mathbb{Z}_2 & \qquad G_{\mathbb{C}'} &= \mathbb{Z}_2 \\ G_{\mathbb{H}} &= SO(3) & \qquad G_{\mathbb{H}'} &= SO(1,2) \\ G_{\mathbb{O}} &= G_{2(-14)} & \qquad G_{\mathbb{O}'} &= G_{2(2)} \end{aligned}$$

Since automorphism group elements leave division algebra multiplication invariant, these groups are subgroups of the corresponding triality group.

For the quaternions, there is an *inner triality automorphism*, Ad_t, corresponding to a 120-degree rotation around the axis formed by averaging the three unit imaginary quaternions. This rotation cycles the three imaginary unit quaternions,

$$t = -\tfrac{1}{2}(e_0 + e_1 + e_2 + e_3) \qquad t^3 = e_0 \qquad t\, e_1\, t^- = e_2 \qquad t\, e_2\, t^- = e_3 \qquad t\, e_3\, t^- = e_1$$

If we instead interpret t as an octonion, we see that Ad_t also preserves octonionic multiplication. If we represent an octonion as an 8-dimensional vector, and use the standard octonionic multiplication table, Ad_t acts on the octonions as a matrix,

$$\mathrm{Ad}_t \qquad \sim \qquad \begin{bmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & 1 & & & & & & \\ & & & & -1/2 & +1/2 & +1/2 & +1/2 \\ & & & & -1/2 & -1/2 & -1/2 & +1/2 \\ & & & & -1/2 & +1/2 & -1/2 & -1/2 \\ & & & & -1/2 & -1/2 & +1/2 & -1/2 \end{bmatrix}$$

If we use the same t in the split-octonions, we get the automorphism:

$$\mathrm{Ad}'_t \qquad \sim \qquad \begin{bmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & 1 & & & & & & \\ & & & & -1/2 & -1/2 & -1/2 & -1/2 \\ & & & & +1/2 & -1/2 & -1/2 & +1/2 \\ & & & & +1/2 & +1/2 & -1/2 & -1/2 \\ & & & & +1/2 & -1/2 & +1/2 & -1/2 \end{bmatrix}$$

This same t does not give an automorphism in the split-quaternions. However, if we have $e'_2 e'_2 = -e'_0$, we can have an inner triality automorphism from $t' = -\frac{1}{2}(e'_0 - \sqrt{3} e'_2)$, which rotates $120°$ in the $e'_0 - e'_2$ plane.

12 Discussion

In this work we have explored the deep relationship between division algebras, Clifford algebras, generalized reflections, triality automorphisms, triality Lie algebras, the magic square of Lie algebras, exceptional Lie algebras, root systems, triality eigenspaces, Vinberg Θ-algebras, and connections with particle physics. From any of these specific subjects, the others can be better understood, so a reader may make their choice of whichever mathematical starting point is more familiar. Although it is expected that current researchers will merely pull useful computational tools from this paper, it is hoped that they will also use this paper to better understand and appreciate the other ways of working with this material for model building.

The true heart of this subject is triality — a real, cyclic, trilinear function of three elements of a vector space — which can be used to define the division algebras and their related split composition algebras. These division algebras lend themselves to the explicit matrix representation of certain Clifford algebras, which have a direct geometric interpretation. With this geometric point of view, we can describe reflections through vectors and, using triality, generalized reflections through spinors. These generalized reflections can be combined to define triality automorphisms, cycling vectors and spinors or three division algebra elements. These vectors and spinors combine with the triality algebras of division algebras to produce the triality Lie algebras, understood as generalizations of $su(3)$. Within these triality Lie algebras, the relationships between vectors, chiral spinors, generalized reflections, and triality automorphisms are described using division algebra products or more explicitly using representative Clifford algebra matrices. We encounter the surprising fact that real Lie algebra automorphisms can include explicit i's, provided the automorphisms commute with the complex conjugation used to define the real form of the Lie algebra. The Cartan-Weyl description of these Lie algebras, and their root systems, can be used to visually appreciate their structure and automorphisms. Although useful, and often visually appealing, the root system description of Lie algebras and their automorphisms elides the signs of structure constants and maps between root vectors. These signs can often be guessed or obtained algorithmically, but are obtained directly by the methods presented in this work. The triality Lie algebras combine in pairs to produce the magic square Lie algebras, which are also invariant under generalized reflections and triality automorphisms. Ultimately, building from division algebras, all exceptional Lie algebras and their automorphisms can be understood and described explicitly using these methods. With an explicit description of triality automorphisms in hand, triality eigenspaces are described, and used to define Vinberg Θ-algebras and 3-symmetric spaces. A canonical triality automorphism for any magic square Lie algebra described using a triality decomposition is generated by the Lie algebra element

$$T = \frac{1}{\sqrt{3}} \left(\gamma_1 + Q_1^- + Q_1^+ \right)$$

the normalized Lie algebra element corresponding to the sum of three unit division algebra elements, vector, negative chiral spinor, and positive chiral spinor. As an inner automorphism, the canonical triality automorphism is the exponentiation of the adjoint action of this generator, $t = e^{-2\pi/3\,\mathrm{Ad}_\mathrm{T}}$. These tools can be used for model building, and possibly to describe the three generations of fermions in the Standard Model.

The relationship to particle physics largely motivates this work, with triality at its center. The strongest objection to E8 Theory has been the mathematical fact that real forms of e_8 necessarily include mirror fermions [33]. While this condemnation is widely accepted by the particle physics community, it aggressively ignores the possibility that mirror fermion degrees of freedom may be inhabited by second and third generation fermions via triality — a possibly wrong idea proposed and explored in our work here. Although the complete details of describing three fermion generations using triality remain unclear, if triality is an interesting area of exploration for physics, which it almost certainly is, then the explicit tools and descriptions provided in this paper are the keys to the castle. It is hoped that other researchers will use this work to further their own explorations within this rich area of mathematical physics.

13 Disclosures and Declarations

13.1 Data Availability Statement

This article contains only theoretical analysis, equations, and derivations. No datasets were generated or analyzed in this work.

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13.3 Competing Interests

The author declares that there are no conflicts of interest regarding the publication of this work.

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