Unification

Groups and representation spaces inside larger groups and representation spaces

GUT,

$$
\begin{aligned}
& su(3)_S + su(2)_W + u(1)_Y \\
& + (3,2)_{+\frac{1}{3}} + (1,2)_{-1} + (3,1)_{+\frac{4}{3}} + (3,1)_{-\frac{2}{3}} + (1,1)_{-2} \qquad \subset \qquad so(10) + 16_{s+} + 16_{s-} \qquad \subset \qquad e_6 \\
& + (\bar{3},2)_{-\frac{1}{3}} + (1,2)_{+1} + (\bar{3},1)_{-\frac{4}{3}} + (\bar{3},1)_{+\frac{2}{3}} + (1,1)_{+2}
\end{aligned}
$$

ToE or GraviGUT, includes gravity (and fermions),

$$
sl(2,\mathbb{C}) + so(10) + 4 \times 10 + 2 \times 16_{s+} + 2 \times 16_{s-} \qquad \subset \qquad so(11,3) + 64_{s+} \qquad \subset \qquad e_{8(-24)}
$$

Should also include three generations of each kind of Dirac fermion, acted on by a finite group,

$$
G_{CPTt} + 8_f \times 3 \qquad \subset \qquad ?
$$

Also, unification of fields and field equations is nice

$$
\underset{\cdot}{\underline{A}} = \frac{1}{2}\underline{\omega} + \underline{e}\phi + \underline{A} + \Psi \qquad\qquad \underset{\cdot}{\underline{\underline{F}}} = d\underset{\cdot}{\underline{A}} + \underset{\cdot}{\underline{A}}\underset{\cdot}{\underline{A}} = \frac{1}{2}\underline{\underline{R}} - \underline{e}\underline{e}\phi^2 + \underline{\underline{T}}\phi - \underline{e}\underline{D}\phi + \underline{\underline{F}}^A + \underline{D}\Psi
$$

$$
S = \int \frac{1}{2}\underset{\cdot}{\underline{\underline{F}}} \star \underset{\cdot}{\underline{\underline{F}}} \sim \int d^4x\,|e|\,\{R + \Lambda + \frac{1}{2}(D\phi)(D\phi) + V(\phi) - \frac{1}{4}F^A F^A + \bar{\Psi} D \Psi\}
$$

Why three generations?

Bosons	Fermions (×3 generations)			Fermion spin states
photon	e neutrino	μ neutrino	τ neutrino	left–handed spin–up
weak bosons	electron	muon	tau	left–handed spin–down
gluons	up quark	charm q	top q	right–handed spin–up
gravitons	down quark	strange q	bottom q	right–handed spin–down
frame–Higgs	up quark	charm q	top q	left–handed spin–up anti
anti–Higgs	down quark	strange q	bottom q	left–handed spin–down anti
charged Higgs	up quark	charm q	top q	right–handed spin–up anti
new bosons	down quark	strange q	bottom q	right–handed spin–down anti

1936, Isidor Isaac Rabi upon discovery of the muon:

"Who ordered that?"

1976, James Bjorken upon discovery of the tau lepton:

"What, another one?"

(apocryphal)

C, P, T, and Triality

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ABSTRACT: Discrete charge, parity, and time symmetries (C, P, and T) of quantized fermion states are extended by a triality symmetry (t), producing the CPTt Group, transforming between three generations of fermions.

KEYWORDS: ToE

Groups

Group Properties	
Ordered group product of elements:	$a\,b = c \quad \in \ G$
Identity element:	$a\,1 = 1\,a = a$
Inverses:	$a\,a^{-} = a^{-}\,a = 1$
Associativity:	$a\,(b\,c) = (a\,b)\,c$

The number of elements in a **finite group** is the **order**.

An n dimensional **Lie group** is a continuum of elements, $g(x) \in G$, parametrized by n real (or complex) parameters, $x \in \Re^n$. It is also a manifold. Near the identity, $g(0) = 1$, Lie group elements may be described by exponentiating n **Lie algebra** generators, $T_A \in \mathfrak{g} = \mathrm{Lie}(G)$,

$$g(x) = e^{x^A T_A} \simeq 1 + x^A T_A$$

Finite collections of Lie group elements (maybe or maybe not connected to the identity) can make an embedded finite group.

A **represenation space** (or **G-module**), V, is a real or complex vector space upon which a **group representation**, $\Pi(G) \subset GL(V)$, or Lie algebra representation, $\pi(\mathfrak{g}) \subset GL(V)$, acts linearly. A representation is **faithful** iff every $\Pi(g)$ is unique and:

$$\Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2) \qquad \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

Quaternion group

Quaternions, e_a, and -1, so 8 elements:

$$\{\pm 1, \pm e_1, \pm e_2, \pm e_3\} \ \in \ G = Q_8$$

Quaternion multiplication does not necessarily commute,

$$e_1 e_2 = e_3 = -e_2 e_1$$

The **center** of a group is the subgroup of elements that commute with everything,

$$\{+1, -1\} \ \in \ C \subset G$$

Group multiplication table, $e_a e_b = M_{ab}{}^c e_c$, with further multiplications by -1 implied:

1	e_1	e_2	e_3
e_1	-1	e_3	$-e_2$
e_2	$-e_3$	-1	e_1
e_3	e_2	$-e_1$	-1

Group representation by Pauli matrices:

$$1 = e_0 = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e_1 = -i\sigma_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad e_2 = -i\sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad e_3 = -i\sigma_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

Pin group

The group of spacetime reflections, $R_u v = v_\perp - v_\|$, is the **pin group**, $G = Pin(1,3) \subset Cl(1,3)$.

We choose chiral matrix representative basis vectors, γ_μ, of $Cl(1,3)$,

$$u = u^\mu \gamma_\mu = \begin{bmatrix} 0 & 0 & u^0 - u^3 & -u^1 + iu^2 \\ 0 & 0 & -u^1 - iu^2 & u^0 + u^3 \\ u^0 + u^3 & u^1 - iu^2 & 0 & 0 \\ u^1 + iu^2 & u^0 - u^3 & 0 & 0 \end{bmatrix} \qquad \gamma = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

A reflection, represented by $\Pi(R_u) = U = u\gamma \in Cl(1,3)$, through a vector, u, acts on vectors (in the vector representation space, $v \in V$) via **adjoint action**, and on **spinors** (in the spinor representation space) via **left action**,

$$v' = R_u\, v = U\, v\, U^- = (u\gamma)\, v\, (u\gamma)^- = -uvu^- = v_\perp - v_\| \qquad \psi' = R_u\, \psi = U\, \psi = (u\gamma)\, \psi$$

Even numbers of reflections generate the **spacetime spin group**, $Spin(1,3) \subset Pin(1,3)$, of Lorentz transformations, with $Spin^+(1,3) \subset Spin(1,3)$ the component connected to the identity. Explicitly:

$$Pin(1,3) = Spin(1,3) \rtimes \{1, T\} = Spin^+(1,3) \rtimes \{1, P, T, PT\} \qquad Spin(1,3) = Spin^+(1,3) \rtimes \{1, PT\}$$

P and T are distinguished reflections, **parity reversal** and **unitary time reversal**,

$$U'_P = -(\gamma_1\gamma)(\gamma_2\gamma)(\gamma_3\gamma) = \gamma_0 \qquad U'_T = \gamma_0\gamma = \gamma_1\gamma_2\gamma_3 \qquad U'_{PT} = U'_P U'_T = \gamma$$

Spin eigenvalues

$Pin(1,3)$ is a 6 dimensional Lie group, with spatial rotation and Lorentz boost generators, $J_\pi = \frac{1}{4}\epsilon_{\pi\rho\sigma}\gamma_\rho\gamma_\sigma$ and $K_\pi = \frac{1}{2}\gamma_0\gamma_\pi$. Two commuting generators span the **Cartan subalgebra**,

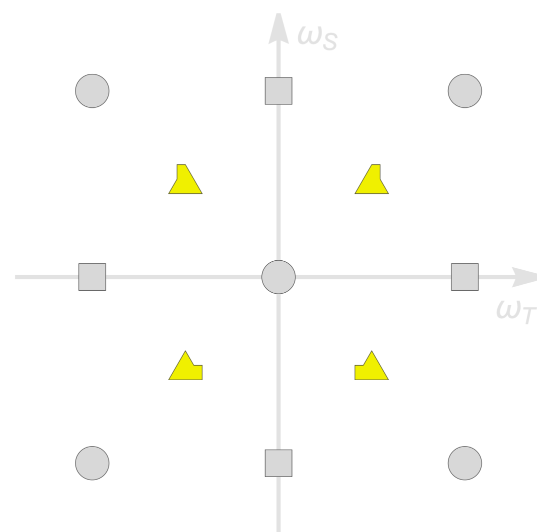
$$J_3 = \frac{1}{4}\gamma_1\gamma_2 = \begin{bmatrix} -\frac{i}{2} & 0 & 0 & 0 \\ 0 & +\frac{i}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & +\frac{i}{2} \end{bmatrix} \qquad K_3 = \frac{1}{2}\gamma_0\gamma_3 = \begin{bmatrix} +\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

These act on vectors, spinors, and the Lie algebra itself. Eigenvectors (**weight vectors**) are particle states and eigenvalues (**weights** or **charges**), **spin** and **boost**, ω_S and ω_T, are conserved in interactions.

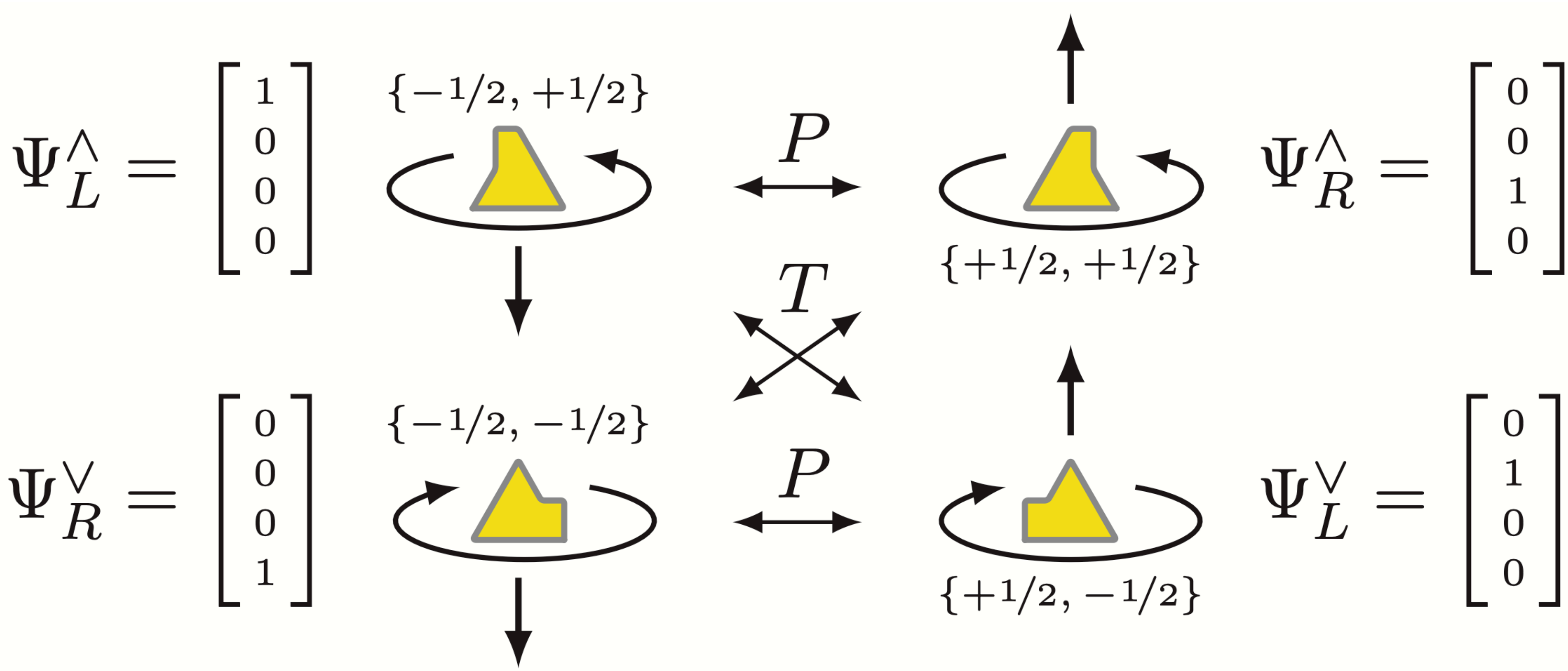
$$J_3 \times v^{\wedge/\vee}_{S/T} = -i\,\omega_S\, v^{\wedge/\vee}_{S/T} \qquad K_3 \times v^{\wedge/\vee}_{S/T} = -\omega_T\, v^{\wedge/\vee}_{S/T}$$

$$J_3\, \psi^{\wedge/\vee}_{L/R} = -i\,\omega_S\, \psi^{\wedge/\vee}_{L/R} \qquad K_3\, \psi^{\wedge/\vee}_{L/R} = -\omega_T\, \psi^{\wedge/\vee}_{L/R}$$

$$\left[J_3, E^{\wedge/\vee}_{L/R}\right] = -i\,\omega_S\, E^{\wedge/\vee}_{L/R} \qquad \left[K_3, E^{\wedge/\vee}_{L/R}\right] = -\omega_T\, E^{\wedge/\vee}_{L/R}$$



Fermion basis states



C, P, and T

Since every fermion has an anti-fermion, there is a **charge symmetry**, C, that transforms between them. This symmetry is not in $Pin(1,3)$, but operates on the complex representation space of $Pin(1,3)$ spinors as an anti-unitary operator,

$$U_C = i\gamma_2 K$$

in which K is complex conjugation. We can combine this with our unitary time operator to get **anti-unitary time conjugation**, and add a phase to parity conjugation,

$$U_T = U_C\, iU_T' = (i\gamma_2 K)(i\gamma_0\gamma) = \gamma_1\gamma_3 K \qquad\qquad U_P = iU_P' = i\gamma_0$$

These three conjugations, C, P, and T, combine to give

$$U_{CP} = -\gamma_0\gamma_2 K \qquad U_{CT} = -i\gamma_1\gamma_2\gamma_3 \qquad U_{PT} = i\gamma_0\gamma_1\gamma_3 K \qquad U_{CPT} = \gamma$$

The **CPT Group**, G_{CPT}, of order 16, thus has multiplication table:

1	C	P	T	CP	CT	PT	CPT
C	$+1$	CP	CT	P	T	CPT	PT
P	CP	-1	PT	$-C$	CPT	$-T$	$-CT$
T	CT	$-PT$	-1	$-CPT$	$-C$	P	CP
CP	P	$-C$	CPT	-1	PT	$-CT'$	$-T$
CT	T	$-CPT$	$-C$	$-PT$	-1	CP	P
PT	CPT	T	$-P$	CT	$-CP$	-1	$-C$
CPT	PT	CT	$-CP$	T'	$-P$	$-C$	-1

Since C commutes, this is identifiable as the split-biquaternion group, $G_{CPT} = Q_8 \times \mathbb{Z}_2$, the direct product of the quaternion group, $Q_8 = \{\pm 1, \pm P, \pm T, \pm PT\}$, and $\mathbb{Z}_2 = \{1, C\}$.

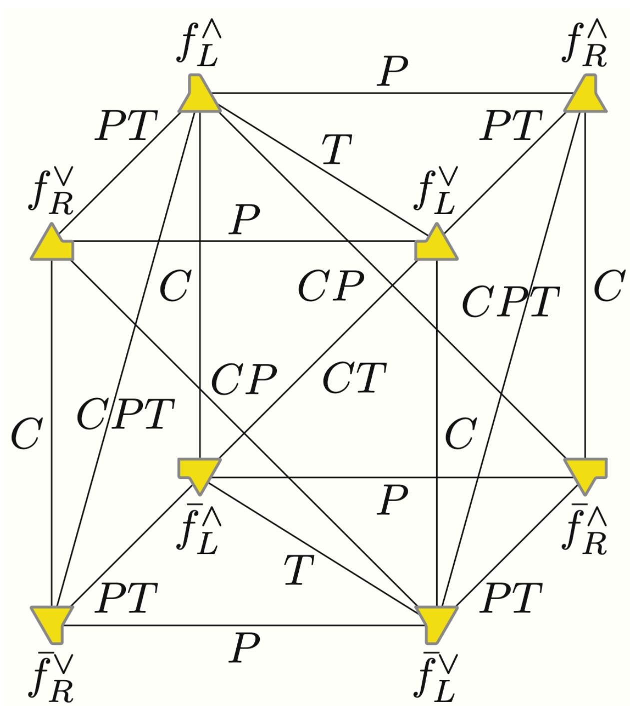
CPT cube

Charge, parity, and time conjugation operators act on fermion states and their weights:

$$(a_{L/R}^{\wedge/\vee})^C = \bar{a}_{L/R}^{\wedge/\vee} \qquad C\,:\,(\omega_T,\omega_S,h,q) \;\mapsto\; (\;\;\omega_T,\;\;\omega_S,\;\;h,-q)$$

$$(a_{L/R}^{\wedge/\vee})^P = -i\,a_{R/L}^{\wedge/\vee} \qquad P\,:\,(\omega_T,\omega_S,h,q) \;\mapsto\; (-\omega_T,\;\;\omega_S,-h,\;\;q)$$

$$(a_{L/R}^{\wedge/\vee})^T = \mp\,a_{L/R}^{\vee/\wedge} \qquad T\,:\,(\omega_T,\omega_S,h,q) \;\mapsto\; (-\omega_T,-\omega_S,\;\;h,\;\;q)$$



Biquaternionic spinors

Equate $Pin(1,3)$ spinor representation space with left-chiral **biquaternions** — complex quaternions,

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad \psi^C = i\gamma_2\psi^* = \begin{bmatrix} -\psi_4^* \\ \psi_3^* \\ \psi_2^* \\ -\psi_1^* \end{bmatrix} \quad \sim \quad \psi_Q = \begin{bmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{bmatrix} = \begin{bmatrix} \psi_L & \bar{\psi}_L \end{bmatrix} \in GL(2,\mathbb{C})$$

Using the Pauli matrix representation of quaternions, $\{e_0 = \sigma_0,\ e_\pi = -i\sigma_\pi\}$, we have the isomorphism to biquaternionic spinors,

$$\psi \qquad \sim \qquad \psi_Q = \psi_Q^0\sigma_0 + \psi_Q^\pi(-i\sigma_\pi) \qquad \sim \qquad \psi_{\mathbb{H}} = \psi_{\mathbb{H}}^\mu e_\mu \ \in \mathbb{C}\otimes\mathbb{H}$$

The action of Lorentz generators (rotations and boosts) on biquaternionic spinors is

$$J_\pi\psi = \left(-\frac{i}{2}\sigma_0\otimes\sigma_\pi\right)\psi \ \sim \ J_\pi\psi_{\mathbb{H}} = \frac{1}{2}e_\pi\psi_{\mathbb{H}} \qquad\qquad K_\pi\psi = \left(\frac{1}{2}\sigma_3\otimes\sigma_\pi\right)\psi \ \sim \ K_\pi\psi_{\mathbb{H}} = \frac{i}{2}e_\pi\psi_{\mathbb{H}}$$

showing $Spin^+(1,3) = SL(2,\mathbb{C}) = \mathbb{C}\otimes\mathbb{H}^{\mathbb{I}}$.

The C, P, and T generators become:

$$\begin{array}{llll} \psi^C = i\gamma_2\psi^* & \psi_Q^C = \psi_Q\sigma_1 & \psi_{\mathbb{H}}^C = i\psi_{\mathbb{H}}e_1 & C \sim ie_1 \\ \psi^P = i\gamma_0\psi & \psi_Q^P = -\sigma_2\psi_Q^*\sigma_1 & \psi_{\mathbb{H}}^P = -\psi_{\mathbb{H}}^* e_3 & P \sim -Ke_3 \\ \psi^T = \gamma_{13}\psi^* & \psi_Q^T = i\sigma_2\psi_Q^* & \psi_{\mathbb{H}}^T = -\psi_{\mathbb{H}}^* e_2 & T \sim -Ke_2 \end{array}$$

with the complex conjugation and quaternion multiplication in C, P, and T acting to the left. These combine to give $CPT \sim -i$.

Quaternion triality and the CPTt Group

How to extend G_{CPT} non-trivially to act on generation-triples of fermions? Introduce the quaternion triality generator:

$$t = -\frac{1}{2}(1 + e_1 + e_2 + e_3) \qquad t^- = t^2 = \frac{1}{2}(-1 + e_1 + e_2 + e_3) \qquad t^3 = 1$$

This can act via the adjoint to cycle imaginary quaternions,

$$\mathrm{ad}_t e_1 = t e_1 t^- = e_2 \qquad \mathrm{ad}_t e_2 = t e_2 t^- = e_3 \qquad \mathrm{ad}_t e_3 = t e_3 t^- = e_1$$

Whether we include the adjoint generator, $\{\mathrm{ad}_t, e_2, e_3\}$, or the t generator itself, $\{t, e_2, e_3\}$, these generators produce the **binary tetrahedral group**, $2T$, of order 24, which is a semi-direct product of subgroups Q_8 and $\mathbb{Z}_3 = \{1, \mathrm{ad}_t, \mathrm{ad}_t^-\} = \{1, t, t^-\}$.

Combining this triality generator with $C \sim ie_1$, $P \sim -Ke_3$, and $T \sim -Ke_2$, we draw several conclusions:

- Triality and $CPT \sim i$ commute.
- The **PTt Group** generated by $\{\mathrm{ad}_t, P, T\}$ is $G_{PTt} = 2T$.
- The **CPTt Group** generated by $\{\mathrm{ad}_t, C, P, T\}$ is $G_{CPTt} = 2T \circ D_4$, of order 96, the central product of the binary tetrahedral group, $2T$, and the dihedral group, $D_4 = \{\pm 1, \pm i, \pm K, \pm iK\}$, of order 8, with a shared central $\mathbb{Z}_2 = \{1, -1\}$.
- Three generations of fermions can be described by three sets of triality-related biquaternionic spinors,

$$\psi^I = \psi_1 \qquad \psi^{II} = \mathrm{ad}_t \psi_2 = t\psi_2 t^- \qquad \psi^{III} = \mathrm{ad}_t^2 \psi_2 = t^- \psi_3 t$$

Note these imply the complex structure in our biquaternionic spinors is triality invariant.

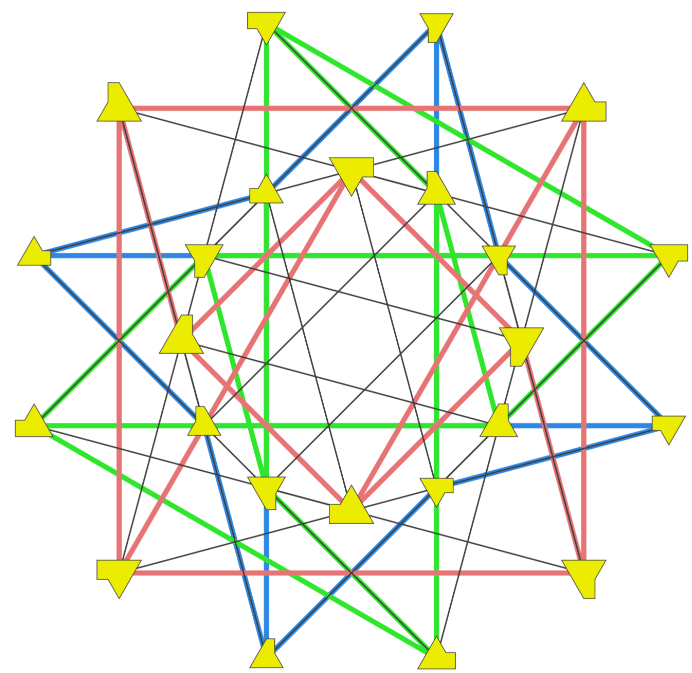
Multi-generational fermion states

To incorporate triality, fermions need minimum of 4 weight coordinates, $\{\omega_t, \omega_S, h, q\}$, with helicity, h, and $\omega_t = 4\omega_S h q$.

Projective representation of charge, parity, time, and triality conjugations in these coords:

$$C \sim \begin{bmatrix} - & & & \\ & + & & \\ & & + & \\ & & & - \end{bmatrix} \qquad P \sim \begin{bmatrix} - & & & \\ & + & & \\ & & - & \\ & & & + \end{bmatrix}$$

$$T \sim \begin{bmatrix} - & & & \\ & - & & \\ & & + & \\ & & & + \end{bmatrix} \qquad t \sim \tfrac{1}{2} \begin{bmatrix} + & - & + & + \\ + & - & - & - \\ + & + & + & - \\ + & + & - & + \end{bmatrix}$$

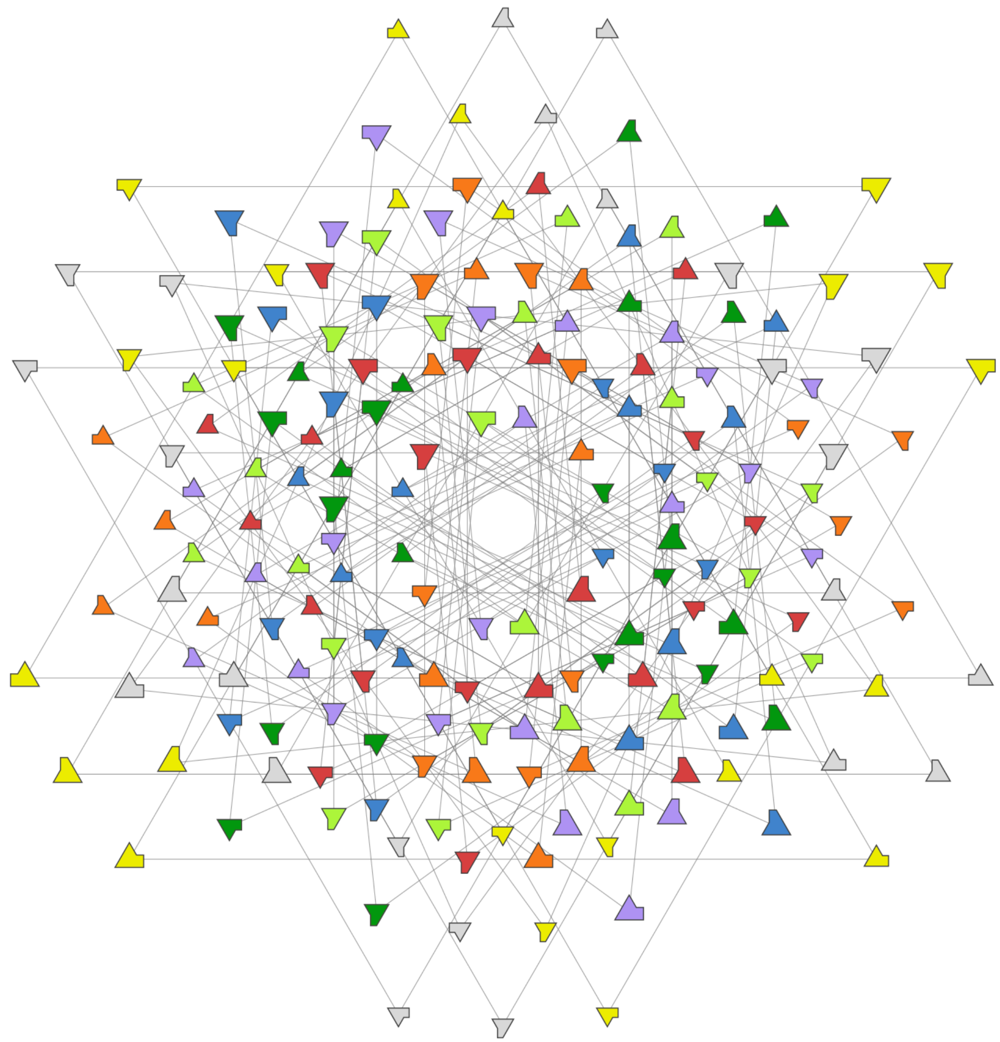


$CPTt$ produces 24-cell of three fermion generations, gen I cube red, gen II cube green, gen II cube blue, related by triality, black.

Note: gen II and gen III fermion charges only make physical sense after transformation by t^2 and t.

Fermions in Exceptional Unification

How many fermion states in the Standard Model? Up or down type fermions, either leptons or 3 colors of quarks, so 8 fermion types times 24-cell for each, gives 192 fermion states.



Only Exceptional Unification accommodates G_{CPTt}. E8 Theory with octo-octonionic $\{\omega_t, \omega_S, U, V, p, x, y, z\}$.