C, P, T, and Triality

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Abstract: Discrete charge, parity, and time symmetries (C, P, and T) of quantized fermion states are extended by a triality symmetry (t), producing the CPTt Group, transforming between three generations of fermions.

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1 Introduction

The Standard Model of particle physics has been wildly successful at explaining physical phenomenon for the past fifty years. Over that time the theory has been filled in with various particles and more accurate measurements of their interactions and masses, but the fundamental structure has endured. Perhaps the most perplexing mystery within the Standard Model is why fermions, described by Dirac spinors, exist in three generations—each generation having identical charges with respect to the fundamental forces but different masses and mixings with respect to the Higgs field. This tri-fold symmetry of fermion generations strongly indicates there is a finite symmetry, triality, that acts in conjunction with the charge, parity, and time reversal symmetries of Dirac spinors in the Standard Model. In this work, we introduce this symmetry in the context of C, P, and T, and work out the group theoretical implications.

Dirac spinors are a foundational part of the Standard Model. They are not, however, an irreducible representation space of the spacetime spin group, $Spin(1,3)$. Rather, Dirac spinors are an irreducible representation space of the spacetime pin group, $Pin(1,3)$, a double cover of the spacetime orthogonal group, $O(1,3)$, and a subgroup of the spacetime Clifford group $Cl^*(1,3)$ (consisting of invertible Clifford algebra elements).[1] The identity component of the spacetime spin group is the spacetime orthochronous spin group, $Spin^+(1,3)$ (the group of rotations obtained by exponentiating $Cl(1,3)$ bivectors), equivalent to $SL(2,\mathbb{C})$, and the double cover of $SO^+(1,3)$. The spacetime orthochronous spin group extends to the spacetime pin group by adding spatial reflections, called parity conjugations (P), and temporal reflections, called time conjugations (T). To summarize, we have:

$$Spin^+(1,3) \otimes \{1, P, T, PT\} = Spin(1,3) \otimes \{1, T\} = Pin(1,3) \subset Cl^*(1,3) \subset Cl(1,3)$$

Because the weak interaction maximally violates P-symmetry (interacting with only left-chiral parts of fermions), many modern authors choose to disregard foundational P-symmetry, building theories up from Weyl spinors in an irreducible representation space of $SL(2,\mathbb{C}) = Spin^+(1,3)$. However, it is very likely that all fermions consist of both left and right-chiral parts that are related by a P-symmetry that is broken but should still be considered foundational.

$Pin(1,3)$ group elements, U, act as $Cl(1,3)$ Clifford algebra elements on Dirac spinors, in conjunction with corresponding active Lorentz transformations, $\Psi(x) \to U\Psi(x')$, and via the Clifford adjoint on Clifford vectors and other Clifford algebra elements, $A(x) \to UA(x')\,U^-$. The $Spin^+(1,3)$ subgroup elements of $Pin(1,3)$ consist of combinations of spatial rotations and Lorentz boosts,

$$U_\theta = e^{-\frac{1}{2}\gamma\gamma_0 n_1 \theta} = \cos\tfrac{\theta}{2} - \gamma\gamma_0 n_1 \sin\tfrac{\theta}{2} \qquad U_\zeta = e^{-\frac{1}{2}\gamma_0 n_2 \zeta} = \cosh\tfrac{\zeta}{2} - \gamma_0 n_2 \sinh\tfrac{\zeta}{2}$$

which combine to make scalar plus bivector plus pseudoscalar ($\gamma = \gamma_0\gamma_1\gamma_2\gamma_3$) elements, $U \in \mathrm{Spin}^+(1,3)$. Alternatively, Clifford rotation elements can be constructed from successive Clifford reflections. Using the pseudoscalar, the reflection of any Clifford element, $A \in Cl(1,3)$, or spinor, Ψ, through a vector, u, can also be written as a Clifford adjoint,

$$A' = R_u A = (u\gamma)A(u\gamma)^- = (u\gamma)A(u^-\gamma) \qquad \Psi' = R_u\Psi = (u\gamma)\Psi$$

Arbitrary Clifford reflections, as elements of $Pin(1,3)$, can be constructed by combining spacetime rotations with a reflection through the unit time Clifford basis vector, $U_T = \gamma_0\gamma = \gamma_1\gamma_2\gamma_3$, called "time conjugation" (T), or reflection through the three unit space Clifford basis vectors, $U_P = -\gamma_1\gamma\gamma_2\gamma\gamma_3\gamma = \gamma_0$, called "parity conjugation" (P)—with the negative sign by convention—or their combination, $U_{PT} = \gamma$. These P and T symmetry conjugations anticommute, $PT = -TP$, and close to form the PT Group (of order 8), a finite subgroup of $Pin(1,3)$. For $Pin(1,3)$ these satisfy $\{U_P^2 = 1, U_T^2 = 1, U_{PT}^2 = -1\}$, while for $Pin(3,1)$ we have $\{U'^2_P = -1, U'^2_T = -1, U'^2_{PT} = -1\}$, so these two pin groups are not isomorphic. Also note that $Spin(1,3) = Spin^+(1,3) \otimes \{1, PT\}$. Since we like working with $Spin^+(1,3)$ because of the isomorphism to $SL(2,\mathbb{C})$, but we would also like our spinors to change sign under P^2, corresponding to a 2π rotation, as they do in $Pin(3,1)$, we can fudge a bit by defining $U'_P = i\gamma_0$, providing the best of both worlds. Physically, we appear to live in a world with underlying $Pin(3,1)$ symmetry, but it's easier to work with $Pin(1,3)$ for calculations.

The spacetime pin group, $Pin(1,3)$, is in $Cl(1,3)$, and can be extended to the Dirac algebra via an antiunitary charge conjugation (C) operator, $U_C = i\gamma_2 K$ (in which K is the complex conjugation operator), acting on complex Dirac spinors—transforming between particles and antiparticles. When we look at quantized fermion fields, T-symmetry becomes more complicated due to the positive energy constraint. A different unitary time conjugation operator, $U_T^Q = \gamma_{13}$, is associated with T-symmetry (T). And this is carried back to the unquantized arena by defining yet a different, antiunitary time conjugation operator on Dirac spinors, $U'_T = -iU_T U_C = \gamma_{13}K$, which we also associate with T-symmetry (T), and we subsequently label the previous unitary T-symmetry, with $U_T = \gamma_0\gamma$, as T_U. Under their complex conjugations and Clifford multiplications, $\{U_C, U'_P, U'_T\}$ generate the CPT Group (of order 16).[2] The projective action of the CPT Group on quantized or unquantized Dirac spinors can be graphically depicted, using weights, as action on a cube. Since the CPT Group is the split-biquaternion group, it is natural to re-identify Dirac spinors as biquaternions, with corresponding C, P, and T actions on them.

In physics, P-symmetry is violated by the weak interaction, and CP-symmetry is violated by the Yukawa interaction with the Higgs field, but CPT-symmetry holds, with $CPT \sim PT_U$ here, so $Spin(1,3) = Spin^+(1,3) \otimes \{1, PT_U\}$ is an unbroken symmetry of nature. In the Standard Model, Dirac spinors are found in triplets, corresponding to three generations of each kind of fermion, each combining with the others according to a mixing matrix to produce mass states. Given this three-fold symmetry, it is natural to introduce a discrete triality symmetry (t) that cycles between generations of Dirac spinors. The main result of this paper is that there is a unique way to nontrivially extend the CPT Group by triality to the CPTt Group (of order 96) while preserving CPT-symmetry. This CPTt Group acts projectively on a 24-cell of Dirac spinor triplets.

2 Gravitational Weights of a Massless Quantum Dirac Spinor

Massless fermions are quantized excitations of Weyl spinors, which are the left or right-chiral halves of Dirac spinors, corresponding to spinor representation spaces of the spacetime Lorentz algebra, $so(1,3)$, described via the $Cl(1,3)$ Clifford algebra. Using Pauli matrices, the Weyl representation of Dirac matrices (the $Cl(1,3)$ Clifford basis vectors) are:

$$\gamma_0 = \sigma_1 \otimes \sigma_0 \qquad \gamma_\pi = -i\sigma_2 \otimes \sigma_\pi$$

These representative matrices multiply to give six Clifford bivector basis generators of $so(1,3)$,

$$J_\pi = \tfrac{1}{4}\epsilon_{\pi\rho\sigma}\gamma_{\rho\sigma} = -\tfrac{i}{2}\sigma_0 \otimes \sigma_\pi \qquad K_\pi = \tfrac{1}{2}\gamma_{0\pi} = \tfrac{1}{2}\sigma_3 \otimes \sigma_\pi$$

corresponding to spatial rotations and spacetime boosts. Identifying the antisymmetric Clifford or matrix product of these with the Lie bracket, the nonzero brackets of the Lorentz algebra are:

$$[J_\pi, J_\rho] = \epsilon_{\pi\rho\sigma} J_\sigma \qquad [J_\pi, K_\rho] = \epsilon_{\pi\rho\sigma} K_\sigma \qquad [K_\pi, K_\rho] = -\epsilon_{\pi\rho\sigma} J_\sigma$$

This Lie algebra structure of $so(1,3)$ is further elucidated by identifying basis elements of a Cartan subalgebra, $\{J_3, K_3\}$, and computing the eigenvalues, j_3 and k_3, with imaginary and real root coordinates, $j_3^{\mathbb{I}}$ and $k_3^{\mathbb{R}}$, and corresponding root vectors, resulting in a Cartan-Weyl basis for the Lie algebra, $\{J_3, K_3, E_L^{\vee/\wedge}, E_R^{\vee/\wedge}\}$, with nonzero brackets,

$$\begin{aligned}
\left[J_3, E_L^{\vee/\wedge}\right] &= (\pm i) E_L^{\vee/\wedge} & \left[K_3, E_L^{\vee/\wedge}\right] &= (\mp 1) E_L^{\vee/\wedge} & E_L^{\vee/\wedge} &= \tfrac{1}{2}\left(\mp J_1 + iJ_2 \pm iK_1 + K_2\right) \\
\left[J_3, E_R^{\vee/\wedge}\right] &= (\pm i) E_R^{\vee/\wedge} & \left[K_3, E_R^{\vee/\wedge}\right] &= (\pm 1) E_R^{\vee/\wedge} & E_R^{\vee/\wedge} &= \tfrac{1}{2}\left(\pm J_1 - iJ_2 \pm iK_1 + K_2\right) \\
[E_L^\vee, E_L^\wedge] &= -i\, J_3 - K_3 & [E_R^\vee, E_R^\wedge] &= -i\, J_3 + K_3
\end{aligned}$$

These are put in Chevalley-Serre form by defining non-orthogonal Cartan basis elements, $H_{L/R} = -iJ_3 \mp K_3$, resulting in the brackets,

$$\left[H_L, E_L^{\vee/\wedge}\right] = \pm 2i E_L^{\vee/\wedge} \qquad [E_L^\vee, E_L^\wedge] = H_L \qquad \left[H_R, E_R^{\vee/\wedge}\right] = \pm 2i E_R^{\vee/\wedge} \qquad [E_R^\vee, E_R^\wedge] = H_R$$

Alternatively, these complex root vectors can be transformed to a real Cartan-Weyl basis, with resulting real structure constants,

$$\begin{aligned}
\left[J_3, E_\pm^{\mathbb{R}}\right] &= (+1) E_\pm^{\mathbb{I}} & \left[K_3, E_\pm^{\mathbb{R}}\right] &= (\pm 1) E_\pm^{\mathbb{R}} & E_\pm^{\mathbb{R}} &= \tfrac{1}{2}\left(\pm J_1 + K_2\right) \\
\left[J_3, E_\pm^{\mathbb{I}}\right] &= (-1) E_\pm^{\mathbb{R}} & \left[K_3, E_\pm^{\mathbb{I}}\right] &= (\pm 1) E_\pm^{\mathbb{I}} & E_\pm^{\mathbb{I}} &= \tfrac{1}{2}\left(\pm J_2 - K_1\right) \\
\left[E_+^{\mathbb{R}}, E_-^{\mathbb{I}}\right] &= \left[E_-^{\mathbb{R}}, E_+^{\mathbb{I}}\right] = -\tfrac{1}{2} J_3 & \left[E_+^{\mathbb{R}}, E_-^{\mathbb{R}}\right] &= \left[E_+^{\mathbb{I}}, E_-^{\mathbb{I}}\right] = \tfrac{1}{2} K_3
\end{aligned}$$

A spacetime Lorentz algebra element represented as a Clifford bivector, $B = B_s^\pi J_\pi + B_t^\pi K_\pi = \frac{1}{2}B^{\mu\nu}\gamma_{\mu\nu}$, acts on a spacetime vector, $v = v^\mu \gamma_\mu$, via anti-symmetric Clifford multiplication,

$$v' = B \times v = \tfrac{1}{2}B^{\mu\nu} v^\rho \left(\gamma_{\mu\nu} \times \gamma_\rho\right) = \tfrac{1}{4}B^{\mu\nu} v^\rho \left(\gamma_\mu \eta_{\nu\rho} - \gamma_\nu \eta_{\mu\rho}\right) = \tfrac{1}{2}B^{\mu\nu} v_\nu \gamma_\mu$$

Using Cartan subalgebra basis elements, $J_3 = \frac{1}{2}\gamma_{12}$ and $K_3 = \frac{1}{2}\gamma_{03}$, the weights and weight vectors of this spacetime vector representation space are:

$$\begin{array}{lll} J_3 \times v_S^{\vee/\wedge} = (\pm i) v_S^{\vee/\wedge} & K_3 \times v_S^{\vee/\wedge} = 0 & v_S^{\vee/\wedge} = \gamma_1 \mp i\gamma_2 \\ J_3 \times v_T^{\vee/\wedge} = 0 & K_3 \times v_T^{\vee/\wedge} = (\pm 1) v_T^{\vee/\wedge} & v_T^{\vee/\wedge} = \gamma_0 \mp \gamma_3 \end{array}$$

A spacetime Dirac spinor, $\psi = \psi^a Q_a$, is acted on by Clifford bivectors via their representative matrices, $\psi' = B\,\psi = \frac{1}{2} B^{\mu\nu} (\gamma_{\mu\nu})^b{}_c \psi^c Q_b$. Using the action of the Cartan subalgebra elements, $J_3 = -\frac{i}{2}\sigma_0 \otimes \sigma_3$ and $K_3 = \frac{1}{2}\sigma_3 \otimes \sigma_3$, the weights and weight vectors of this spinor representation space of the spacetime Lorentz algebra are:

$$\begin{array}{llll} J_3\, \psi_L^{\vee/\wedge} = (\pm i/2) \psi_L^{\vee/\wedge} & K_3\, \psi_L^{\vee/\wedge} = (\mp 1/2) \psi_L^{\vee/\wedge} & \psi_L^{\wedge} = Q_1 & \psi_R^{\wedge} = Q_3 \\ J_3\, \psi_R^{\vee/\wedge} = (\pm i/2) \psi_R^{\vee/\wedge} & K_3\, \psi_R^{\vee/\wedge} = (\pm 1/2) \psi_R^{\vee/\wedge} & \psi_L^{\vee} = Q_2 & \psi_R^{\vee} = Q_4 \end{array}$$

The roots of $so(1,3)$ correspond to gravitational spin connection states, $\omega_{L/R}^{\wedge/\vee}$, vector weights correspond to gravitational frame states, $e_{S/T}^{\wedge/\vee}$, and spinor weights correspond to massless fermion states, $f_{L/R}^{\wedge/\vee}$. Since the spin operator is $S_z = iJ_3$, the corresponding spin quantum number is $\omega_S = s_z = -j_3^{\mathbb{I}}$; and similarly for the boost quantum number we define $\omega_T^{\mathbb{R}} = -k_3^{\mathbb{R}}$. (Real weight components are labeled with $\mathbb{R}$ to distinguish them from more typical imaginary weight components.)

$so(1,3)$		$k_3^{\mathbb{R}}$	$j_3^{\mathbb{I}}$	$\omega_T^{\mathbb{R}}$	ω_S
○	$\omega_L^{\wedge/\vee}$	± 1	∓ 1	∓ 1	± 1
○	$\omega_R^{\wedge/\vee}$	∓ 1	∓ 1	± 1	± 1
□	$e_S^{\wedge/\vee}$	0	∓ 1	0	± 1
□	$e_T^{\wedge/\vee}$	∓ 1	0	± 1	0
▲ ▲	$f_L^{\wedge/\vee}$	$\pm 1/2$	$\mp 1/2$	$\mp 1/2$	$\pm 1/2$
▲ ▲	$f_R^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$\pm 1/2$	$\pm 1/2$

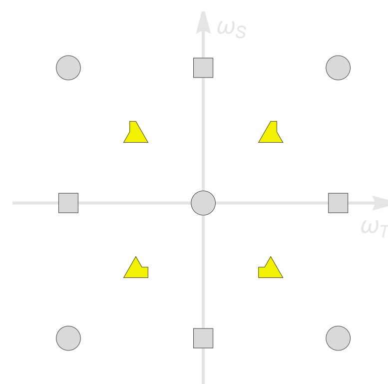


Table 1. Roots and weights of $so(1,3)$.

A Dirac spinor corresponds to a fermion or anti-fermion with specific momentum and spin. Massive fermions can be treated as massless fermions interacting with a Higgs field. Massless fermions have definite helicity, with aligned or anti-aligned spin and momentum. Solutions to the massless Dirac equation, $0 = i\gamma^\mu \partial_\mu \Psi$, are constructed from Pauli spinor helicity states, $p_u \chi_\pm = \pm\chi_\pm$, corresponding to a momentum direction,

$$p_u = p_u^\pi \sigma_\pi = \sigma_1 \sin(\theta)\cos(\phi) + \sigma_2 \sin(\theta)\sin(\phi) + \sigma_3 \cos(\theta)$$

Explicitly, the momentum direction representative matrix and helicity states are:

$$p_u = \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \qquad \chi_+ = \begin{bmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{bmatrix} \qquad \chi_- = \begin{bmatrix} e^{-i\phi/2}\sin\frac{\theta}{2} \\ -e^{i\phi/2}\cos\frac{\theta}{2} \end{bmatrix}$$

which are normalized to satisfy $\chi_a^\dagger \chi_b = \delta_{ab}$ and the identity $\chi_\pm(-p) = \pm i \chi_\mp(p)$. Massless Dirac solutions consist of positive or negative energy parts, with left or right helicity, for any specified momentum,

$$\Psi_+ = u_p^{L/R} e^{-ip_\mu x^\mu} \qquad\qquad \Psi_- = v_p^{L/R} e^{+ip_\mu x^\mu}$$

in which the spacetime four-momentum is:

$$p^\mu = \left(E, E\, p_u^1, E\, p_u^2, E\, p_u^3\right)$$

and the positive or negative energy Dirac spinor helicity states are:

$$u_p^L = \begin{bmatrix} \chi_- \\ 0 \end{bmatrix} \qquad u_p^R = \begin{bmatrix} 0 \\ \chi_+ \end{bmatrix} \qquad v_p^L = \begin{bmatrix} \xi_- \\ 0 \end{bmatrix} \qquad v_p^R = \begin{bmatrix} 0 \\ \xi_+ \end{bmatrix}$$

which satisfy $h\, u_p^{L/R} = \mp\frac{1}{2} u_p^{L/R}$ and $h\, v_p^{L/R} = \mp\frac{1}{2} v_p^{L/R}$, with $h = p_u^\pi S_\pi = \frac{1}{2}\left(\sigma_0 \otimes p_u\right)$. The negative energy Dirac spinor helicity eigenstates are defined to be the charge conjugates,

$$v_p^L = (u_p^R)^C = i\gamma_2 u_p^{R\,*} = \begin{bmatrix} \epsilon \chi_+^* \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_- \\ 0 \end{bmatrix} \qquad v_p^R = (u_p^L)^C = i\gamma_2 u_p^{L\,*} = \begin{bmatrix} 0 \\ -\epsilon \chi_-^* \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_+ \end{bmatrix} \tag{2.1}$$

with charge conjugate Weyl spinors defined as $\xi_\pm = \mp \epsilon \chi_\mp^* = -\chi_\pm$, in which $\epsilon = -i\sigma_2$ is the skew matrix. As well as this charge conjugation identity, Dirac spinor helicity eigenstates also satisfy identities related to parity,

$$i\gamma_0\, u_{-p}^{L/R} = \pm u_p^{R/L} \qquad\qquad i\gamma_0\, v_{-p}^{L/R} = \mp v_p^{R/L}$$

and time reversal,

$$\gamma_{13}\, u_{-p}^{L/R} = -i u_p^{L/R\,*} \qquad\qquad \gamma_{13}\, v_{-p}^{L/R} = +i v_p^{L/R\,*}$$

Although we usually think of a massless Weyl spinor as determined by its momentum, it is equally valid to consider a Weyl spinor state as primary, and use that to determine its momentum. Via spectral decomposition, we have:

$$2\, \chi_\mp \chi_\mp^\dagger = \sigma_0 \mp p_u = p_{0\, L/R}$$

in which we have determined the left or right null four-vector and momentum direction from a left or right helicity state. If we generalize this to allow for arbitrary Weyl spinors,

$$2\, \psi_{L/R} \psi_{L/R}^\dagger = p_{L/R} = p^0 \sigma_0 \mp p^\pi \sigma_\pi = E\left(\sigma_0 \mp p_u^\pi \sigma_\pi\right)$$

we obtain complex null four-momenta of the corresponding energy, satisfying $0 = \det\left(p_{L/R}\right) = p^\mu p^\nu \eta_{\mu\nu}$. This relation leads to efficient spinor-helicity methods for computing scattering amplitudes.

Dirac basis spinors, $\psi_L^{\wedge/\vee} = Q_{1/2}$ and $\psi_R^{\wedge/\vee} = iQ_{3/4}$, correspond to a left handed fermion traveling in the $\mp\hat{z}$ direction or a right handed fermion traveling the $\pm\hat{z}$ direction. These basis spinors correspond to four basis helicity states,

$$\chi_-(-\hat{z}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \chi_-(+\hat{z}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \chi_+(+\hat{z}) = \begin{bmatrix} i \\ 0 \end{bmatrix} \qquad \chi_+(-\hat{z}) = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

and the corresponding eight Dirac spinor helicity states, $u_\pm^{L/R}$ and $v_\pm^{L/R}$.

A massless quantum Dirac spinor in Minkowski spacetime,

$$\underset{\cdot}{\hat{\Psi}} = \int \frac{d^3p}{(2\pi)^3(2E)} \left(\underset{\cdot}{\hat{a}}_p^{L/R} u_p^{L/R} e^{-ip_\mu x^\mu} + \underset{\cdot}{\hat{b}}_p^{R/L\,\dagger} v_p^{L/R} e^{+ip_\mu x^\mu} \right)$$

and its adjoint,

$$\bar{\underset{\cdot}{\hat{\Psi}}} = \underset{\cdot}{\hat{\Psi}}^\dagger \gamma^0 = \int \frac{d^3p}{(2\pi)^3(2E)} \left(\underset{\cdot}{\hat{a}}_p^{L/R\,\dagger} \bar{u}_p^{L/R} e^{+ip_\mu x^\mu} + \underset{\cdot}{\hat{b}}_p^{R/L} \bar{v}_p^{L/R} e^{-ip_\mu x^\mu} \right)$$

include creation and annihilation operators for particles, $\underset{\cdot}{\hat{a}}_p^{L/R}$, and antiparticles, $\underset{\cdot}{\hat{b}}_p^{L/R}$, of left and right chirality, for all possible momenta. If we consider only the basis states of momentum in the positive, $+$, or negative, $-$, $\hat{z}$ direction, the massless quantum Dirac spinor along z is:

$$\underset{\cdot}{\hat{\Psi}}_z = \begin{bmatrix} \underset{\cdot}{\hat{a}}_-^L e^{-iE(t+z)} - \underset{\cdot}{\hat{b}}_-^{R\,\dagger} e^{+iE(t+z)} \\ \underset{\cdot}{\hat{a}}_+^L e^{-iE(t-z)} - \underset{\cdot}{\hat{b}}_+^{R\,\dagger} e^{+iE(t-z)} \\ i\underset{\cdot}{\hat{a}}_+^R e^{-iE(t-z)} - i\underset{\cdot}{\hat{b}}_+^{L\,\dagger} e^{+iE(t-z)} \\ i\underset{\cdot}{\hat{a}}_-^R e^{-iE(t+z)} - i\underset{\cdot}{\hat{b}}_-^{L\,\dagger} e^{+iE(t+z)} \end{bmatrix}$$

Similarly, the massless quantum Dirac spinor adjoint along z is:

$$\bar{\underset{\cdot}{\hat{\Psi}}}_z = \begin{bmatrix} -i\underset{\cdot}{\hat{a}}_+^{R\,\dagger} e^{+-} + i\underset{\cdot}{\hat{b}}_+^L e^{--} & -i\underset{\cdot}{\hat{a}}_-^{R\,\dagger} e^{++} + i\underset{\cdot}{\hat{b}}_-^L e^{-+} & \underset{\cdot}{\hat{a}}_-^{L\,\dagger} e^{++} - \underset{\cdot}{\hat{b}}_-^R e^{-+} & \underset{\cdot}{\hat{a}}_+^{L\,\dagger} e^{+-} - \underset{\cdot}{\hat{b}}_+^R e^{--} \end{bmatrix}$$

These are acted upon by the Lorentz algebra, $so(1,3)$, with Cartan subalgebra basis elements chosen to be the (anti-Hermitian) spin, $J_3 = \frac{1}{2}\gamma_{12} = -\frac{i}{2}\sigma_0 \otimes \sigma_3 = -i\,S_z$, and (Hermitian) boost, $K_3 = \frac{1}{2}\gamma_{03} = \frac{1}{2}\sigma_3 \otimes \sigma_3$, bivectors of the $Cl(1,3)$ Clifford algebra. Typically, such as for the anti-Hermitian rotation operator, $O = J_3$, there is a corresponding anti-Hermitian operator on the infinite-dimensional unitary representation space operators of quantum field theory, such as $\hat{O} = \hat{J}_3$, satisfying:

$$\left[\hat{O}, \underset{\cdot}{\hat{\Psi}}\right] = O\,\underset{\cdot}{\hat{\Psi}} \qquad \left[\hat{J}_3, \underset{\cdot}{\hat{\Psi}}\right] = J_3\,\underset{\cdot}{\hat{\Psi}} \tag{2.2}$$

and, for the adjoint,

$$\left[\hat{O}^\dagger, \bar{\underset{\cdot}{\hat{\Psi}}}\right] = -\bar{\underset{\cdot}{\hat{\Psi}}} \left(\gamma_0 O^\dagger \gamma^0\right) \qquad \left[\hat{J}_3, \bar{\underset{\cdot}{\hat{\Psi}}}\right] = -\bar{\underset{\cdot}{\hat{\Psi}}}\,J_3$$

For the Hermitian boost operator, K_3, we take the corresponding operator, $\hat{K}_3$ on the infinite-dimensional unitary representation space to be anti-Hermitian to preserve quantum unitarity, and so we have:

$$\left[\hat{K}_3, \underset{\cdot}{\hat{\Psi}}\right] = K_3\,\underset{\cdot}{\hat{\Psi}} \qquad \left[\hat{K}_3, \bar{\underset{\cdot}{\hat{\Psi}}}\right] = \bar{\underset{\cdot}{\hat{\Psi}}} \left(\gamma_0 K_3^\dagger \gamma^0\right) = -\bar{\underset{\cdot}{\hat{\Psi}}}\,K_3$$

These formulas allow us to find the spin and boost eigenvalues (weights), j_3 and k_3, of the fermion annihilation and creation operators,

$$
\begin{aligned}
\left[\hat{J}_3, \hat{a}^L_\mp\right] &= (\mp i/2)\,\hat{a}^L_\mp & \qquad \left[\hat{J}_3, \hat{b}^{R\,\dagger}_\mp\right] &= (\mp i/2)\,\hat{b}^{R\,\dagger}_\mp \\
\left[\hat{J}_3, \hat{a}^R_\pm\right] &= (\mp i/2)\,\hat{a}^R_\pm & \qquad \left[\hat{J}_3, \hat{b}^{L\,\dagger}_\pm\right] &= (\mp i/2)\,\hat{b}^{L\,\dagger}_\pm \\[1em]
\left[\hat{K}_3, \hat{a}^L_\mp\right] &= (\pm 1/2)\,\hat{a}^L_\mp & \qquad \left[\hat{K}_3, \hat{b}^{R\,\dagger}_\mp\right] &= (\pm 1/2)\,\hat{b}^{R\,\dagger}_\mp \\
\left[\hat{K}_3, \hat{a}^R_\pm\right] &= (\mp 1/2)\,\hat{a}^R_\pm & \qquad \left[\hat{K}_3, \hat{b}^{L\,\dagger}_\pm\right] &= (\mp 1/2)\,\hat{b}^{L\,\dagger}_\pm \\[1em]
\left[\hat{J}_3, \hat{a}^{R\,\dagger}_\pm\right] &= (\pm i/2)\,\hat{a}^{R\,\dagger}_\pm & \qquad \left[\hat{J}_3, \hat{b}^L_\pm\right] &= (\pm i/2)\,\hat{b}^L_\pm \\
\left[\hat{J}_3, \hat{a}^{L\,\dagger}_\mp\right] &= (\pm i/2)\,\hat{a}^{L\,\dagger}_\mp & \qquad \left[\hat{J}_3, \hat{b}^R_\mp\right] &= (\pm i/2)\,\hat{b}^R_\mp \\[1em]
\left[\hat{K}_3, \hat{a}^{R\,\dagger}_\pm\right] &= (\mp 1/2)\,\hat{a}^{R\,\dagger}_\pm & \qquad \left[\hat{K}_3, \hat{b}^L_\pm\right] &= (\mp 1/2)\,\hat{b}^L_\pm \\
\left[\hat{K}_3, \hat{a}^{L\,\dagger}_\mp\right] &= (\pm 1/2)\,\hat{a}^{L\,\dagger}_\mp & \qquad \left[\hat{K}_3, \hat{b}^R_\mp\right] &= (\pm 1/2)\,\hat{b}^R_\mp
\end{aligned}
$$

Summarizing this structure, the table of spin and boost quantum numbers, $\omega_S = s_z = -j^{\mathbb{I}}_3$ and $\omega^{\mathbb{R}}_T = -k^{\mathbb{R}}_3$, of the annihilation and creation operators of a massless quantum Dirac spinor—with a relabeling for particle spin and helicity—is:

4_s			$\omega^{\mathbb{R}}_T$	ω_S	h	q
	$a^\wedge_L$	$\hat{a}^L_-$	$-1/2$	$+1/2$	$-1/2$	$+1/2$
	$a^\vee_L$	$\hat{a}^L_+$	$+1/2$	$-1/2$	$-1/2$	$+1/2$
	$a^\wedge_R$	$\hat{a}^R_+$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$a^\vee_R$	$\hat{a}^R_-$	$-1/2$	$-1/2$	$+1/2$	$+1/2$
	$\bar{a}^\wedge_L$	$\hat{b}^L_-$	$-1/2$	$+1/2$	$-1/2$	$-1/2$
	$\bar{a}^\vee_L$	$\hat{b}^L_+$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{a}^\wedge_R$	$\hat{b}^R_+$	$+1/2$	$+1/2$	$+1/2$	$-1/2$
	$\bar{a}^\vee_R$	$\hat{b}^R_-$	$-1/2$	$-1/2$	$+1/2$	$-1/2$

$4^\dagger_s$		$\omega^{\mathbb{R}}_T$	ω_S	h	q
$a^{\wedge\,\dagger}_L$	$\hat{a}^{L\,\dagger}_-$	$-1/2$	$-1/2$	$+1/2$	$-1/2$
$a^{\vee\,\dagger}_L$	$\hat{a}^{L\,\dagger}_+$	$+1/2$	$+1/2$	$+1/2$	$-1/2$
$a^{\wedge\,\dagger}_R$	$\hat{a}^{R\dagger}_+$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
$a^{\vee\,\dagger}_R$	$\hat{a}^{R\dagger}_-$	$-1/2$	$+1/2$	$-1/2$	$-1/2$
$\bar{a}^{\wedge\,\dagger}_L$	$\hat{b}^{L\,\dagger}_-$	$-1/2$	$-1/2$	$+1/2$	$+1/2$
$\bar{a}^{\vee\,\dagger}_L$	$\hat{b}^{L\,\dagger}_+$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
$\bar{a}^{\wedge\,\dagger}_R$	$\hat{b}^{R\dagger}_+$	$+1/2$	$-1/2$	$-1/2$	$+1/2$
$\bar{a}^{\vee\,\dagger}_R$	$\hat{b}^{R\dagger}_-$	$-1/2$	$+1/2$	$-1/2$	$+1/2$

Table 2. The weights of the annihilation and creation operators of a charged massless quantum Dirac spinor, 4_s, of $so(1,3)$ or $so(3,1)$.

The helicity quantum number is $h = p_z s_z = 2\,\omega^{\mathbb{R}}_T \omega_S$, and the $q = \pm 1/2$ quantum number is for whatever internal charge the particle has. Note that the helicity, spin, and charge, but not boost, quantum numbers of a creation operator are opposite that of the corresponding annihilation operator; and there is a weight match between annihilating particles and creating anti-particles, such as $\bar{a}^{\wedge\,\dagger}_L = a^\vee_R$. The spin and boost quantum numbers of the annihilation of a massless fermion match those of a Dirac spinor (Table 1), as do those of the corresponding anti-fermion.

3 Charge, Parity, Time, Triality, Biquaternionic Spinors, and the CPTt Group

The charge, parity, and time conjugates of Dirac solutions are also Dirac solutions, and correspond to conjugations of a massless quantum Dirac spinor. The charge conjugate, $\Psi^C = i\gamma_2 \Psi^*$, is:

$$\begin{aligned}
\hat{\underset{.}{\Psi}}^C &= \int \frac{d^3p}{(2\pi)^3(2E)} \left(\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^C u_p^{L/R} e^{-ip_\mu x^\mu} + \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^C v_p^{L/R} e^{+ip_\mu x^\mu} \right) \\
&= i\gamma_2 \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{.}{a}}{}_p^{L/R} u_p^{L/R} e^{-ip_\mu x^\mu} + \hat{\underset{.}{b}}{}_p^{R/L\,\dagger} v_p^{L/R} e^{+ip_\mu x^\mu} \right)^* \\
&= \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{.}{a}}{}_p^{L/R\,\dagger} v_p^{R/L} e^{+ip_\mu x^\mu} + \hat{\underset{.}{b}}{}_p^{R/L} u_p^{R/L} e^{-ip_\mu x^\mu} \right)
\end{aligned}$$

using massless Dirac solution identities, $i\gamma_2 u_p^{L/R\,*} = v_p^{R/L}$ and $i\gamma_2 v_p^{L/R\,*} = u_p^{R/L}$. The equivalent charge conjugation transformations of the creation and annihilation operators, using the corresponding operation on the infinite-dimensional representation, $\hat{\underset{.}{\Psi}}^C = \hat{C}\hat{\underset{.}{\Psi}}\hat{C}^-$, are thus:

$$\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^C = \hat{\underset{.}{b}}{}_p^{L/R} \qquad\qquad \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^C = \hat{\underset{.}{a}}{}_p^{R/L\,\dagger} \tag{3.1}$$

The parity conjugate, $\Psi^P = i\gamma_0 \Psi(t, -x)$, is

$$\begin{aligned}
\hat{\underset{.}{\Psi}}^P &= i\gamma_0 \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{.}{a}}{}_{-p}^{L/R} u_{-p}^{L/R} e^{-ip_\mu x^\mu} + \hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger} v_{-p}^{L/R} e^{+ip_\mu x^\mu} \right) \\
&= \int \frac{d^3p}{(2\pi)^3(2E)} \left(\pm \hat{\underset{.}{a}}{}_{-p}^{L/R} u_p^{R/L} e^{-ip_\mu x^\mu} \mp \hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger} v_p^{R/L} e^{+ip_\mu x^\mu} \right)
\end{aligned}$$

using $i\gamma_0\, u_{-p}^{L/R} = \pm u_p^{R/L}$ and $i\gamma_0\, v_{-p}^{L/R} = \mp v_p^{R/L}$. The equivalent parity conjugation transformations of the creation and annihilation operators are thus:

$$\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^P = \mp \hat{\underset{.}{a}}{}_{-p}^{R/L} \qquad\qquad \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^P = \pm \hat{\underset{.}{b}}{}_{-p}^{L/R\,\dagger} \tag{3.2}$$

The time conjugate, $\Psi^T = \gamma_{13} \Psi(-t, x)$, of a massless quantum Dirac spinor corresponds to an antiunitary operator on Fock space,

$$\begin{aligned}
\hat{\underset{.}{\Psi}}^T &= \hat{T}' \hat{\underset{.}{\Psi}} \hat{T}'^- = \int \frac{d^3p}{(2\pi)^3(2E)} \left(\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^T u_p^{L/R\,*} e^{+ip_\mu x^\mu} + \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^T v_p^{L/R\,*} e^{-ip_\mu x^\mu} \right) \\
&= \gamma_{13} \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{.}{a}}{}_{-p}^{L/R} u_{-p}^{L/R} e^{+ip_\mu x^\mu} + \hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger} v_{-p}^{L/R} e^{-ip_\mu x^\mu} \right) \\
&= \int \frac{d^3p}{(2\pi)^3(2E)} \left(-i \hat{\underset{.}{a}}{}_{-p}^{L/R} u_p^{L/R\,*} e^{+ip_\mu x^\mu} + i \hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger} v_p^{L/R\,*} e^{-ip_\mu x^\mu} \right)
\end{aligned}$$

using $\gamma_{13}\, u_{-p}^{L/R} = -i u_p^{L/R\,*}$ and $\gamma_{13}\, v_{-p}^{L/R} = +i v_p^{L/R\,*}$. The time conjugation transformations of the creation and annihilation operators for particles and antiparticles is thus:

$$\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^T = -i\, \hat{\underset{.}{a}}{}_{-p}^{L/R} \qquad\qquad \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^T = +i\, \hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger} \tag{3.3}$$

Applying time conjugation twice, we get:

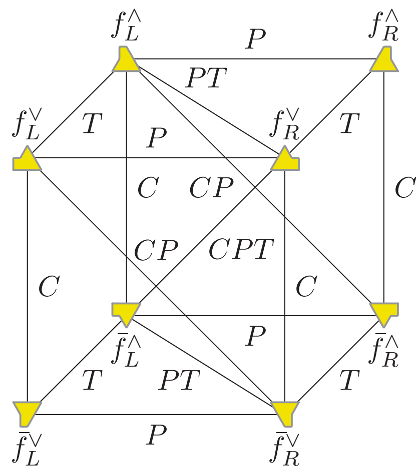
$$\left(\hat{\underset{.}{a}}{}_p^{L/R}\right)^{T^2} = +i \left(\hat{\underset{.}{a}}{}_{-p}^{L/R}\right)^T = -\hat{\underset{.}{a}}{}_p^{L/R} \qquad\qquad \left(\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}\right)^{T^2} = -i \left(\hat{\underset{.}{b}}{}_{-p}^{R/L\,\dagger}\right)^T = -\hat{\underset{.}{b}}{}_p^{R/L\,\dagger}$$

and so $\left(\hat{a}^{L/R}_{-p}\right)^T = +i\,\hat{a}^{L/R}_p$ and $\left(\hat{b}^{R/L\dagger}_{-p}\right)^T = -i\,\hat{b}^{R/L\dagger}_p$, which isn't entirely obvious. The existence of this antiunitary time conjugation operator, squaring to minus one, implies our fermion representation space is quaternionic.[4]

Applied to the weight vectors of a massless quantum Dirac spinor, these conjugations give:

$$\begin{aligned} (a^\wedge_L)^C &= \bar{a}^\wedge_L & C &: (\omega^{\mathbb{R}}_T, \omega_S, h, q) \mapsto (\ \ \omega^{\mathbb{R}}_T, \ \ \omega_S, \ \ h, -q) \\ (a^\wedge_L)^P &= -\,a^\wedge_R & P &: (\omega^{\mathbb{R}}_T, \omega_S, h, q) \mapsto (-\omega^{\mathbb{R}}_T, \ \ \omega_S, -h, \ \ q) \\ (a^\wedge_L)^T &= -i\,a^\vee_L & T &: (\omega^{\mathbb{R}}_T, \omega_S, h, q) \mapsto (-\omega^{\mathbb{R}}_T, -\omega_S, \ \ h, \ \ q) \end{aligned} \tag{3.4}$$

Plotting fermion weights, (ω_S, h, q), and their conjugation relationships, we get the CPT cube:



Composition of the C, P, and T operators produces the CPT Group, $G_{CPT} = Q_8 \times \mathbb{Z}_2$, of order 16, equivalent to the split-biquaternion group. To understand this equivalence, we can identify the charge conjugation operator, $C \sim I$, with a split-complex number, $I^2 = 1$, which commutes with parity and time conjugation operators identified with unit quaternions, $P \sim e_3$, $T \sim e_2$, and $PT \sim e_3 e_2 = -e_1$. The resulting compositional multiplication table is:

1	C	P	T	CP	CT	PT	CPT
C	$+1$	CP	CT	P	T	CPT	PT
P	CP	-1	PT	$-C$	CPT	$-T$	$-CT$
T	CT	$-PT$	-1	$-CPT$	$-C$	P	CP
CP	P	$-C$	CPT	-1	PT	$-CT$	$-T$
CT	T	$-CPT$	$-C$	$-PT$	-1	CP	P
PT	CPT	T	$-P$	CT	$-CP$	-1	$-C$
CPT	PT	CT	$-CP$	T	$-P$	$-C$	-1

Table 3. The CPT Group multiplication table—with further multiplications by -1 implied.

Because fermions exist in three generations, we can introduce a natural fourth discrete conjugation operator, triality (t), that maps between generations and satisfies $t^3 = 1$. One nontrivial extension of the CPT Group, $G_{CPT} = Q_8 \times \mathbb{Z}_2$, to a group, $G_{CPTt'}$, acting on three generations of fermions, can be constructed by identifying a triality generator element, such as $t' \sim -\frac{1}{2}(1+e_1+e_2+e_3)$.[3] This choice of triality generator, t', commutes with the split-complex generator, $C \sim I$, and cycles quaternion basis elements, such as $t' e_1 t'^2 = e_2$. This triality generator extends the PT group, $G_{PT} = Q_8$, to the binary-tetrahedral group, $G_{PTt'} = 2T$, and including charge conjugation via the split-complex generator gives the CPTt' Group, $G_{CPTt'} = 2T \times \mathbb{Z}_2$, the split-binary-tetrahedral group, of order 48. Although this is mathematically correct, this is not the only choice of group extension by triality. The Standard Model is not invariant under charge conjugation, C, so it is not expected that our t symmetry should commute with C. It is the case that the Standard Model is invariant under CPT, so what we need is to have different C, P, and T representatives that don't commute with our t, such that the resulting CPT generator does commute with t. Rather than guess at such new group representatives, we can revisit the operation of C, P, and T on Dirac spinors, and translate these to operations on biquaternionic spinors.

A Dirac spinor describes both a fermion and an anti-fermion, via positive and negative energy Dirac solutions, (2.1). This suggests a re-arrangement of degrees of freedom, using the charge conjugate,

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} \qquad \Psi^C = i\gamma_2 \Psi^* = \begin{bmatrix} -\Psi_4^* \\ \Psi_3^* \\ \Psi_2^* \\ -\Psi_1^* \end{bmatrix} \qquad \Psi_Q = \begin{bmatrix} \Psi_1 & -\Psi_4^* \\ \Psi_2 & \Psi_3^* \\ \Psi_3 & \Psi_2^* \\ \Psi_4 & -\Psi_1^* \end{bmatrix} \sim \begin{bmatrix} \psi_{\mathbb{H}L} \\ \psi_{\mathbb{H}R} \end{bmatrix}$$

in which all Dirac spinor degrees of freedom can inhabit either the left or right-chiral *Dirac biquaternions*, $\psi_{\mathbb{H}L}$ or $\psi_{\mathbb{H}R}$. Here we make use of the representation of quaternions, $e_\mu \in \mathbb{H}$, using Pauli matrices, $\{e_0 \sim \sigma_0,\, e_\pi \sim -i\sigma_\pi\}$, and the definition of biquaternions as quaternions with complex coefficients, $\psi_{\mathbb{H}L} \in \mathbb{C} \otimes \mathbb{H}$.

$$\psi_{\mathbb{H}L} = \psi^\mu_{\mathbb{H}L} e_\mu \sim \psi_{QL} = \begin{bmatrix} \Psi_1^{\mathbb{R}} + i\Psi_1^{\mathbb{I}} & -\Psi_4^{\mathbb{R}} + i\Psi_4^{\mathbb{I}} \\ \Psi_2^{\mathbb{R}} + i\Psi_2^{\mathbb{I}} & \Psi_3^{\mathbb{R}} - i\Psi_3^{\mathbb{I}} \end{bmatrix} = \begin{bmatrix} \psi_L \; \bar{\psi}_L \end{bmatrix} \qquad \psi_{QR} = i\sigma_2 \psi^*_{QL} \sigma_1 \sim \psi_{\mathbb{H}R} = i\psi^*_{\mathbb{H}L} e_3$$

$$\begin{aligned} \psi_{\mathbb{H}L} =& \tfrac{1}{2}\left(\Psi_1^{\mathbb{R}} + \Psi_3^{\mathbb{R}} - i\Psi_1^{\mathbb{I}} + i\Psi_3^{\mathbb{I}}\right) e_0 + \tfrac{1}{2}\left(\Psi_2^{\mathbb{I}} + \Psi_4^{\mathbb{I}} + i\Psi_2^{\mathbb{R}} - i\Psi_4^{\mathbb{R}}\right) e_1 \\ &+ \tfrac{1}{2}\left(-\Psi_2^{\mathbb{R}} - \Psi_4^{\mathbb{R}} + i\Psi_2^{\mathbb{I}} - i\Psi_4^{\mathbb{I}}\right) e_2 + \tfrac{1}{2}\left(\Psi_1^{\mathbb{I}} + \Psi_3^{\mathbb{I}} + i\Psi_1^{\mathbb{R}} - i\Psi_3^{\mathbb{R}}\right) e_3 \end{aligned}$$

Describing the biquaternions and their representation requires juggling several conjugations. Since the Pauli matrices satisfy $\sigma^*_\mu = \sigma_2 \bar{\sigma}_\mu \sigma_2$, we can define a similar conjugation for biquaternions, $\psi^*_{QL} \sim -e_2 \psi^*_{\mathbb{H}L} e_2$; and since the Pauli matrices are Hermitian, we also have $\psi^\dagger_{QL} \sim \tilde{\psi}^*_{\mathbb{H}L}$, using complex and quaternionic conjugation, $\{\tilde{e}_0 = e_0, \tilde{e}_\pi = -e_\pi\}$. The invariant bilinear form on the biquaternions directly relates to the bilinear Dirac spinor scalar,

$$\begin{aligned} (\psi_{\mathbb{H}L}, \psi_{\mathbb{H}L}) =& \left(\Psi_1^{\mathbb{R}}\Psi_3^{\mathbb{R}} + \Psi_2^{\mathbb{R}}\Psi_4^{\mathbb{R}} + \Psi_1^{\mathbb{I}}\Psi_3^{\mathbb{I}} + \Psi_2^{\mathbb{I}}\Psi_4^{\mathbb{I}}\right) + i\left(-\Psi_1^{\mathbb{R}}\Psi_3^{\mathbb{I}} - \Psi_2^{\mathbb{R}}\Psi_4^{\mathbb{I}} + \Psi_1^{\mathbb{I}}\Psi_3^{\mathbb{R}} + \Psi_2^{\mathbb{I}}\Psi_4^{\mathbb{R}}\right) \\ =& \tilde{\psi}_{\mathbb{H}L}\psi_{\mathbb{H}L} = \det(\psi_{QL}) \\ \bar{\Psi}\Psi =& 2\left(\Psi_1^{\mathbb{R}}\Psi_3^{\mathbb{R}} + \Psi_2^{\mathbb{R}}\Psi_4^{\mathbb{R}} + \Psi_1^{\mathbb{I}}\Psi_3^{\mathbb{I}} + \Psi_2^{\mathbb{I}}\Psi_4^{\mathbb{I}}\right) = 2\,\Re(\tilde{\psi}_{\mathbb{H}L}\psi_{\mathbb{H}L}) \end{aligned}$$

We can now calculate C, P, and T symmetry conjugations for biquaternionic fermions,

$$\begin{array}{llll} \Psi^C = i\gamma_2 \Psi^* & \psi^C_{QL} = \psi_{QL}\sigma_1 & \psi^C_{\mathbb{H}L} = i\psi_{\mathbb{H}L} e_1 & C \sim ie_1 \\ \Psi^P = i\gamma_0 \Psi & \psi^P_{QL} = -\sigma_2 \psi^*_{QL}\sigma_1 & \psi^P_{\mathbb{H}L} = -\psi^*_{\mathbb{H}L} e_3 & P \sim -Ke_3 \\ \Psi^T = \gamma_{13} \Psi^* & \psi^T_{QL} = i\sigma_2 \psi^*_{QL} & \psi^T_{\mathbb{H}L} = -\psi^*_{\mathbb{H}L} e_2 & T \sim -Ke_2 \end{array} \tag{3.5}$$

in which we use the antiunitary time conjugation operator, U'_T, and correctly reproduce the CPT Group action on fermions, with quaternionic multiplication from the right. From these C, P, and T generators we have $CPT \sim -i$.

We can now add a triality generator, $t \sim -\frac{1}{2}(1 + e_1 + e_2 + e_3)$, to those of our new C, P, and T representatives and see that this does not commute with C and does commute with CPT. With the addition of t, since we have $PT \sim e_2 e_3 = e_1$, and t cycles quaternions, we can also construct an expression for the complex conjugation generator in terms of other generators, $K \sim tPTttT$. This K generator commutes with P, T, and t, but anti-commutes with C. It is antiunitary, corresponding to complex conjugation of a Dirac spinor, $\Psi^K = K\Psi = \Psi^*$, and corresponds to *creation conjugation* on the infinite-dimensional representation space generators of QFT,

$$\begin{aligned} \hat{\underset{\cdot}{\Psi}}^K &= \hat{K}\hat{\underset{\cdot}{\Psi}}\hat{K}^- = \int \frac{d^3p}{(2\pi)^3(2E)} \left(\left(\hat{\underset{\cdot}{a}}{}_p^{L/R}\right)^K u_p^{L/R*} e^{+ip_\mu x^\mu} + \left(\hat{\underset{\cdot}{b}}{}_p^{R/L\dagger}\right)^K v_p^{L/R*} e^{-ip_\mu x^\mu} \right) \\ &= \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{\cdot}{a}}{}_p^{L/R} u_p^{L/R} e^{-ip_\mu x^\mu} + \hat{\underset{\cdot}{b}}{}_p^{R/L\dagger} v_p^{L/R} e^{+ip_\mu x^\mu} \right)^* \\ &= \int \frac{d^3p}{(2\pi)^3(2E)} \left(\hat{\underset{\cdot}{a}}{}_p^{L/R\dagger} u_p^{L/R*} e^{+ip_\mu x^\mu} + \hat{\underset{\cdot}{b}}{}_p^{R/L} v_p^{L/R*} e^{-ip_\mu x^\mu} \right) \end{aligned}$$

with

$$\left(\hat{\underset{\cdot}{a}}{}_p^{L/R}\right)^K = \hat{\underset{\cdot}{a}}{}_p^{L/R\dagger} \qquad \left(\hat{\underset{\cdot}{b}}{}_p^{L/R}\right)^K = \hat{\underset{\cdot}{b}}{}_p^{L/R\dagger}$$

Creation conjugation changes particle annihilation into particle creation, mapping positive energy states to nonphysical negative energy states. This is not considered a symmetry of nature—but it is part of our symmetry group.

The CPTt Group, G_{CPTt}, a finite group of order 96, is generated by:

$$C \sim ie_1 \qquad P \sim -Ke_3 \qquad T \sim -Ke_2 \qquad t \sim -\tfrac{1}{2}(1 + e_1 + e_2 + e_3) \tag{3.6}$$

It contains the eight unit quaternions, $\{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$, comprising a quaternion subgroup, Q_8, and the unit complex numbers and conjugation operator, $\{\pm 1, \pm i, \pm K, \pm iK\}$, comprising a dihedral subgroup, D_4, of order 8. It also contains the sixteen quaternionic Hurwitz integers, $\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$, which extend Q_8 to the binary tetrahedral group, $2T$, of order 24. The remaining 64 elements of G_{CPTt} are compositions of these, with $2T$ and D_4 both normal subgroups of G_{CPTt} combining non-trivially due to their shared -1. The PT subgroup, $G_{PT} = Q_8$, which extends by t to $G_{PTt} = 2T$, commutes with K, so there is a $\mathbb{Z}_2$ subgroup, comprised of $\{1, K\}$, such that $2T \times \mathbb{Z}_2$ is a subgroup of G_{CPTt}. Consulting GAP, of 231 finite groups of order 96, only one monolithic group has both a $2T \times \mathbb{Z}_2$ and D_4 subgroup—this is the CPTt Group, $G_{CPTt} = (2T \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ (GAP ID $[96, 190]$).[5]

Just as the CPT Group acts on 8 quantized Dirac fermion states projectively represented as the CPT cube in three dimensions, with vertices corresponding to fermion weights, (ω_S, h, q), the CPTt Group acts on three generations of fermion states projectively represented as a 24-cell, which lives naturally in four dimensions. For the four weight coordinates of the 24-cell we can choose $(\omega_t, \omega_S, h, q)$, in which $\omega_t = 4\omega_S hq$ is a *Euclidean boost weight*, such that the eight weights of one fermion correspond to the weights of an octonion under $so(8)$. In these coordinates the projective group action changes the signs of the first generation fermions, by matrices, C_I, P_I, and T_I, and the second and third generation particles have weights related to the first via multiplication times a *triality matrix*, t. The resulting CPTt Group projective representation generators are:

$$C_I = \begin{bmatrix} - & & & \\ & + & & \\ & & + & \\ & & & - \end{bmatrix} \quad P_I = \begin{bmatrix} - & & & \\ & + & & \\ & & - & \\ & & & + \end{bmatrix} \quad T_I = \begin{bmatrix} - & & & \\ & - & & \\ & & + & \\ & & & + \end{bmatrix} \quad t = \frac{1}{2}\begin{bmatrix} + & - & + & + \\ + & - & - & - \\ + & + & + & - \\ + & + & - & + \end{bmatrix}$$

The weights of the second and third generation particles are necessarily nonsensical under direct interpretation, but are correct—matching those of the first generation—when considered under triality. Similarly, the action of C, P, and T on the second (and similarly on the third) generation particles are by the matrices $C_{II} = tC_I t^2$, $P_{II} = tP_I t^2$, and $T_{II} = tT_I t^2$. To better understand what triality is and where this triality matrix comes from, we need to understand division algebras.[6]

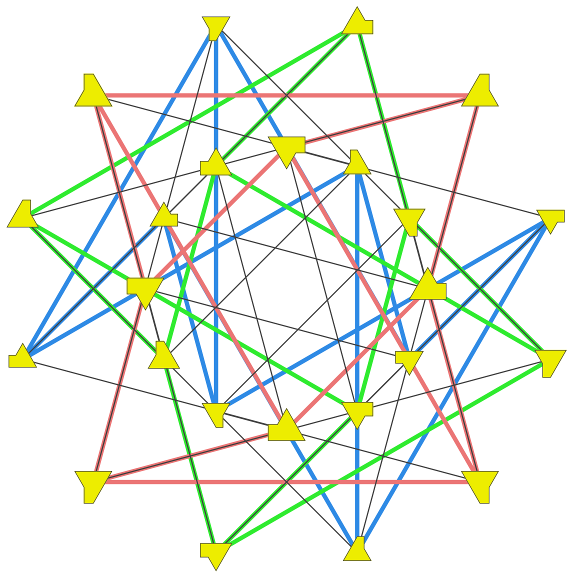


Figure 1. The 24 elementary particle states of three generations of massless quantum Dirac fermion states (such as electron, muon, and tau) represented as a 24-cell, acted on by the CPTt Group. The CPT cube of the 8 first generation states is shown with red edges, the second generation CPT cube in green, and third generation CPT cube in blue. The three generations are linked by triality, t, shown in black, with second and third generation fermion states shown with smaller glyphs.

4 Discussion

The fundamental symmetries of a massless quantum Dirac spinor are the charge (3.1), parity (3.2), and time (3.3) conjugations of the corresponding creation and annihilation operators. Charge conjugation swaps particles and anti-particles, parity conjugation reflects momentum and swaps left and right helicities (preserving spin), and time conjugation reflects momentum and swaps spin (preserving helicity). These conjugations correspond directly to operations on Dirac spinors or the corresponding biquaternionic spinors, and to projective actions on their weights. Time conjugation must be handled especially carefully, via a unitary $U_T^Q = \gamma_{13}$ on Dirac spinors or an antiunitary $U'_T = \gamma_{13} K$ operator that preserves the group structure. The finite group generated by these C, P, and T conjugations is the CPT Group, $G_{CPT} = Q_8 \times \mathbb{Z}_2$, equivalent to the split-biquaternion group. This group acts projectively on the 8 weights of fermion states in the CPT cube.

Using an isomorphism to biquaternionic Dirac spinors, the $C = ie_1$, $P = -Ke_3$, and $T = -Ke_2$ generators (3.5) of the CPT Group are extended by a quaternionic triality generator, $t = \frac{1}{2}(1 + e_1 + e_2 + e_3)$, commuting with the CPT generator, $CPT = -i$, to produce the monolithic CPTt Group, $G_{CPTt} = (2T \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, of order 96. This group acts projectively on the 24 weights (the vertices of a 24-cell) corresponding to three generations of a fermion type and its corresponding CPT cubes. It is important to note that the weights (spins and charges) of the second and third generation fermions described this way are nonsensical when considered as weights of the Lorentz algebra acting on the first generation—they only make sense as weights related to first-generation weights by triality.

The identification of triality, t, as a partner to C, P, and T symmetries, and the extension to the CPTt Group, seems likely to be of fundamental importance in the Standard Model and its unification with gravity. Although it is possible to trivially extend the CPT Group to a composite CPTt Group, such as extending the CPT Group by S_4, such extension seems unlikely to present the rich and varied generational mixing in the Standard Model, and we prefer a more unified model, with a monolithic CPTt Group. In the complete Standard Model there are 8 fermion types: neutrinos, electron-type leptons, three colors of up-type quarks, and three colors of down-type quarks. In a unified theory (which includes right-handed neutrinos) these must correspond to 8 disjoint 24-cells. The only current proposal for a unified theory that meets this criteria, with a triality symmetry acting between 192 distinct fermion weights grouped into 8 disjoint 24-cells, is E8 Theory.[7, 8]

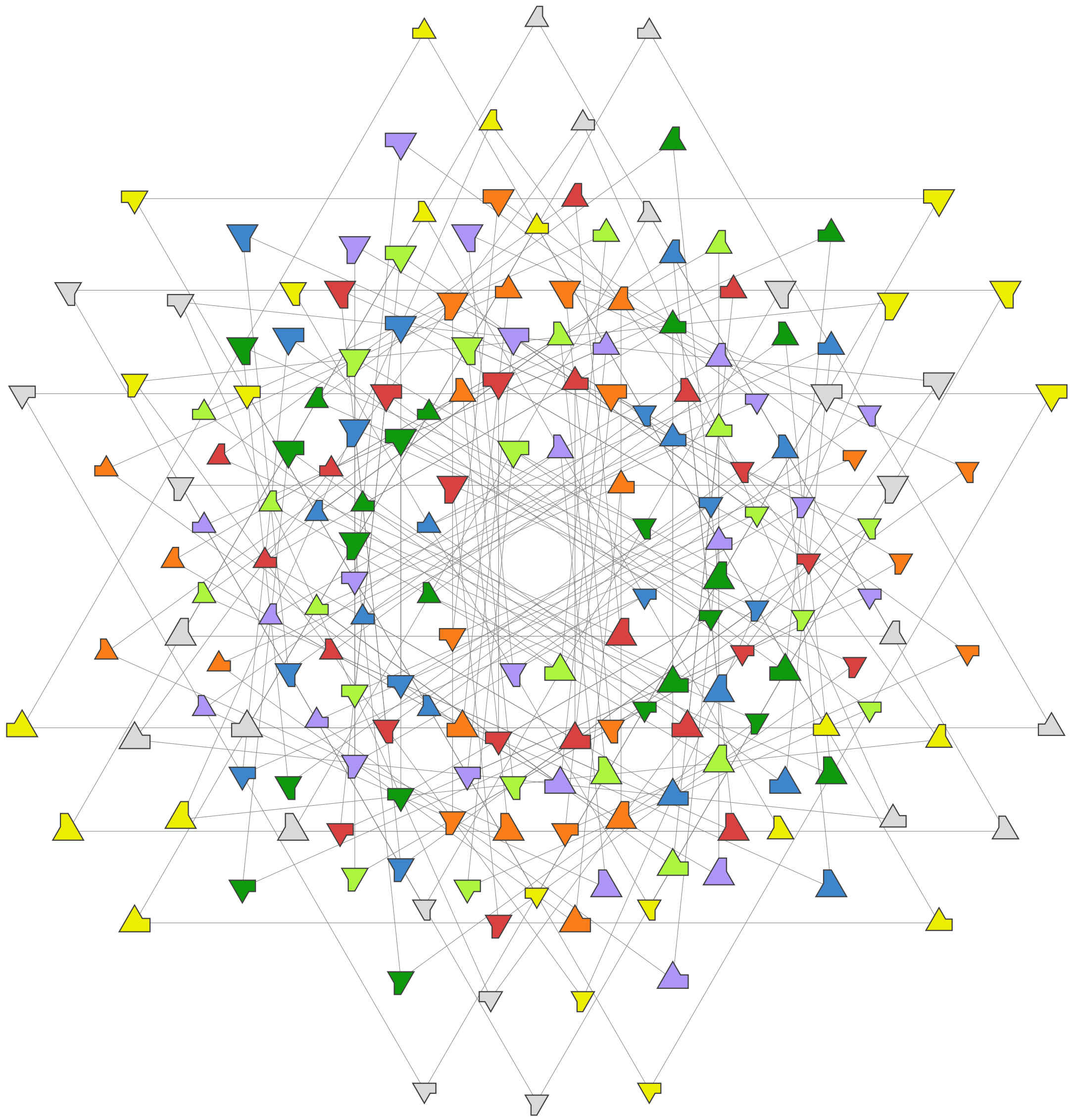


Figure 2. The 192 elementary particle states of three generations of massless quantum Dirac fermion states, including 8 disjoint 24-cells, with each 24-cell corresponding to a different type of fundamental fermion: neutrinos (gray), electron-type leptons (yellow), up-type quarks (red, green, and blue) and down-type quarks (orange, chartreuse, and purple). Each 24-cell includes a triality-related triplet (connected by gray lines, with second and third generation fermions shown with smaller glyphs) of 8 fermion states (cubes, not shown) related by C, P, and T conjugations.

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