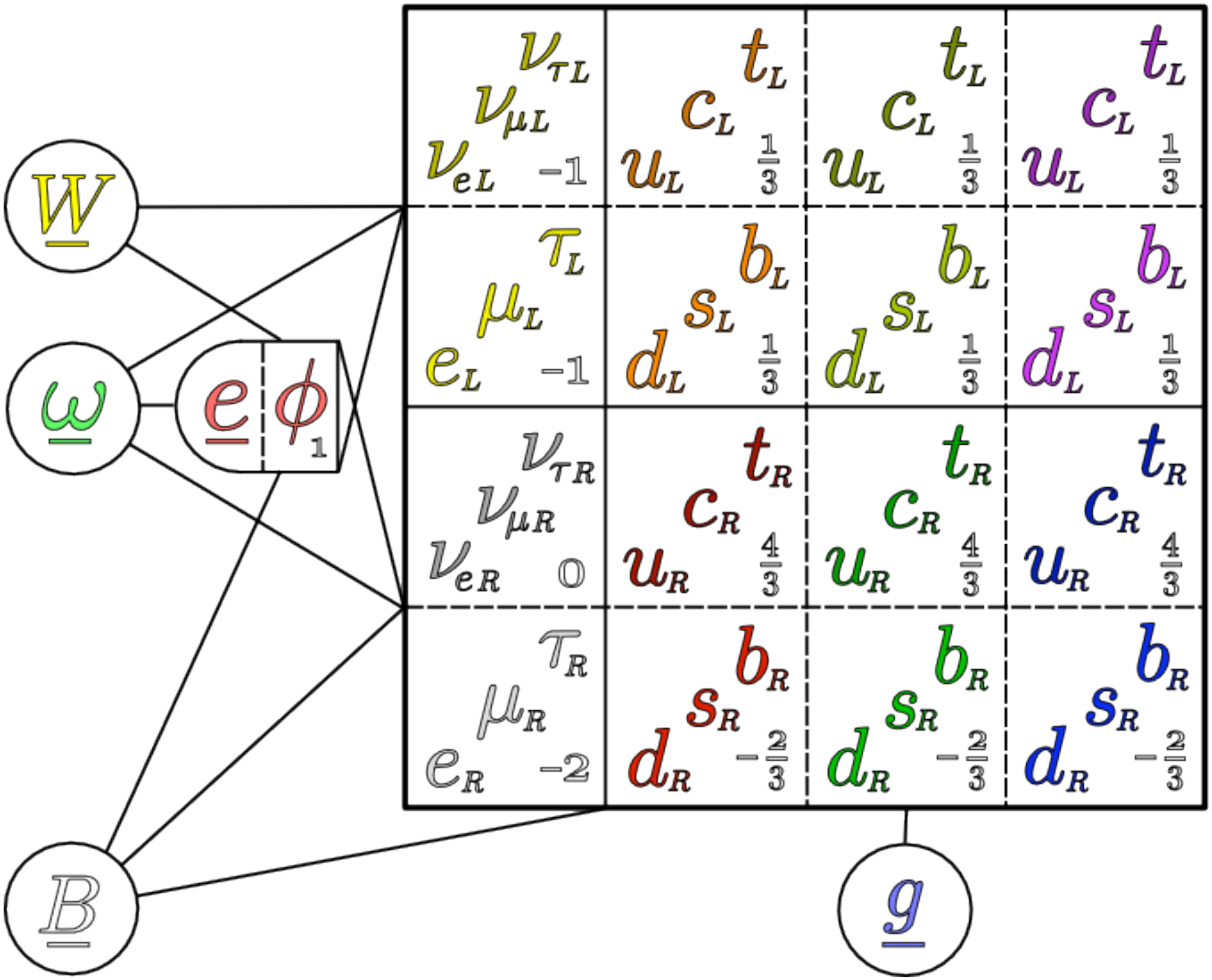
An Exceptionally Simple Theory of Everything



http://deferentialgeometry.org Garrett Lisi FQXi

Everything as a principal bundle connection

$$\underline{\omega} = d\underline{x}^k \tfrac{1}{2} \omega_k{}^{\mu\nu} \gamma_{\mu\nu} \in \underline{C}l^2(3,1) \qquad \underline{e} = d\underline{x}^k (e_k)^\mu \gamma_\mu \in \underline{C}l^1(3,1)$$

$$\underline{W} = d\underline{x}^k W_k{}^\pi \tfrac{i}{2} \sigma_\pi \in s\underline{u}(2) \qquad \begin{bmatrix} \phi_+ \\ \phi_0 \end{bmatrix} \qquad \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix} \qquad \begin{bmatrix} e^\wedge_L \\ e^\vee_L \\ e^\wedge_R \\ e^\vee_R \end{bmatrix}$$

$$\underline{B} = d\underline{x}^k B_k i \in \underline{u}(1) \qquad Y$$

$$\underline{g} = d\underline{x}^k g_k{}^A \tfrac{i}{2} \lambda_A \in s\underline{u}(3) \qquad [u^r, u^g, u^b]$$

$$\updownarrow$$

$$\underline{A}_{\cdot} = \tfrac{1}{2}\underline{\omega} + \tfrac{1}{4}\underline{e}\phi + \underline{W} + \underline{B} + \underline{g} + (\nu_{\cdot e} + e_{\cdot} + u_{\cdot} + d_{\cdot})$$
$$+ (\nu_{\cdot\mu} + \mu_{\cdot} + c_{\cdot} + s_{\cdot}) + (\nu_{\cdot\tau} + \tau_{\cdot} + t_{\cdot} + b_{\cdot})$$

$$\underline{\underline{F}}_{\cdot} = \underline{d}\underline{A}_{\cdot} + \frac{1}{2}[\underline{A}_{\cdot}, \underline{A}_{\cdot}]$$

Review of some representation theory

Cartan subalgebra: $C = C^a T_a \;\subset\; \mathrm{Lie}(G)$

Built from a maximal commuting set of R generators,

$$\left[T_a, T_b\right] = T_a T_b - T_b T_a = 0 \qquad \forall \quad 1 \leq a, b \leq R$$

Root vectors, V_β, are eigenvectors of C in the Lie bracket,

$$[C, V_\beta] = \alpha_\beta V_\beta = \sum_a i C^a \alpha_{a\beta} V_\beta$$

Roots, $\alpha_{a\beta}$, are the eigenvalue coefficients. The pattern of roots in R dimensions corresponds to the Lie algebra,

$$[V_\beta, V_\gamma] = V_\delta \quad \Leftrightarrow \quad \alpha_\beta + \alpha_\gamma = \alpha_\delta$$

Weight vectors and **weights** are eigenvectors and eigenvalue coefficients of C acting on some representation space,

$$C\, V_\beta = \alpha_\beta V_\beta$$

Weight vectors are particles, weights are their quantum numbers.

Gluon and quark weights

$$g = g^A T_A = g^A \frac{i}{2}\lambda_A = \frac{i}{2}\begin{bmatrix} g^3+\frac{1}{\sqrt{3}}g^8 & g^1-ig^2 & g^4-ig^5 \\ g^1+ig^2 & -g^3+\frac{1}{\sqrt{3}}g^8 & g^6-ig^7 \\ g^4+ig^5 & g^6+ig^7 & -\frac{2}{\sqrt{3}}g^8 \end{bmatrix}$$

Cartan subalgebra: $C = g^3 T_3 + g^8 T_8$ (the diagonal)

Roots and root vectors:

$$[C, V_{g^{g\bar{b}}}] = i\left(\left(-\frac{1}{2}\right)g^3 + \left(\frac{\sqrt{3}}{2}\right)g^8\right)V_{g^{g\bar{b}}} \qquad V_{g^{g\bar{b}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

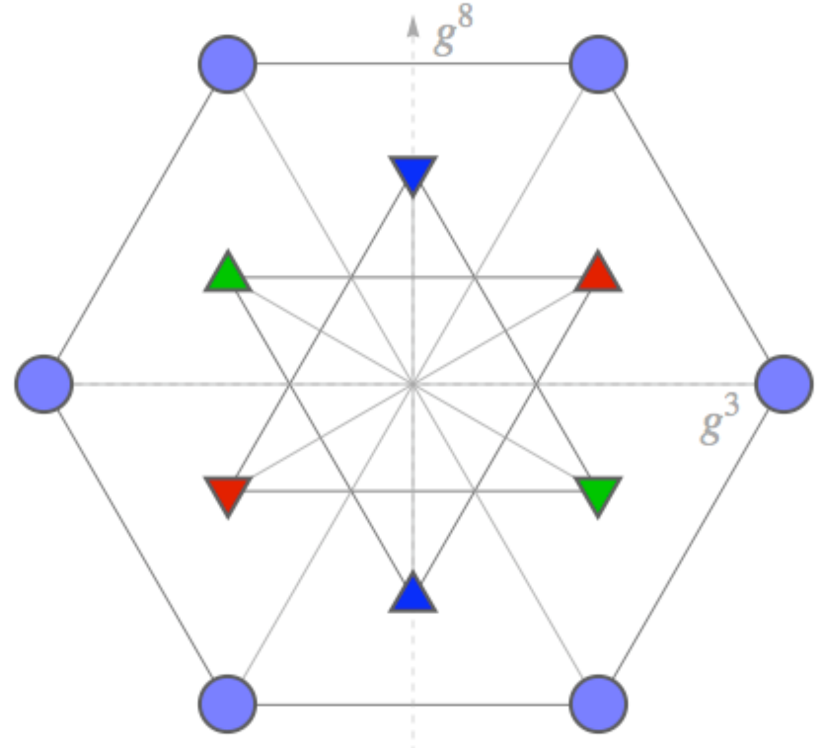
for the $g^{g\bar{b}}$ gluon. Weights and weight vectors:

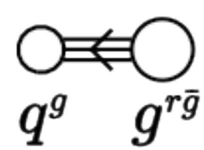
$$C\,V_{q^r} = i\left(\left(\frac{1}{2}\right)g^3 + \left(\frac{1}{2\sqrt{3}}\right)g^8\right)V_{q^r} \qquad V_{q^r} = [1, 0, 0]$$

for a red quark, q^r, and for their duals acted on by $-C^T$, the anti-quarks.

Strong G2

$G2$	V_β	g^3	g^8
$g^{r\bar{g}}$	$(T_2 - iT_1)$	1	0
$g^{\bar{r}g}$	$(-T_2 - iT_1)$	-1	0
$g^{r\bar{b}}$	$(T_5 - iT_4)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$g^{\bar{r}b}$	$(-T_5 - iT_4)$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$g^{\bar{g}b}$	$(-T_7 - iT_6)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$g^{g\bar{b}}$	$(T_7 - iT_6)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
q^r	$[1,0,0]$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$
q^g	$[0,1,0]$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$
q^b	$[0,0,1]$	0	$-\frac{1}{\sqrt{3}}$
$\bar{q}^r$	$[1,0,0]$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
$\bar{q}^g$	$[0,1,0]$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
$\bar{q}^b$	$[0,0,1]$	0	$\frac{1}{\sqrt{3}}$





Exceptional Lie brackets

The 14 Lie algebra elements of the smallest exceptional Lie group, $G2$:

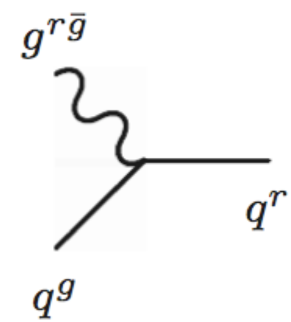
$$\begin{aligned} g2 &= su(3) + 3 + \bar{3} \\ &\qquad \underline{g} \quad + \underset{\cdot}{q} + \underset{\cdot}{\bar{q}} \in \underset{-\cdot}{g2} \end{aligned}$$

The structure of $G2$ implies a Lie bracket equivalent to a fundamental action,

$$[g, q] = \left[g^A T_A, q^B T_B\right] = g\, q = \begin{bmatrix} \frac{i}{2} g^3 + \frac{i}{2\sqrt{3}} g^8 & g^{r\bar{g}} & g^{r\bar{b}} \\ g^{\bar{r}g} & -\frac{i}{2} g^3 + \frac{i}{2\sqrt{3}} g^8 & g^{g\bar{b}} \\ g^{\bar{r}b} & g^{\bar{g}b} & -\frac{i}{\sqrt{3}} g^8 \end{bmatrix} \begin{bmatrix} q^r \\ q^g \\ q^b \end{bmatrix}$$

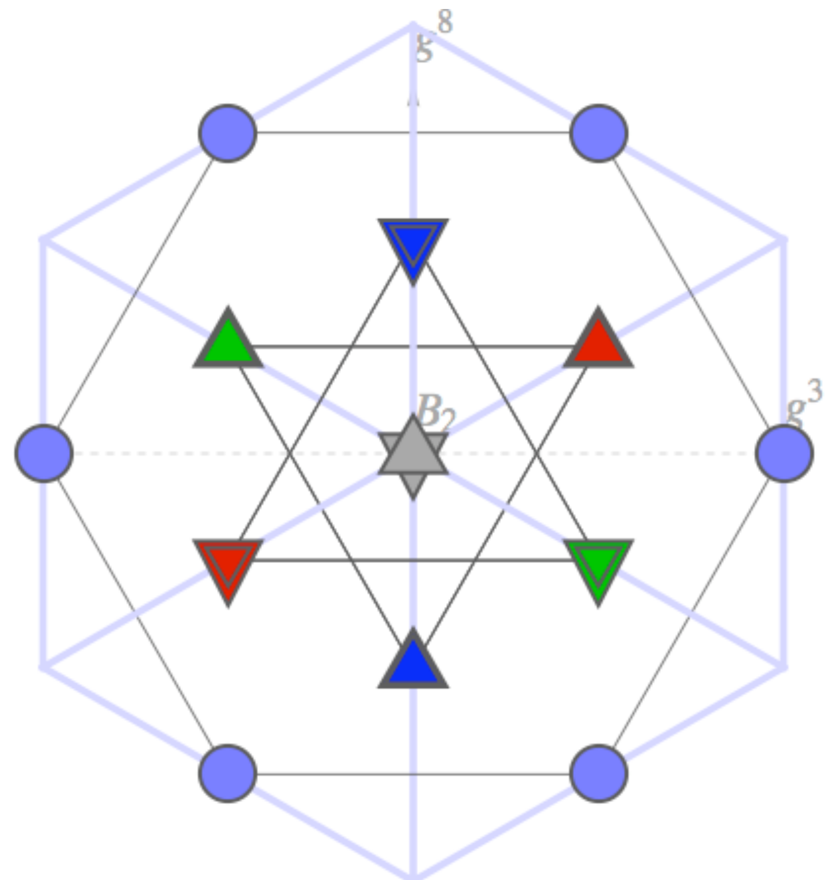
corresponding to the strong interactions, such as

$$\left[g^{r\bar{g}}, q^g\right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = q^r$$



G2 in SO(7)

$G2+$		x	y	z	g^3	g^8	$\frac{\sqrt{2}}{\sqrt{3}}B_2$
	$g^{r\bar{g}}$	-1	1	0	1	0	0
	$g^{\bar{r}g}$	1	-1	0	-1	0	0
	$g^{r\bar{b}}$	-1	0	1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0
	$g^{\bar{r}b}$	1	0	-1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0
	$g^{\bar{g}b}$	0	1	-1	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0
	$g^{g\bar{b}}$	0	-1	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0
	q_I^r	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$
	q_I^g	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$
	q_I^b	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$
	$\bar{q}_I^r$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	$\bar{q}_I^g$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	$\bar{q}_I^b$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	q_{II}	∓ 1			"	"	$\pm\frac{1}{3}$
	q_{III}	$\pm 1 \quad \pm 1$			"	"	$\mp\frac{2}{3}$
	l	$\mp\frac{1}{2}$	$\mp\frac{1}{2}$	$\mp\frac{1}{2}$	0	0	$\pm\frac{1}{2}$



Gravitational SO(3,1)

$$\omega = \tfrac{1}{2}\omega^{\mu\nu}\gamma_{\mu\nu} = \begin{bmatrix} \omega_L & \\ & \omega_R \end{bmatrix}$$

$$\omega_{L/R} = \begin{bmatrix} i\omega^3_{L/R} & \omega^\wedge_{L/R} \\ \omega^\vee_{L/R} & -i\omega^3_{L/R} \end{bmatrix}$$

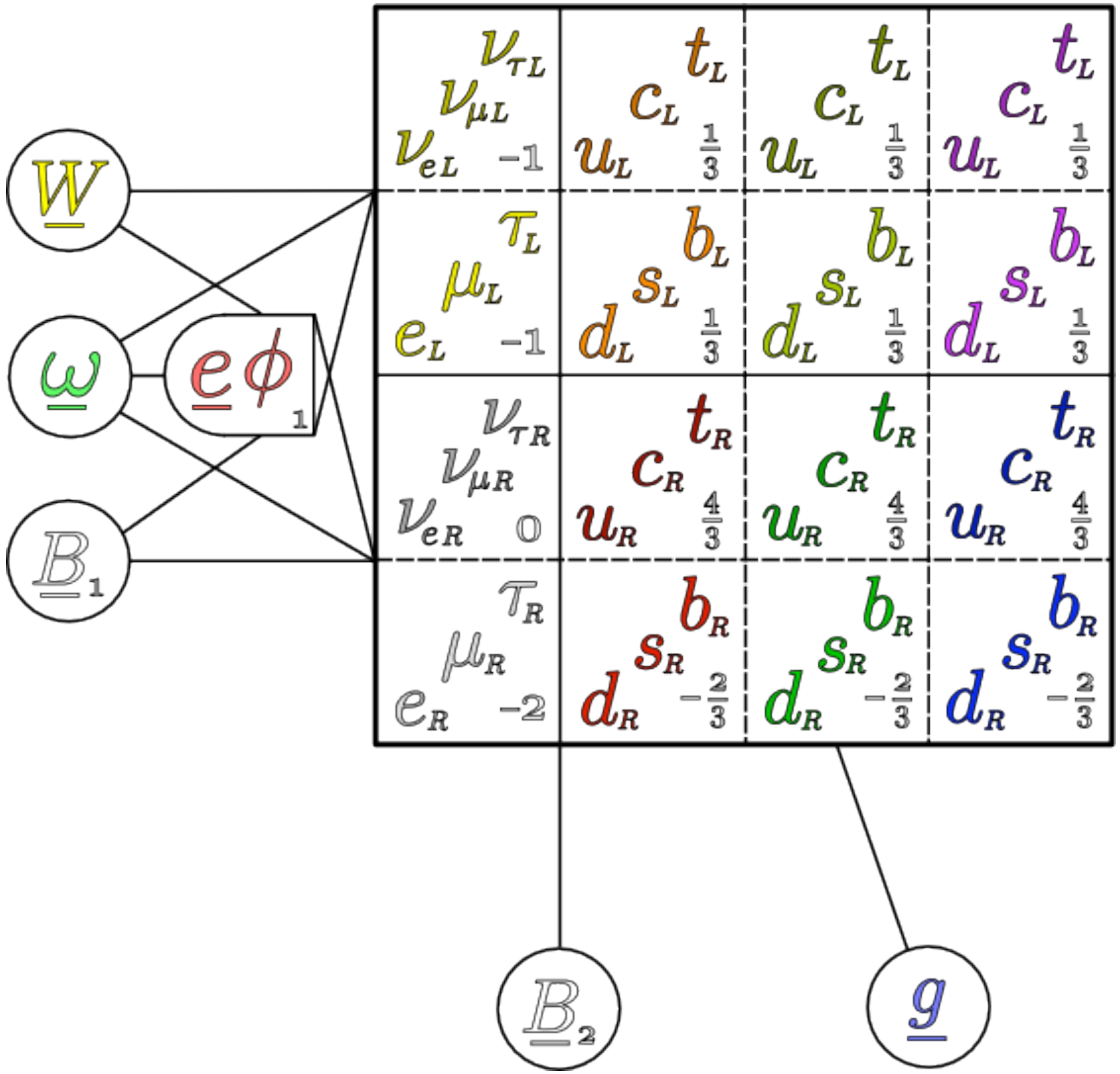
$$e = e^\mu \gamma_\mu = \begin{bmatrix} & e_R \\ e_L & \end{bmatrix}$$

$$e_{L/R} = \begin{bmatrix} e_T^{\wedge/\vee} & \mp e_S^\wedge \\ \mp e_S^\vee & e_T^{\vee/\wedge} \end{bmatrix}$$

$$f = \begin{bmatrix} f_L \\ f_R \end{bmatrix} \qquad f_{L/R} = \begin{bmatrix} f^\wedge_{L/R} \\ f^\vee_{L/R} \end{bmatrix}$$

$SO(3,1)$		ω^3_L	ω^3_R
●	$\omega^\wedge_L$	2	0
●	$\omega^\vee_L$	-2	0
●	$\omega^\wedge_R$	0	2
●	$\omega^\vee_R$	0	-2
■	$e^\wedge_S$	1	1
■	$e^\vee_S$	-1	-1
■	$e^\wedge_T$	-1	1
■	$e^\vee_T$	1	-1
▲	$f^\wedge_L$	1	0
▲	$f^\vee_L$	-1	0
▲	$f^\wedge_R$	0	1
▲	$f^\vee_R$	0	-1

Pati-Salam model plus gravity



$$\big(SO(3,1) + 4 \times 4 + SU(2)_L + SU(2)_R\big) + \big(U(1) + SU(3)\big)$$

Electroweak SU(2) and U(1)

$$W = \begin{bmatrix} \frac{i}{2}W^3 & W^+ \\ W^- & -\frac{i}{2}W^3 \end{bmatrix} \quad B_1 = \begin{bmatrix} \frac{i}{2}B_1^3 & B_1^+ \\ B_1^- & -\frac{i}{2}B_1^3 \end{bmatrix}$$

$$\left[\begin{bmatrix} W & \\ & B_1 \end{bmatrix}, \begin{bmatrix} & \phi_B \\ \phi_W & \end{bmatrix} \right]$$

$$\phi_{W/B} = \begin{bmatrix} -\phi_{0/1} & \phi_+ \\ \phi_- & \phi_{1/0} \end{bmatrix}$$

$$\begin{bmatrix} W & \\ & B_1 \end{bmatrix} \quad \begin{bmatrix} \nu_{eL} \\ e_L \\ \nu_{eR} \\ e_R \end{bmatrix} \quad \begin{bmatrix} u_L \\ d_L \\ u_R \\ d_R \end{bmatrix}$$

$$\left(\frac{\sqrt{3}}{\sqrt{5}}B_1^3 - \frac{\sqrt{2}}{\sqrt{5}}B_2\right) = \left(\frac{\sqrt{3}}{\sqrt{5}}\right)\frac{1}{2}Y \;\rightarrow\; g_1 = \frac{\sqrt{3}}{\sqrt{5}}$$

$SO(4)$		W^3	B_1^3	$\frac{\sqrt{2}}{\sqrt{3}}B_2$	$\frac{1}{2}Y$	Q
	W^+	1	0	0	0	1
	W^-	-1	0	0	0	-1
	B_1^+	0	1	0	1	1
	B_1^-	0	-1	0	-1	-1
	ϕ_+	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1
	ϕ_-	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
	ϕ_0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	ϕ_1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0
	ν_{eL}	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
	e_L	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1
	ν_{eR}	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
	e_R	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	-1
	u_L	$\frac{1}{2}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
	d_L	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
	u_R	0	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
	d_R	0	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Graviweak SO(7,1)

$$H_1 = (\tfrac{1}{2}\omega + \tfrac{1}{4}e\phi + W + B_1)$$
$$\in so(3,1) + 4\times 4 + \big(su(2) + su(2)\big)$$
$$= Cl^2(7,1) = so(7,1) = d4$$

$$(\nu_e + e) \in S^{8+}$$

$$H_1\,(\nu_e + e) =$$

$$\begin{bmatrix} \frac{1}{2}\omega_L + \frac{i}{2}W^3 & W^+ & -\frac{1}{4}e_R\phi_1 & \frac{1}{4}e_R\phi_+ \\ W^- & \frac{1}{2}\omega_L - \frac{i}{2}W^3 & \frac{1}{4}e_R\phi_- & \frac{1}{4}e_R\phi_0 \\ -\frac{1}{4}e_L\phi_0 & \frac{1}{4}e_L\phi_+ & \frac{1}{2}\omega_R + \frac{i}{2}B_1^3 & B_1^+ \\ \frac{1}{4}e_L\phi_- & \frac{1}{4}e_L\phi_1 & B_1^- & \frac{1}{2}\omega_R - \frac{i}{2}B_1^3 \end{bmatrix} \begin{bmatrix} \nu_{eL} \\ e_L \\ \nu_{eR} \\ e_R \end{bmatrix}$$

$D4$	$\frac{1}{2}\omega_L^3$	$\frac{1}{2}\omega_R^3$	W^3	B_1^3
$\omega_L^{\wedge/\vee}$	± 1	0	0	0
$\omega_R^{\wedge/\vee}$	0	± 1	0	0
$W^\pm$	0	0	± 1	0
$B_1^\pm$	0	0	0	± 1
$e_T^{\wedge/\vee}\phi_+$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$e_S^{\wedge/\vee}\phi_+$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$e_T^{\wedge/\vee}\phi_-$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$e_S^{\wedge/\vee}\phi_-$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$e_T^{\wedge/\vee}\phi_0$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$e_S^{\wedge/\vee}\phi_0$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$e_T^{\wedge/\vee}\phi_1$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$e_S^{\wedge/\vee}\phi_1$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\nu_{eL}^{\wedge/\vee}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	0
$e_L^{\wedge/\vee}$	$\pm\frac{1}{2}$	0	$-\frac{1}{2}$	0
$\nu_{eR}^{\wedge/\vee}$	0	$\pm\frac{1}{2}$	0	$\frac{1}{2}$
$e_R^{\wedge/\vee}$	0	$\pm\frac{1}{2}$	0	$-\frac{1}{2}$

Graviweak F4

A **triality** rotation, T, of $D4$:

$$\begin{bmatrix} \frac{1}{2}\omega'^3_L \\ \frac{1}{2}\omega'^3_R \\ W'^3 \\ B'^3_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\omega^3_L \\ \frac{1}{2}\omega^3_R \\ W^3 \\ B^3_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega^3_R \\ B^3_1 \\ W^3 \\ \frac{1}{2}\omega^3_L \end{bmatrix}$$

$$T\,T\,T\,\omega^\wedge_R = T\,T\,\omega^\wedge_L = T\,B^+_1 = \omega^\wedge_R$$

Roots invariant under this T:

$$\{W^+,\, W^-,\, e^\wedge_S\phi_+,\, e^\wedge_S\phi_0,\, e^\vee_S\phi_-,\, e^\vee_S\phi_1\}$$

Rotations to triality-equivalent vector and negative chiral spinor representation spaces:

$$T\,S^{8+} = V^8 \qquad T\,V^8 = S^{8-} \qquad T\,S^{8-} = S^{8+}$$

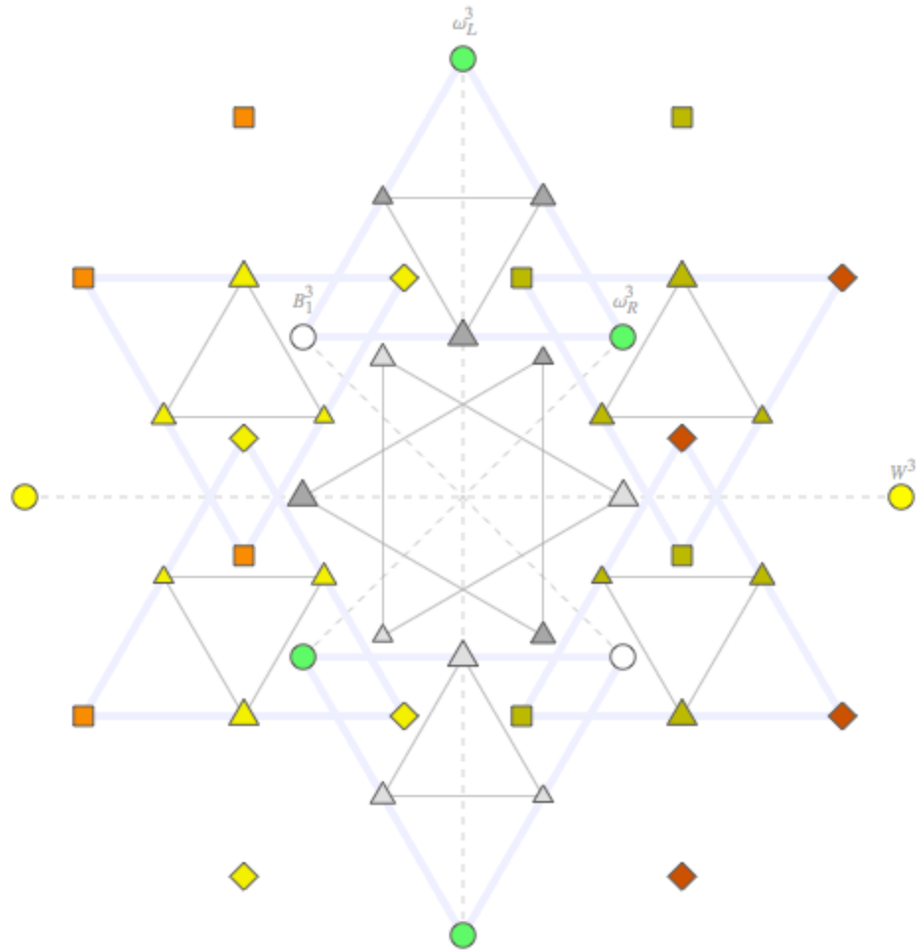
Three generations, related by triality:

$$T\,e^\wedge_L = \mu^\wedge_L \qquad T\,\mu^\wedge_L = \tau^\wedge_L \qquad T\,\tau^\wedge_L = e^\wedge_L$$

V^8	$\frac{1}{2}\omega^3_L$	$\frac{1}{2}\omega^3_R$	W^3	B^3_1
tri	$\frac{1}{2}\omega^3_R$	B^3_1	W^3	$\frac{1}{2}\omega^3_L$
$\nu^{\wedge/\vee}_{\mu L}$	0	0	$\frac{1}{2}$	$\pm\frac{1}{2}$
$\mu^{\wedge/\vee}_L$	0	0	$-\frac{1}{2}$	$\pm\frac{1}{2}$
$\nu^{\wedge/\vee}_{\mu R}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0
$\mu^{\wedge/\vee}_R$	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	0

S^{8-}	$\frac{1}{2}\omega^3_L$	$\frac{1}{2}\omega^3_R$	W^3	B^3_1
tri	B^3_1	$\frac{1}{2}\omega^3_L$	W^3	$\frac{1}{2}\omega^3_R$
$\nu^{\wedge/\vee}_{\tau L}$	0	$\pm\frac{1}{2}$	$\frac{1}{2}$	0
$\tau^{\wedge/\vee}_L$	0	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0
$\nu^{\wedge/\vee}_{\tau R}$	$\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\tau^{\wedge/\vee}_R$	$-\frac{1}{2}$	0	0	$\pm\frac{1}{2}$

F4 root system



F4 and G2 together

$$
\begin{array}{lcll}
F4 & : & (\tfrac{1}{2}\omega^3_L, \tfrac{1}{2}\omega^3_R, W^3, B^3_1) & \left\{ \begin{array}{l} \text{graviweak interactions} \\ \text{three generations} \end{array} \right. \\
G2 & : & \qquad\qquad\qquad\qquad\qquad (B_2, g^3, g^8) & \left\{ \begin{array}{l} \text{strong interactions} \\ \text{anti-particles} \end{array} \right. \\
\hline
E8 & : & (\tfrac{1}{2}\omega^3_L, \tfrac{1}{2}\omega^3_R, W^3, B^3_1, w, B_2, g^3, g^8) & \{ \text{ everything}
\end{array}
$$

Breakdown of E8 to the standard model and gravity:

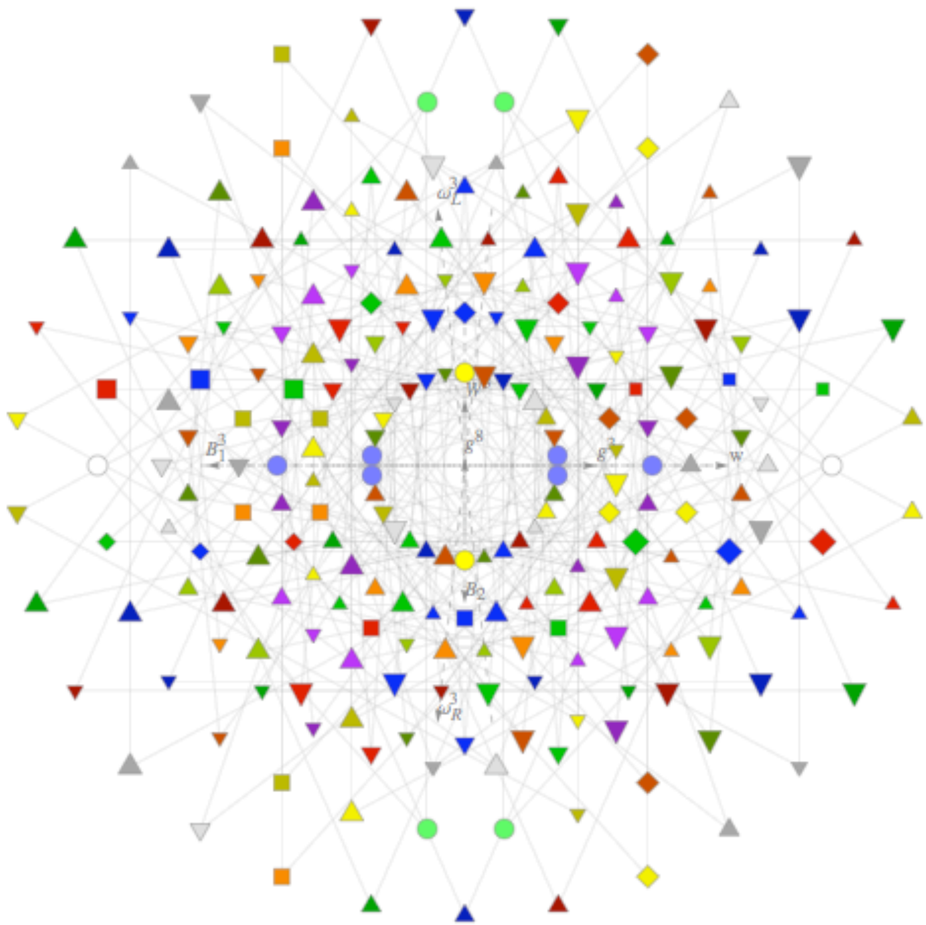
$$
\begin{aligned}
e8 &= f4 + g2 + 26 \times 7 \\
&= so(7,1) + su(3) + (S^{8+} + V^8 + S^{8-}) \times (1+1+3+\bar{3}) + 3 \times (3+\bar{3}) + 2
\end{aligned}
$$

$$
A = \left(\tfrac{1}{2}\omega + \tfrac{1}{4}e\phi + W + B_1\right) + g + 3 \times \Psi + x\Phi + B_2 + w
$$

Two new quantum numbers and some non-standard particles:

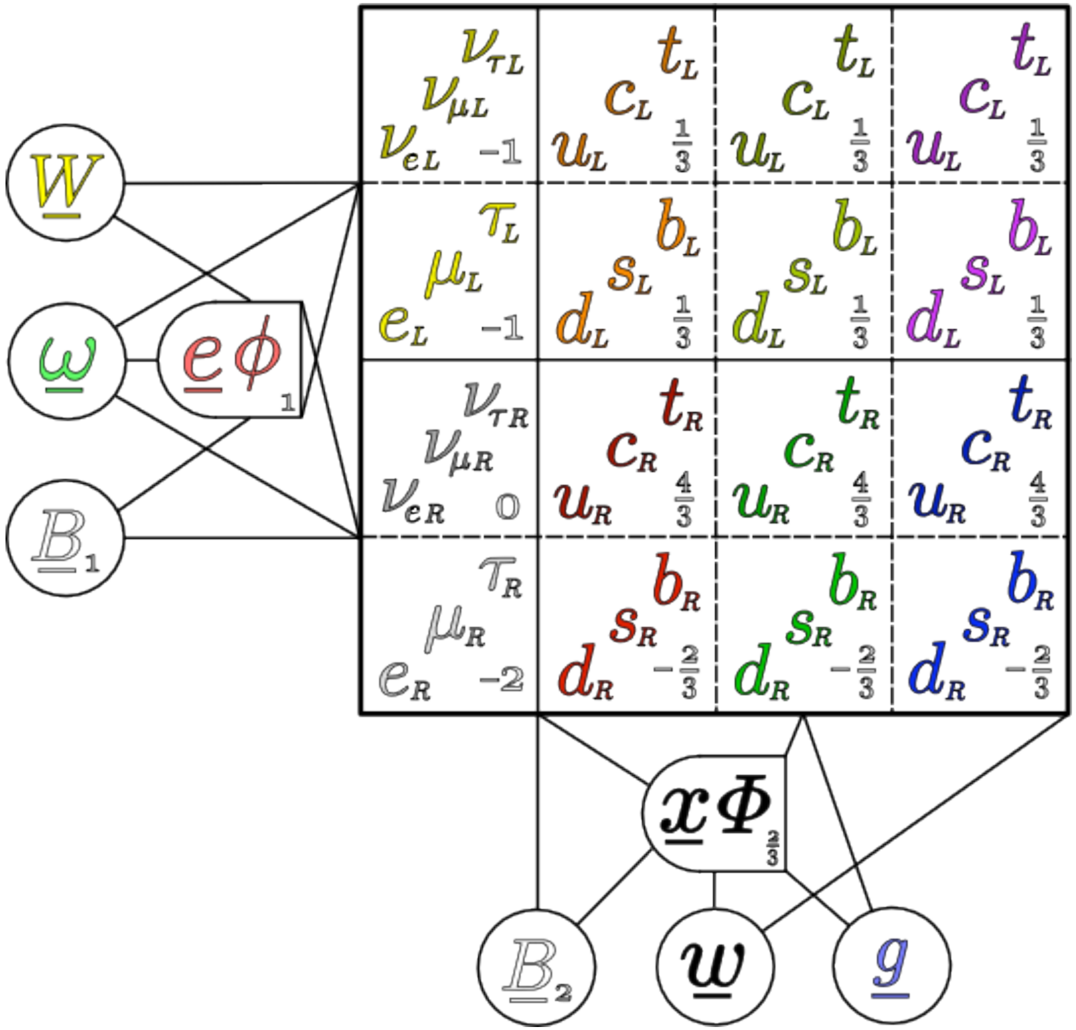
$$
\{\, w \quad (B^3_1 + B_2) \quad B^\pm_1 \quad x_{1/2/3}\Phi^{r/g/b} \quad x_{1/2/3}\bar{\Phi}^{r/g/b} \,\}
$$

E8 root system



E.S.T.o.E.: Each E8 vertex corresponds to an elementary particle field.

E8 periodic table



"E8 is perhaps the most beautiful structure in all of mathematics, but it's very complex." — Hermann Nicolai

E8 connection

$$\underset{\cdot}{\underline{A}} = \underline{H}_1 + \underline{H}_2 + \underset{\cdot}{\Psi}_I + \underset{\cdot}{\Psi}_{II} + \underset{\cdot}{\Psi}_{III} \quad \in \quad \underset{\cdot}{\underline{e8}}$$

$$\begin{aligned}
\underline{H}_1 &= \tfrac{1}{2}\underline{\omega} + \tfrac{1}{4}\underline{e}\phi + \underline{W} + \underline{B}_1 && \in \quad s\underline{o}(7,1) \\
& \underline{\omega} && \in \quad s\underline{o}(3,1) \\
& \underline{e}\phi = (\underline{e}_1 + \underline{e}_2 + \underline{e}_3 + \underline{e}_4) \times (\phi_{+/0} + \phi_{-/1}) && \in \quad \underline{4} \times (2 + \bar{2}) \\
& \underline{W} + \underline{B}_1 && \in \quad s\underline{u}(2) + s\underline{u}(2) \\
\underline{H}_2 &= \underline{w} + \underline{B}_2 + \underline{x}\Phi + \underline{g} && \in \quad s\underline{o}(7,1)\,? \\
& \underline{w} + \underline{B}_2 && \in \quad \underline{u}(1) + \underline{u}(1) \\
& \underline{x}\Phi = (\underline{x}_I + \underline{x}_{II} + \underline{x}_{III}) \times (\Phi^{r/g/b} + \Phi^{\bar{r}/\bar{g}/\bar{b}}) && \in \quad \underline{3} \times (3 + \bar{3}) \\
& \underline{g} && \in \quad s\underline{u}(3) \\
\underset{\cdot}{\Psi}_I &= \underset{\cdot}{\nu}_e + \underset{\cdot}{e} + \underset{\cdot}{u} + \underset{\cdot}{d} && \in \quad S^{8+} \times S^{8+} \\
\underset{\cdot}{\Psi}_{II} &= \underset{\cdot}{\nu}_\mu + \underset{\cdot}{\mu} + \underset{\cdot}{c} + \underset{\cdot}{s} && \in \quad V^8 \times V^8 \\
\underset{\cdot}{\Psi}_{III} &= \underset{\cdot}{\nu}_\tau + \underset{\cdot}{\tau} + \underset{\cdot}{t} + \underset{\cdot}{b} && \in \quad S^{8-} \times S^{8-}
\end{aligned}$$

E8 curvature

$$\underset{=\cdot}{F} = \underset{-}{d}\underset{-\cdot}{A} + \underset{-\cdot}{A}\underset{-\cdot}{A} = \underset{=}{F}_1 + \underset{=}{F}_2 + \underset{-}{D}\left(\underset{\cdot}{\Psi}_I + \underset{\cdot}{\Psi}_{II} + \underset{\cdot}{\Psi}_{III}\right) \quad \in \quad \underset{-\cdot}{e8}$$

$$\underset{=}{F}_1 = \tfrac{1}{2}\left(\underset{=}{R} - \tfrac{1}{8}\underset{-}{e}\underset{-}{e}\phi^2\right) + \tfrac{1}{4}\left(\underset{=}{T}\phi - \underset{-}{e}\underset{-}{D}\phi\right) + \left(\underset{=}{F}_{B_1} + \underset{=}{F}_W\right) \quad \in \quad s\underset{-}{o}(7,1)$$

$$\underset{=}{R} = \underset{-}{d}\underset{-}{\omega} + \tfrac{1}{2}\underset{-}{\omega}\underset{-}{\omega} \quad \in \quad s\underset{-}{o}(3,1)$$

$$\underset{=}{T}\phi - \underset{-}{e}\underset{-}{D}\phi = \left(\underset{-}{d}\underset{-}{e} + \tfrac{1}{2}[\underset{-}{\omega}, \underset{-}{e}]\right)\phi - \underset{-}{e}\left(\underset{-}{d}\phi + [\underset{-}{B}_1 + \underset{-}{W}, \phi]\right) \quad \in \quad \underset{-}{4} \times (2 + \bar{2})$$

$$\underset{=}{F}_{B_1} + \underset{=}{F}_W = (\underset{-}{d}\underset{-}{B}_1 + \underset{-}{B}_1\underset{-}{B}_1) + (\underset{-}{d}\underset{-}{W} + \underset{-}{W}\underset{-}{W}) \quad \in \quad s\underset{-}{u}(2) + s\underset{-}{u}(2)$$

$$\underset{=}{F}_2 = \left(\underset{=}{F}_w + \underset{=}{F}_{B_2} + \underset{-}{x}\Phi\underset{-}{x}\Phi\right) + \left((\underset{-}{D}\underset{-}{x})\Phi - \underset{-}{x}\underset{-}{D}\Phi\right) + \underset{=}{F}_g \quad \in \quad s\underset{-}{o}(7,1)\,?$$

$$\underset{=}{F}_w + \underset{=}{F}_{B_2} = \underset{-}{d}\underset{-}{w} + \underset{-}{d}\underset{-}{B}_2 \quad \in \quad \underset{-}{u}(1) + \underset{-}{u}(1)$$

$$(\underset{-}{D}\underset{-}{x})\Phi - \underset{-}{x}\underset{-}{D}\Phi = \left(\underset{-}{d}\underset{-}{x} + [\underset{-}{w} + \underset{-}{B}_2, \underset{-}{x}]\right)\Phi - \underset{-}{x}\left(\underset{-}{d}\Phi + [\underset{-}{g}, \Phi]\right) \in \quad \underset{-}{3} \times (3 + \bar{3})$$

$$\underset{=}{F}_g = \underset{-}{d}\underset{-}{g} + \underset{-}{g}\underset{-}{g} \quad \in \quad s\underset{-}{u}(3)$$

$$\underset{-}{D}\underset{\cdot}{\Psi} = \left(\underset{-}{d} + \frac{1}{2}\underset{-}{\omega} + \frac{1}{4}\underset{-}{e}\phi\right)\underset{\cdot}{\Psi} + \underset{-}{W}\underset{\cdot}{\Psi}_L + \underset{-}{B}_1\underset{\cdot}{\Psi}_R + \underset{\cdot}{\Psi}\left(\underset{-}{w} + \underset{-}{B}_2 + \underset{-}{x}\Phi\right) + \underset{\cdot}{\Psi}_q\,\underset{-}{g}$$

Action for everything

Modified BF action, using $\dot{\underline{\underline{B}}} = \underline{\underline{B}} + \dot{\underline{\underline{\underline{B}}}}$:

$$S = \int \left\langle \dot{\underline{\underline{B}}}\, \underline{\underline{F}}_{\cdot} + \underset{\sim}{\Phi}(\underline{H}_1, \underline{H}_2, \underline{\underline{B}}) \right\rangle$$

$$= \int \left\langle \dot{\underline{\underline{\underline{B}}}}\, \underline{D} \underset{\cdot}{\Psi} + \underline{\underline{B}}\,\underline{\underline{F}} + \tfrac{\pi G}{4} \underline{\underline{B}}_G \underline{\underline{B}}_G \gamma + \underline{\underline{B}}' * \underline{\underline{B}}' \right\rangle$$

$$= \int \left\langle \dot{\underline{\underline{\underline{B}}}}\, \underline{D} \underset{\cdot}{\Psi} + \underset{\sim}{e} \frac{1}{16\pi G} \phi^2 \left(R - \frac{3}{2}\phi^2\right) + \frac{1}{4} \underline{\underline{F}}' * \underline{\underline{F}}' \right\rangle$$

Cosmological constant from the Higgs VEV: $\Lambda = \frac{3}{4}\phi^2$

Implies frame VEV is de Sitter: $\underline{\underline{R}} = \frac{\Lambda}{6} \underline{e}\,\underline{e}$ $\qquad R = 4\Lambda$

Vacuum expectation value of the curvature vanishes: $\underline{\underline{F}}_{\cdot} = 0$

Gravitational part of the action

$$S_G = \int \left\langle \underset{=}{B}_G \underset{=}{F}_G + \tfrac{\pi G}{4} \underset{=}{B}_G \underset{=}{B}_G \gamma \right\rangle \qquad \underset{=}{F}_G = \tfrac{1}{2}\left(\underset{=}{R} - \tfrac{1}{8} \underset{-}{e}\,\underset{-}{e}\,\phi^2\right) \in \underset{-}{so}(3,1)$$

$$\delta \underset{=}{B}_G \to \underset{=}{B}_G = \frac{1}{\pi G}\left(\underset{=}{R} - \tfrac{1}{8} \underset{-}{e}\,\underset{-}{e}\,\phi^2\right)\gamma \qquad \gamma = \gamma_1\gamma_2\gamma_3\gamma_4$$

$$S_G = \frac{1}{\pi G} \int \left\langle \underset{=}{F}_G \underset{=}{F}_G \gamma \right\rangle = \frac{1}{4\pi G} \int \left\langle \left(\underset{=}{R} - \tfrac{1}{8} \underset{-}{e}\,\underset{-}{e}\,\phi^2\right)\left(\underset{=}{R} - \tfrac{1}{8} \underset{-}{e}\,\underset{-}{e}\,\phi^2\right)\gamma \right\rangle$$

$$\left\langle \underset{=}{R}\,\underset{=}{R}\,\gamma \right\rangle = \underset{-}{d}\left\langle \left(\underset{-}{\omega}\,\underset{-}{d}\,\underset{-}{\omega} + \tfrac{1}{3}\underset{-}{\omega}\,\underset{-}{\omega}\,\underset{-}{\omega}\right)\gamma \right\rangle \qquad \leftarrow \text{Chern-Simons}$$

$$\frac{1}{4!}\left\langle \underset{-}{e}\,\underset{-}{e}\,\underset{-}{e}\,\underset{-}{e}\,\gamma \right\rangle = -\underset{\sim}{e} \qquad \leftarrow \text{volume element}$$

$$\left\langle \underset{-}{e}\,\underset{-}{e}\,\underset{=}{R}\,\gamma \right\rangle = -\underset{\sim}{e}\,R \qquad \leftarrow \text{curvature scalar}$$

$$S_G = \frac{1}{16\pi G} \int \underset{\sim}{e}\,\phi^2 \left(R - \tfrac{3}{2}\phi^2\right) \qquad \text{cosmological constant:} \quad \Lambda = \tfrac{3}{4}\phi^2$$

Fermionic part of the action

Choosing the anti-Grassmann 3-form to be $\underset{\equiv}{\dot{B}} = \underset{\sim}{e} \dot{\Psi} \vec{e}$ gives the massive Dirac action in curved spacetime:

$$S_f = \int \langle \underset{\equiv}{\dot{B}} \underset{=}{F}_{\cdot} \rangle = \int \langle \underset{\equiv}{\dot{B}} \underset{-}{D} \underset{\cdot}{\Psi} \rangle$$

$$= \int \langle \underset{\sim}{e} \dot{\Psi} \vec{e} \big(\underset{-}{d} \underset{\cdot}{\Psi} + \underset{-}{H}_1 \underset{\cdot}{\Psi} + \underset{\cdot}{\Psi} \underset{-}{H}_2 \big) \rangle$$

$$= \int \langle \underset{\sim}{e} \dot{\Psi} \vec{e} \big((\underset{-}{d} + \tfrac{1}{2} \underset{-}{\omega} + \tfrac{1}{4} \underset{-}{e} \phi + \underset{-}{W} + \underset{-}{B}_1) \underset{\cdot}{\Psi} + \underset{\cdot}{\Psi} (\underset{-}{w} + \underset{-}{B}_2 + \underset{-}{x} \Phi + \underset{-}{g}) \big) \rangle$$

$$= \int \underset{\sim}{d^4 x} \, |e| \, \langle \dot{\Psi} \gamma^\mu (e_\mu)^i \big(\partial_i \underset{\cdot}{\Psi} + \tfrac{1}{4} \omega_i{}^{\mu\nu} \gamma_{\mu\nu} \underset{\cdot}{\Psi} + W_i \underset{\cdot}{\Psi} + B_{1i} \underset{\cdot}{\Psi}$$

$$+ \underset{\cdot}{\Psi} w_i + \underset{\cdot}{\Psi} B_{2i} + \underset{\cdot}{\Psi} x_i \Phi + \underset{\cdot}{\Psi} g_i \big) + \dot{\Psi} \, \phi \, \underset{\cdot}{\Psi} \rangle$$

The $\dot{\Psi} \, \phi \, \underset{\cdot}{\Psi}$ is the standard Higgs mass term.

The $\dot{\Psi} \gamma^\mu \underset{\cdot}{\Psi} x_\mu \Phi$ term... I don't understand yet — promising for CKM.

E.S.T.o.E. summary

Everything in an $E8$ principal bundle connection,

$$\underset{\frown}{A} = \underset{\frown}{e8}$$

Periodic table of interactions (Feynman vertices) from curvature,

$$\underset{\Rightarrow}{F} = \underset{\rightarrow}{d}\underset{\frown}{A} + \frac{1}{2}\left[\underset{\frown}{A}, \underset{\frown}{A}\right]$$

described by the $E8$ root polytope. Three generations through triality,

$$T\,e = \mu \qquad T\,\mu = \tau \qquad T\,\tau = e$$

Pati-Salam $SU(2)_L \times SU(2)_R \times SU(4)$GUT and MM gravity together,

$$S = \int \left\langle \dot{\underset{=}{B}} \underset{\Rightarrow}{F} + \frac{\pi G}{4} \underset{=}{B}_G \underset{=}{B}_G \gamma + \underset{=}{B}' * \underset{=}{B}' \right\rangle$$

No (or few) free parameters — masses from Higgs VEV's,

$$g_1 = \sqrt{\frac{3}{5}} \qquad g_2 = 1 \qquad g_3 = 1 \qquad \Lambda = \frac{3}{4}\phi^2 \qquad \phi_0, \phi_1, \Phi \ldots$$

Everything is pure geometry, and it's very beautiful!

E.S.T.o.E. discussion

- Quantization
  - Coupling constants run. Large Λ compatible with UV fixed point.
  - Just a connection — amenable to LQG, spin foams, etc.
    - McKay correspondence $\to$ finite groups, braids?
- Understand triality-generation relationship better
  - Possible collapse or mixing to graviweak $SL(2,\mathbb{C})$.
  - The role of $\underline{x}\Phi$ and symmetry breaking.
  - Getting the CKMPMNS matrix would be nice.
- Why is the action what it is?
  - Pulling $\underline{e}$ out and putting it into $\underline{\underline{F}} * \underline{\underline{F}}$ and $\dot{\underline{\underline{\underline{B}}}}$ seems weird.
    - Is the four dimensional base manifold emergent?
- Natural explanation for QM as a bonus?

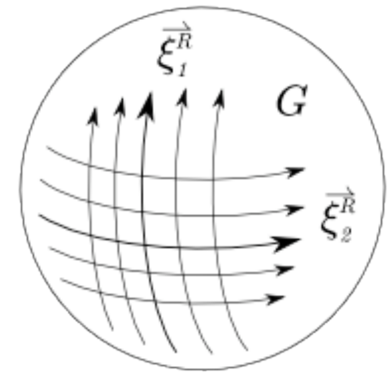
What this theory will mean, if it all works:

- Naturally combines standard model with gravity — it's a T.o.E.
- Our universe is very pretty!

Geometry of Yang-Mills theory

Start with a **Lie group manifold** (*torsor*), G, coordinatized by y^p.
Two sets of invariant vector fields (*symmetries*, ***Killing vector*** *fields*):

$$\vec{\xi}^{L}_{A}(y)\,\underline{d}g = T_A\, g(y) \qquad \vec{\xi}^{R}_{A}(y)\,\underline{d}g = g(y)\, T_A$$



Lie derivative: $[\vec{\xi}^{R}_{A}, \vec{\xi}^{R}_{B}] = C_{AB}{}^{C}\vec{\xi}^{R}_{C}$

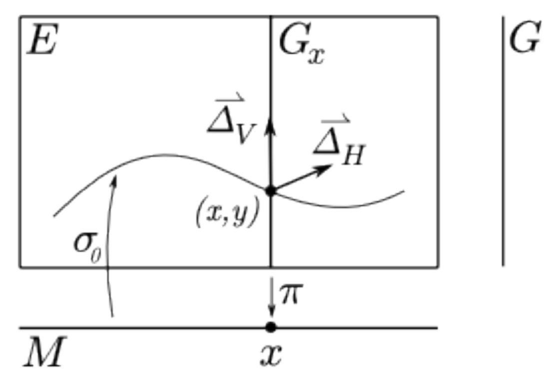
Lie bracket: $[T_A, T_B] = C_{AB}{}^{C} T_C$

Killing form (***Minkowski metric***): $g_{AB} = C_{AC}{}^{D} C_{BD}{}^{C}$

Maurer-Cartan form (***frame***): $\underline{\mathcal{I}} = \underline{d}y^p (\xi^{R}_{p})^{A} T_A$

Entire space of a **principal bundle**: $E \sim M \times G$
Ehresmann principal bundle connection over patches of E:

$$\underline{\vec{\mathcal{E}}}(x,y) = \underline{d}x^i A_i{}^{B}(x)\, \vec{\xi}^{L}_{B}(y) + \underline{d}y^p \vec{\partial}_p$$



Gauge field **connection** over M:

$$\underline{A}(x) = \sigma_0^* \underline{\vec{\mathcal{E}}}\, \underline{\mathcal{I}} = \underline{d}x^i A_i{}^{B}(x)\, T_B$$

BRST gauge fixing

$\underset{\cdot}{\delta}\underset{\sim}{L} = 0$ under **gauge transformation**: $\quad \underset{\cdot}{\delta}\underline{A} = -\underline{\nabla}\underset{\cdot}{C} = -\underline{d}\underset{\cdot}{C} - \left[\underline{A}, \underset{\cdot}{C}\right]$

Account for gauge part of $\underline{A}$ by introducing **Grassmann** valued **ghosts**,

$\underset{\cdot}{C} \in \mathrm{L\underset{\cdot}{i}e}(G)_g$, **anti-ghosts**, $\underset{\sim}{\dot{B}}$, **partners**, $\underset{\sim}{\lambda}$, and **BRST transformation**:

$$
\begin{aligned}
\underset{\cdot}{\delta}\underline{A} &= -\underline{\nabla}\underset{\cdot}{C} & \qquad \underset{\cdot}{\delta}\underset{\cdot}{C} &= -\tfrac{1}{2}\left[\underset{\cdot}{C}, \underset{\cdot}{C}\right] \\
\underset{\cdot}{\delta}\underline{\underline{B}} &= \left[\underline{\underline{B}}, \underset{\cdot}{C}\right] & \qquad \underset{\cdot}{\delta}\underset{\sim}{\dot{B}} &= \underset{\sim}{\lambda} \\
\underset{\cdot}{\delta}\underset{\sim}{\lambda} &= 0
\end{aligned}
$$

This satisfies $\underset{\cdot}{\delta}\underset{\sim}{L} = 0$ and $\underset{\cdot}{\delta}\underset{\cdot}{\delta} = 0$.

Choose a **BRST potential**, $\underset{\sim}{\dot{\Psi}} = \left\langle \underset{\sim}{\dot{B}}\,\underline{A} \right\rangle$, to get new Lagrangian:

$$\underset{\sim}{L}' = \underset{\sim}{L} + \underset{\cdot}{\delta}\underset{\sim}{\dot{\Psi}} = \underset{\sim}{L} + \left\langle \underset{\sim}{\lambda}\,\underline{A}_g \right\rangle + \left\langle \underset{\sim}{\dot{B}}\,\underline{\nabla}\,\underset{\cdot}{C} \right\rangle$$

BRST partners act as Lagrange multipliers; **effective Lagrangian**:

$$\underset{\sim}{L}^{\mathrm{eff}} = \underset{\sim}{L}[\underline{\underline{B}}', \underline{A}'] + \left\langle \underset{\sim}{\dot{B}}\,\underline{\nabla}'\underset{\cdot}{C} \right\rangle$$