BRST Reduction and Quantization of Constrained Hamiltonian Systems

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Abstract

We discuss the BRST reduction and quantization of classical gauge theories in the case where the gauge transformations are given by the action of a Lie group on a phase space. After discussing reduction of gauge systems in general, including the procedure of Marsden-Weinstein reduction using the momentum mapping (which is not well known to physicists), we develop the classical BRST formalism in a mathematically rigorous manner, using Marsden-Weinstein reduction. It turns out that the algebra of "functions" on the extended phase space (including ghosts and ghost momenta) has a beautiful and mathematically rigorous formulation in terms of the exterior algebras of the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. This leads in a natural way to the existence of a Poisson structure on the ghosts and ghost momenta, which in the physics literature usually is postulated.

BRST quantization in the case of Lie group gauge transformations is then treated by specifying exactly the quantum ghost algebra and exploring the quantum BRST condition $\hat{\Omega}\,|\psi\rangle = 0$. It turns out that BRST quantization runs into problems similar to those encountered in for instance Dirac's quantization scheme. This in our eyes defeats the purpose of BRST quantization.

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Chapter 1

Introduction

Constrained Hamiltonian systems are systems of which the dynamics are required to satisfy a set of conditions called constraints. It is a classical result by Dirac that all gauge systems are constrained Hamiltonian systems. When quantizing constrained systems, several problems arise. For example, because of the existence of unphysical degrees of freedom, the usual inner product becomes infinite.

Over time a number of ways of dealing with constraints and their quantization have been proposed. In general, there are two approaches: one can first try to eliminate all gauge freedom in the classical theory (which is called reduction), and then quantize the remaining system. Or one can try to quantize the complete system including the unphysical degrees of freedom, and impose some sort of "quantum constraint" in order to eliminate the gauge freedom in the quantum theory. This is done in the approach taken by Dirac.

Another example of the second approach is the BRST[1] method of quantizing and reduction. The existence of the BRST symmetry was first discovered [3, 13] within the context of the Faddeev-Popov path integral for Yang-Mills theories. Later on it was recognized that this symmetry exists for any gauge invariant system, and it was realized that it could be used as a method for reducing classical gauge theories.

In the BRST method, one first enlarges the classical phase space by introducing even more unphysical degrees of freedom, called the ghosts, and their conjugates, the ghost momenta. The reduced system can then be found by constructing the cohomology of a nilpotent operator called the (classical) BRST operator. When quantizing the resulting system, one then has to impose a condition on the quantum state space, the so called BRST condition.

The main purpose of this paper is to give a the BRST procedure a sound

[1]Named after its discoverers Becchi, Rouet and Stora [3] and, independently of the former three, Tyutin [13].

mathematical background. In the physics literature it is often not clear what is the precise origin of the ghosts and their momenta, and important properties of the extended phase space and its Poisson structure are often given in an ad hoc fashion. However, it turns out to be possible to give the concepts emerging in the BRST procedure a precise mathematical meaning.

After treating a number of ways of dealing classically with constrained systems, we give a precise derivation of the BRST reduction process. Then we turn to quantizing these systems, with an emphasis on the investigation of the quantum BRST condition.

Our contribution is giving a pedagogical treatment of the somewhat impenetrable mathematical literature, in such a fashion that mathematical rigour has not been sacrificed, yet the conceptual ideas have come to the foreground. We hope that this discussion will open the eyes of physicists to the beauty of a mathematically oriented treatment of the BRST formalism. (This work might be contrasted with the treatment given in [8], which is mathematically incomprehensible, and as a result is also conceptually hard to grasp.)

Part I

Reduction of Constrained Systems

Chapter 2

Constrained Systems

2.1 Introduction

We will first set up the theory of constrained Hamiltonian systems in the way which is the most familiar to physicists, using coordinates q^i for positions and p_i for momenta. Our treatment is mainly based on [8, 12]. In the next chapter we will give a geometrical reformulation of this subject.

2.2 The Lagrange and Hamilton formalisms

The classical motions of a physical system are those that make the action given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt$$

stationary under variations $\delta q^n(t)$ of the Lagrangian variables q^i, $i = 1, \ldots, n$, which vanish at the endpoints t_1 and t_2. The equations of motions are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i},$$

for $i = 1, \ldots, n$. This is the Lagrange formalism.

Another way to describe the system is in the Hamilton formalism. In moving from the Lagrange to the Hamilton formalism the first step is to define momenta p_i conjugate to $\dot{q}^i$ by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{2.1}$$

and the Hamiltonian H by

$$H\left(q^i, p_i\right) = p_i \dot{q}^i\left(q^j, p_j\right) - L\left(q^i, \dot{q}^i\left(q^j, p_j\right)\right).$$

The equations of motion in the Hamilton formalism are then given by

$$\begin{aligned}\frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i};\\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i}.\end{aligned} \tag{2.2}$$

In general, the time development of any phase space function F is given by its Poisson bracket with the Hamiltonian,

$$\dot{F}\left(q^i, p_i\right) = \{F, H\},$$

the Poisson bracket being defined as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}. \tag{2.3}$$

2.3 Primary and Secondary Constraints

When the velocities as functions of the coordinates and momenta are non-invertible, the momenta are not all independent, and there exist relations

$$\phi_m(q, p) = 0, \tag{2.4}$$

for $m = 1, \ldots M$, that follow from the definition of the momenta. These relations are called *primary constraints.* We will assume (2.4) defines a submanifold embedded in phase space, the *primary constraint surface.*

The Hamiltonian H, defined by

$$H = \dot{q}^n p_n - L, \tag{2.5}$$

is not uniquely determined as a function of the p's and q's. From (2.5) we find

$$\delta H = \dot{q}^n \delta p_n - \frac{\partial L}{\partial q^n}\delta q^n. \tag{2.6}$$

It can be shown [8] that if $\lambda_n \delta q^n + \mu^n \delta p_n = 0$ then

$$\begin{aligned}\lambda_n &= u^m \frac{\partial \phi_m}{\partial q^n};\\ \mu^n &= u^m \frac{\partial \phi_m}{\partial p_n}.\end{aligned}$$

Applying this result to (2.6) we find for the equations of motion:

$$\begin{aligned}&\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n};\\ &\dot{p}_n = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n};\\ &\phi_m(q, p) = 0.\end{aligned}$$

These equations can also be derived by demanding

$$\delta \int_{t_1}^{t_2} \left(\dot{q}^n p_n - H - u^m \phi_m\right) = 0,$$

for arbitrary variations $\delta q^n, \delta p_n, \delta u^m$ with the restriction $\delta q^n(t_1) = \delta q(t_2) = 0$. So we see that the u^m's enter the action as Lagrange multipliers for the constraints. It is clear that the theory is invariant under $H \to H + c^m \phi_m$ since this only leads to a redefinition of the u^m's.

The time development of an arbitrary function of the p's and q's $F(q,p)$ is now given by

$$\dot{F} = \{F, H\} + u^m \{F, \phi_m\}. \tag{2.7}$$

A requirement for the primary constraints is that they should be preserved in time. This leads to

$$\{\phi_m, H\} + u^{m'} \{\phi_m, \phi_{m'}\} = 0. \tag{2.8}$$

Equation (2.8) can either reduce to a relation independent of the u's or it may impose a restriction on the u's. In the former case, if this relation is also independent of the primary constraints it is called a *secondary constraint*. Secondary constraints are thus a consequence of the equations of motion, while primary constraints are only a consequence of the non-invertibility of the velocities as functions of the p's and q's.

If there is a secondary constraint, say $\chi(q,p)$, we must impose a new consistency condition

$$\{\chi, H\} + u^m \{\chi, \phi_m\} = 0.$$

This can give either new secondary constraints or merely restricts the choice of u's, etc. Eventually we will end up with a set of secondary constraints

$$\phi_k = 0,$$

for $k = M+1, \ldots, M+K$, where K is the total number of secondary constraints. The reason for this notation is that the distinction between primary and secondary constraints is not an important one, and it will be useful to denote both primary and secondary constraints by

$$\phi_j = 0, \qquad j = 1, \ldots, M+K = J. \tag{2.9}$$

The equations ϕ_j are called dependent if there exist one or more functions C^i_j such that

$$\phi_j = C^i{}_j \phi_i.$$

If this is the case the constraints, as well as the system as a whole, are called “reducible”. If the ϕ_j are independent, they are called “irreducible”.

2.4 Weak and Strong Equations

It is convenient to introduce the *weak equality symbol* $\approx$ to indicate an equality which is in general only true on the constraint surface $\phi_j = 0$ (or "on shell"). Thus, two functions F and G that coincide on the constraint surface are said to be weakly equal, $F \approx G$. If a relation holds throughout phase space, the symbol $=$ is used and the relation is called *strong*.

2.5 The Total Hamiltonian

Once we have found a complete set of constraints (2.9), it is useful to look at the restrictions on the u's. These are

$$\{\phi_j, H\} + u^m \{\phi_j, \phi_m\} \approx 0. \tag{2.10}$$

A general solution is

$$u^m = U^m + V^m,$$

where U^m is a particular solution of (2.10) and V^m is the most general solution of the homogeneous system

$$V^m \{\phi_j, \phi_m\} \approx 0,$$

which is a linear combination of the independent solutions ${V_a}^m$, $a = 1, \ldots A$. Thus the most general solution of (2.10) is given by

$$u^m \approx U^m + v^a (t) {V_a}^m,$$

where the coefficients $v^a (t)$ are totally arbitrary.

We can now rewrite (2.7) in the form

$$\dot{F} \approx \{F, H' + v^a \phi_a\},$$

where we have defined

$$\begin{aligned} H' &= H + u^m \phi_m; \\ \phi_a &= {V_a}^m \phi_m. \end{aligned} \tag{2.11}$$

The function $H_T = H' + v^a \phi_a$ is called the *total Hamiltonian.* In terms of the total Hamiltonian the equations of motion become

$$\dot{F} \approx \{F, H_T\}.$$

2.6 First-Class and Second-Class Constraints

An important classification of constraints is the distinction between *first-class* and *second-class* constraints. A constraint ϕ_i is said to be first-class if its Poisson bracket with every constraint vanishes weakly:

$$\{\phi_i, \phi_j,\} \approx 0, \qquad j = 1, \ldots J,$$

or equivalently

$$\{\phi_i, \phi_j\} = C_{ij}{}^k \phi_k,$$

where the $C_{ij}{}^k$ are certain functions on phase space. Later on we will assume them to be constants, in which case one is in the "group case". From the previous section it is clear the ϕ_a appearing in (2.11) are first-class. Moreover, they form a complete set of first class primary constraints. Any constraint which is not first-class is second-class.

From now on will confine ourselves to the case were there exist only first-class constraints, and we will denote them by G_a, which is the usual symbol found in the literature on this subject.

The notion of first-class constraints can be extended to general functions on phase space: a function F is first-class if

$$\{F, G_a\} \approx 0, \qquad a = 1, \ldots A.$$

2.7 Gauge Transformations

Since the functions v^a appearing in the total Hamiltonian are arbitrary, not all the canonical variables are observable. Since the v^a are functions of time, the value of the canonical variables at a time t_2 will depend on the choice of the v^a in the interval $t_1 \leq t \leq t_2$. In particular, if $t_2 = t_1 + \delta t$, the change of a dynamical variable F due to two different choices v^a and v'^a takes the form

$$\delta F = \delta v^a \{F, G_a\}, \tag{2.12}$$

with $\delta v^a = (v^a - v'^a)\, \delta t$. This transformation does not alter the physical state at t_2. *The first-class primary constraints generate gauge transformations.*

Furthermore, the following two results hold:

1. The Poisson bracket $\{G_a, G_b\}$ of any two first-class primary constraints generates a gauge transformation.

2. The Poisson bracket $\{G_a, H'\}$ of any first-class primary constraint G_a with the first-class Hamiltonian H' generates a gauge transformation.

This may indicate that some secondary first-class constraints may also act as gauge generators, since $\{G_a, G_b\}$ and $\{G_a, H'\}$ will be linear combinations of first-class constraints. There is no reason to expect these to be primary, however. Though it is not always true that every first-class secondary constraint is a gauge generator (the so called "Dirac conjecture") we only look at theories in which *all first-class constraints generate gauge transformations.*

2.8 Gauge invariant functions

Two phase space functions that coincide on the constraint surface $G_a = 0$, denoted by Σ, cannot be distinguished. So, relevant functions are not all the smooth phase space functions but functions which are smooth on Σ. The space $C^\infty(\Sigma)$ of smooth functions on Σ can be characterized as follows. The functions that vanish on Σ form an ideal in $C^\infty(P)$, the space of smooth functions on the whole phase space P. Given this ideal, denoted by $\mathcal{N}$, one can consider the quotient algebra containing the equivalence classes of phase space functions whose difference is an element of $\mathcal{N}$. This quotient is just $C^\infty(\Sigma)$.

A classical observable is, by definition, a function on the constraint surface that is gauge invariant. This can be expressed as

$$\{F, G_a\} \approx 0, \tag{2.13}$$

i.e. observables are first-class.

2.9 The Extended Hamiltonian

The motion generated by the total Hamiltonian H_T contains only the first-class primary constraints. To account for all the gauge freedom one has to add all first-class constraints. This leads to the *extended Hamiltonian* H_E:

$$H_E = H' + \lambda^a G_a.$$

For the time evolution of gauge invariant functions, it doesn't matter which Hamiltonian one uses, because of (2.13). For any other variable or function, we must use H_E to account for all the gauge freedom.

The equations of motion for the extended Hamilton formalism can also be derived by an action principle, namely

$$\begin{aligned} \delta S_E &= 0, \\ S_E &= \int \left(\dot{q}^n p_n - H - \lambda^a G_a\right), \end{aligned}$$

which reduces to

$$\begin{aligned}\dot{F} &\approx \{F, H_E\}\,;\\ G_a &\approx 0,\end{aligned}$$

as it should.

Chapter 3

Geometric Reduction

3.1 Introduction

We will now treat the subject of constrained Hamiltonian systems and reduction using a geometrical formalism, based on [1, 5, 6]. There are several reasons for doing this. First of all, it is not always possible to use global coordinates on a configuration space. Even if this can be done, the phase space doesn't have to be a cotangent bundle of this configuration space. And lastly, the geometrical approach provides a more intuitive and conceptually clear way of dealing with Hamiltonian dynamics (through the flow of vector fields).

3.2 Tangent and Cotangent Bundle as Phase Spaces

In general, we can take the configuration space M, the space of which the q^i's are the coordinates, to be an n-dimensional differentiable manifold. For describing dynamics, i.e. the time evolution of the system, this space is however unsuitable. One needs at least first-order differential equations, which geometrically are vector fields. The obvious candidate for a space on which we can define vector fields is the tangent bundle TM of M, which we can identify with the velocity phase space, with local coordinates q^i and $\dot{q}^i$.[1] The Lagrangian L is a function on TM,

$$L : TM \rightarrow \mathbb{R}.$$

Hamiltonian mechanics takes place on the cotangent bundle T^*M which is the momentum phase-space with coordinates (q^i, p_i) and the Hamiltonian H is a function on T^*M,

$$H : T^*M \rightarrow \mathbb{R}.$$

[1] Here $\dot{q}^i$ is nothing but a convenient notation; the $\dot{q}$'s are completely independent from the q's, i.e. the dot here does not denote the time derivative.

The transformation from TM to T^*M (i.e.. from the Lagrange to the Hamilton formalism) is given by the *fiber derivative* FL:

$$\begin{aligned} FL &: TM \to T^*M; \\ FL\left(q^i, \dot{q}^i\right) &= \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}_i}\right). \end{aligned}$$

FL maps the fibers of TM into fibers of T^*M. In general FL will only be bijective in the unconstrained case.

3.3 Hamilton Equations

If we introduce local coordinates $\left(q^i, p_i\right)$ on the phase space $S = T^*M$, the general form of a vector field X on S is given by

$$X = \alpha^i \frac{\partial}{\partial q^i} + \beta_i \frac{\partial}{\partial p_i},$$

where α^i and β_i are functions on S. An integral curve γ of X is a curve

$$\begin{aligned} \gamma &: \mathbb{R} \to S; \\ t &\to \gamma(t) = \left(q^i(t), p_i(t)\right), \end{aligned}$$

which satisfies

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)).$$

In local coordinates this reduces to

$$\begin{aligned} \dot{q}^i &= \alpha^i; \\ \dot{p}_i &= \beta_i. \end{aligned}$$

Comparing this with the original Hamilton equations of motion (2.2) we can conclude that the physical trajectories are integral curves of the so called *Hamiltonian vector field* X_H,

$$X_H \equiv \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

This leads us to the *symplectic form.*

Definition 1 *A symplectic manifold is a pair (W, ω), where W is a manifold and ω a closed, non-degenerate antisymmetric 2-form called the* symplectic form. *This means in particular*

1. $d\omega = 0$;

2. $\omega(X,Y)=0\ ,\ \forall\ Y\in TW \Longleftrightarrow X=0;$

3. $\omega(X,Y)=-\omega(Y,X)\,.$

In coordinates (q^i,p_i) on S we introduce the canonical symplectic form

$$\omega = dq^i \wedge dp_i.$$

It is then easily shown that

$$\begin{aligned} i_{X_H} dq^i &= \frac{\partial H}{\partial p_i}; \\ i_{X_H} dp_i &= -\frac{\partial H}{\partial q^i}, \end{aligned}$$

and so

$$\begin{aligned} i_{X_H}\omega &= i_{X_H}\left(dq^i \wedge dp_i\right) = \left(i_{X_H} dq^i\right) dp_i - \left(i_{X_H} dp_i\right) dq^i \\ &= \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i = dH, \end{aligned}$$

which can be rewritten as

$$i_{X_H}\omega\,(.,Y) = \omega\,(X_H,Y) = dH\,(Y) = \mathcal{L}_Y H, \tag{3.1}$$

for any vector field Y. This replaces the original Hamilton equations of motion. Now, ω is symplectic on $S=T^*M$, so given a Hamiltonian (3.1) has a unique solution in S, namely

$$X_H = (dH)^\sharp\,.$$

3.4 Poisson Brackets

When we define the Poisson bracket for two functions $F,G\in C^\infty(S)$ by

$$\{F,G\} = \omega\,(X_F,X_G)\,, \tag{3.2}$$

it is easy to show that within a chart (q^i,p_i), where $X_F = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i}$ and similarly for G, (3.2) becomes

$$\{F,G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i},$$

which is the standard expression (2.3) for the Poisson bracket.

3.5 Geometric Description of Constraints

We will now reformulate the concept of constrained Hamiltonian systems, introduced in the previous chapter, in a geometrical setting [5, 6].

In the previous sections we have assumed the fiber derivative FL to be bijective (i.e., the unconstrained case), in which case $FL\,(TM) = T^*M$ is automatically a symplectic manifold if endowed with the canonical symplectic form $\omega = dq^i \wedge dp_i$. In the general case where constraints might occur FL will fail to be bijective, and $FL\,(TM)$ will be a submanifold $P \subset T^*M$, which doesn't have to be symplectic.

In any case, we can still define a closed two-form ω which in turn defines a mapping from TP to T^*P given by

$$\Omega_\omega : TP \to T^*P : X \to \Omega_\omega\,(X) = i_X\omega. \tag{3.3}$$

For finite-dimensional manifolds there are then two possibilities: either ω is symplectic, in which case Ω_ω is an isomorphism and (P, ω) is a symplectic manifold, or ω is degenerate, Ω_ω is neither injective nor surjective and (P, ω) is called a *presymplectic manifold*. Equation (3.3) allows us to write (3.1) as

$$\Omega_\omega\,(X_H) = dH. \tag{3.4}$$

Thus for a given Hamiltonian H (3.4) doesn't always have a solution if (P, ω) is presymplectic. There may however be a submanifold of P on which (3.4) does have a solution. There exists a systematic algorithm to find this submanifold, a precise mathematical formulation of which will be given in the next section. In general the process is as follows.

We start with a presymplectic manifold (P, ω) and a Hamiltonian H. Since Ω_ω is not surjective, the image $\Omega_\omega\,(T_uP)$ of the fiber T_uP is in general a subspace of T_u^*P, and the 1-form dH is in general not in the range of Ω_ω. We can now restrict ourselves to a subset M_1 of P where dH is in the range of Ω_ω. We'll assume M_1 is a manifold and call it a constraint manifold. This gives us a map

$$j : M_1 \to P,$$

and the embedding of the tangent space of M_1 :

$$j_* : TM_1 \to TP.$$

The image of this mapping is denoted by $\underline{TM}_1$:

$$\underline{TM}_1 = \{X \in TP\lceil_M \quad | \quad X = j_*Y \quad , \quad Y \in TM_1\}.$$

On M_1 we have a solution X_H of the equations of motion, but in general X_H is an element of $TP\lceil_M$, not of $\underline{TM}_1$. Elements of $\underline{TM}_1$ are called tangent to M_1. If X_H is not tangent to M_1 an integral curve of it will leave M_1, and the system will evolve to a point where (3.3) doesn't have a solution. Since this is unacceptable, we restrict ourselves to a submanifold M_2 of M_1 where X_H is tangent to M_1. M_2 is the next constraint manifold. On M_2 we have a solution X_H which is tangent to M_1 but not necessarily to M_2 itself. So we again restrict ourselves tot the next constraint manifold $M_3 \subset M_2$ where X_H is tangent to M_2. On M_3 we can have the same problem, and we are forced to make a chain of submanifolds which hopefully ends at some submanifold M_k on which X_H is tangent to M_k on all points of M_k.

3.6 The Constraint Algorithm

We will now turn to the mathematical formulation.

Definition 2 *The* symplectic complement $TM^{\perp}$ *of* $TM \subset TP$ *with respect to* ω *is defined by*

$$TM^{\perp} = \{Y \in TP\lceil_M \quad | \quad \langle Y \,|\Omega_\omega(X)\rangle \equiv \omega(X,Y) = 0 \quad , \quad \forall\, X \in \underline{TM}\}$$

We'll also need the following theorem [6].

Theorem 3 *Let* $\alpha \in T^*P$, *then*

$$\alpha \in \Omega_\omega(\underline{TM}) \Longleftrightarrow \left\langle TM^{\perp} \right| \alpha\rangle = 0. \tag{3.5}$$

With this we can construct the algorithm. For a given Hamiltonian H we want to investigate wether or not the equation

$$\Omega_\omega(X) = i_X\omega = dH \tag{3.6}$$

has solutions. A necessary condition is that dH must be an element of $\Omega_\omega(TP)$, thus according to (3.5) we must have

$$\left\langle TP^{\perp} \right| dH\rangle = 0. \tag{3.7}$$

We now take M_1 to be the submanifold of P on which (3.7) holds:

$$M_1 = \left\{p \in P \quad | \quad \left\langle TP^{\perp} \right| dH\rangle(p) = 0\right\}.$$

On M_1 we have a solution X_H of (3.6), but it need not be tangent to M_1. For a solution to be tangent to M_1 we must have

$$dH \in \Omega_\omega(\underline{TM_1}),$$

or, using the theorem,

$$\left\langle TM_1^{\perp} \middle| \, dH \right\rangle = 0. \tag{3.8}$$

So we introduce a submanifold M_2 of M_1,

$$M_2 = \left\{ m \in M_1 \quad | \quad \left\langle TM_1^{\perp} \middle| \, dH \right\rangle (m) = 0 \right\},$$

on which (3.8) holds. This way we find a chain of submanifolds

$$M_l = \left\{ m \in M_{l-1} \quad | \quad \left\langle TM_{l-1}^{\perp} \middle| \, dH \right\rangle (m) = 0 \right\}.$$

If finally we end up with a manifold M_k with the property

$$\left\langle TM_k^{\perp} \middle| \, dH \right\rangle = 0$$

for all its points the chain stops. M_k is called the *final constraint manifold.*

3.7 Gauge Freedom and the Reduced Phase Space

Given the final constraint manifold $M_k \subset P$ on which

$$i_{X_H}\omega\lceil_{M_K} = dH\lceil_{M_k}, \quad X_H \in TP, \; dH \in T^*P, \tag{3.9}$$

has a solution X_H tangent to M_k, we can pull back (3.9) to M_k using

$$\begin{aligned} j &: M_k \to P; \\ \omega_k &= j^*\omega; \\ H_k &= j^*H_k, \end{aligned}$$

and obtain

$$i_{X_{H_k}}\omega_k = dH_k, \quad X_{H_k} \in TM_k, \; dH_k \in T^*M_k. \tag{3.10}$$

Every solution X_H of (3.9) tangent to M_k can be written as

$$X_H = j_*\overline{X},$$

where $\overline{X}$ is a solution of (3.10). The converse is in general not true; if $\overline{X}$ is a solution of (3.10) then $j_*\overline{X}$ is not always a solution of (3.9). So (3.9) and (3.10) are different.

To discuss the difference between the two equations, we look at the non-uniqueness of the solution of both equations. We first consider (3.9). To any solution X_H we may add an element Z of $\ker \Omega_\omega \cap \underline{TM_k}$ to obtain another solution tangent to M_k.

This arbitrariness is called gauge freedom, and X_H and $X_H + Z$ are gauge equivalent vectors. Gauge freedom thus means that an initial condition, a point $m(0)$ of M_k, will evolve to different points $m_{X_H}(t)$, depending on the choice for the solution of (3.9).

We now make the assumption that *this freedom has no physical significance*, so X_H and $X_H + Z$ describe the same physics. This leads to the following.

Definition 4 *Points on M_k are* physically equivalent *or* gauge equivalent *if they can be reached from the same initial condition by integral curves of solutions of (3.9) in the same amount of "time", i.e. the variable t parametrizing the integral curves.*

Definition 5 *A gauge vector field is a vector field the integral curves of which consist of physically equivalent points.*

The collection

$$G_1 = \ker \Omega_\omega \cap \underline{TM_k}$$

considered above consists of gauge vector fields. These are in general not the only possible gauge vectors. It can be shown that:

- If X is a solution of (3.9) tangent to M_k and Z a gauge vector field then their commutator $[X, Z]$ is also a gauge vector field.
- If Z_1, Z_2 are gauge vector fields then $[Z_1, Z_2]$ is a gauge vector field.

The gauge vector fields obtained in this way can be elements of G_1, though they don't have to be. Starting from G_1 we can construct a series of sets $G_2, G_3, \ldots, G_n$ of gauge vector fields by defining

$$G_n = G_{n-1} + [X, G_{n-1}] + [G_{n-1}, G_{n-1}], \quad n = 2, 3, \ldots,$$

with X a solution of (3.9) tangent to M_k. Obviously $G_{n-1} \subset G_n$, so by this construction we enlarge in every step the collection of gauge vector fields until for some n_f we have

$$G_{n_f - 1} = G_{n_f}.$$

The process stops, and we have $G_n = G_{n_f}$ for $n > n_f$. G_{n_f} is then the complete set of gauge vector fields for (3.9).

Now consider the pullback equation (3.10). We can add to a solution X_{H_k} any element $\overline{Z}$ of $\overline{G}_1 = \ker \Omega_{\omega_k}$. We could apply to $\overline{G}_1$ the same procedure as above, but $\overline{G}_1$ turns out to be stable under this process. Hence the set of gauge vectors of (3.10) is $\overline{G}_1$. Since G_1, the initial gauge freedom of (3.9), belongs to $j_*\overline{G}_1 = TM_k^\perp \cap \underline{TM}_k$ and $\overline{G}_1$ is stable, all G_n and in particular the complete set of gauge vector fields of (3.9) G_{n_f} will belong to $TM_k^\perp \cap \underline{TM}_k$. So the gauge freedom of (3.9) is contained in that of (3.10). We have found that every solution of (3.9) corresponds to a solution of (3.10), and equally so for gauge freedom, but not necessarily vice versa. In most cases however,

$$G_{n_f} = TM_k^\perp \cap \underline{TM}_k, \tag{3.11}$$

and both equations will have corresponding solutions. From now on we will assume (3.11) to be true.

We now introduce the following terminology: M_k is called

1. *isotropic* if $\underline{TM}_k \subset TM_k^\perp$;
2. *coisotropic* or *first class* if $TM_k^\perp \subset \underline{TM}_k$;
3. *second class* or *weakly symplectic* if $TM_k \cap TM_k^\perp = \{0\}$;
4. *mixed* in all other cases.

This leads to the following statements regarding gauge freedom:

1. An isotropic constraint manifold has as its gauge vectors *all* tangent vectors. This means that all points are gauge equivalent and that there is no dynamics on M_k. In terms of coordinates and functions, this means that the constraint functions form a basis for the space of functions on all of phase space.
2. A first class manifold has maximal gauge freedom: all potential gauge vectors (i.e. vectors in $TM_k^\perp$) are actual gauge vectors (i.e. belong to $\underline{TM}_k$). In the language of the previous chapter, this means that all constraints are first-class.
3. In a second class manifold there is *no* gauge freedom, all point are physically inequivalent. This is the same as saying that all constraints are second-class.

In this paper we will only look at first class constraint manifolds.

3.8 The Reduced Phase Space

To eliminate the gauge freedom on a first class manifold we want to identify gauge equivalent points of M_k. We have to find the set L_m of all points that are equivalent to a given point m of M_k. One constructs L_m by taking the union of all integral curves of vector fields Z going through m. One says that M_k is "foliated" by the "leaves" L_m of gauge equivalent points.

We can then form the quotient space $R = M_k/\sim$ of M_k by the following equivalence relation: two points m_1 and m_2 are equivalent if they lie on the same leaf L (i.e. if they are gauge equivalent). R is the space of leaves L_M of M_k.

There exists a canonical projection

$$\pi : M_k \to R,$$

which assigns to each point of M_k the leaf to which it belongs. We assume R is a smooth manifold and π is a smooth mapping. R is called the reduced phase space of the presymplectic manifold (P, ω) and Hamiltonian H.

A two-form ω_R on R is introduced as follows. Take X and Y to be two vectors tangent to a point r of R. This point corresponds to a leaf L in M_k. Take a point m on L and two tangent vectors $\tilde{X}$ and $\tilde{Y}$ in $T_m M_k$ that project on X and Y:

$$\begin{aligned} X &= \pi_* \tilde{X}; \\ Y &= \pi_* \tilde{Y}. \end{aligned}$$

Then ω_R is given by

$$\omega_R(X, Y) = \omega_k\left(\tilde{X}, \tilde{Y}\right),$$

where $\omega_k = j^*\omega$, the pullback of ω onto M_k. It can be shown that ω_R defined this way is symplectic.

Because of the way the final constraint manifold has been constructed we have

$$\left\langle TM_k^{\perp} \right| dH_k \rangle = 0.$$

This means H_k is constant along the leaves of the foliation, since $TM_k^{\perp}$ is the tangent space to the leaves. In other words, H_k is gauge invariant and defines a function H_R on R by

$$H_R(L_m) = H_k(m).$$

We now have constructed a symplectic form ω_R and a Hamiltonian H_R on the reduced phase space R. The equations of motion read

$$i_{X_{H_R}}(\omega_R) = dH_R,$$

with a unique solution X_{H_R}. R is the true phase space of the presymplectic system (P, ω, H).

3.9 Gauge Vectors and Constraints

We have now two descriptions of gauge freedom on constraint manifolds: the first in terms of the Poisson bracket with a constraint function (as given by (2.12)), the second in terms of gauge vectors. These two are of course related: the gauge vectors are the Hamiltonian vector fields for the functions ϕ_a.

Chapter 4

Lie Gauge Groups

4.1 Introduction

From now on, we will assume the gauge transformations to be actions of a Lie group G on phase space. Some essential elements from Lie group theory and Lie group actions on manifolds are treated in appendix A. This chapter closely follows [1], where we've added the perspective of constrained systems.

4.2 The Momentum Mapping

As we have seen in the previous chapters, we can think of gauge transformations as the flow of gauge vectors. In the case of a Lie group action, these gauge vectors are just the infinitesimal generators of the Lie group action. Choosing an element ξ of $\mathfrak{g}$, the Lie algebra of G, we get a in each point x in phase space a gauge vector

$$X^{\xi}(x) = \frac{d}{dt}\Phi(\exp t\xi, x)\bigg|_{t=0}.$$

Now, given a gauge vector field X^{ξ}, one can look for the constraint associated with it, i.e. the function whose Hamiltonian vector field is just X^{ξ}. Such a function, denoted by $\hat{J}(\xi) : P \to \mathbb{R}$, by definition obeys

$$d\hat{J}(\xi) = i_{X^{\xi}}\omega.$$

We can view $\hat{J}$ as a map from $\mathfrak{g}$ to $C^{\infty}(P)$:

$$\begin{aligned} \hat{J} : \mathfrak{g} &\to C^{\infty}(P); \\ \xi &\to \hat{J}(\xi). \end{aligned}$$

Now define a map $J : P \to \mathfrak{g}^*$ (the dual of the Lie algebra $\mathfrak{g}$) by

$$J(x) \cdot \xi = \hat{J}(\xi)(x).$$

So J is a map from P to the space of linear functionals on $\mathfrak{g}$. Under certain conditions, which will be given below, J is called a *momentum mapping* for the action of G. Since the l.h.s. of this equation is linear in ξ, so is the r.h.s. By choosing a basis ξ_i in $\mathfrak{g}$ we can write for every $\chi \in \mathfrak{g}$

$$\hat{J}(\chi) = \hat{J}\left(\chi^i \xi_i\right) = \chi^i \hat{J}(\xi_i) \equiv \chi^i \hat{J}_i.$$

The functions $\hat{J}_i$ thus form a basis for the constraints.

Now we give the formal definition of the momentum mapping.

Definition 6 *Let (P, ω) be a connected symplectic manifold and $\Phi : G \times P \to P$ a symplectic action of the Lie group G on P, that is, for each $g \in G$, the map $\Phi_g : P \to P$ is symplectic*[1]*. We say that a mapping from P onto $\mathfrak{g}^*$,*

$$J : P \to \mathfrak{g}^*$$

is a momentum mapping for the action provided that, for every $\xi \in \mathfrak{g}$,

$$d\hat{J}(\xi) = i_{X^\xi}\omega,$$

where $\hat{J}(\xi) : P \to \mathbb{R}$ is defined by $\hat{J}(\xi)(x) = J(x) \cdot \xi$ and X^ξ is the infinitesimal generator of the action corresponding to ξ. In other words, J is a momentum mapping provided that

$$X_{\hat{J}(\xi)} = X^\xi$$

for all $\xi \in \mathfrak{g}$. The collection (P, ω, Φ, J) is called a Hamiltonian G-space.

Definition 7 *A momentum mapping J is called Ad^*-*equivariant *if*

$$J(\Phi_g(x)) = Ad^*_{g^{-1}} J(x)$$

for every $g \in G$.

An Ad^*-equivariant momentum mapping obeys [1]

$$\left\{\hat{J}(\xi), \hat{J}(\zeta)\right\} = \hat{J}([\xi, \zeta]).$$

This is just the first-class condition, since

$$\begin{aligned} \left\{\hat{J}_i, \hat{J}_j\right\} &= \left\{\hat{J}(\xi_i), \hat{J}(\xi_j)\right\} \\ &= \hat{J}\left([\xi_i, \xi_j]\right) \\ &= C_{ij}{}^k \hat{J}(\xi_k) \\ &= C_{ij}{}^k \hat{J}_k. \end{aligned}$$

So when J is Ad^*-equivariant, all constraints are first-class.

[1] Let (P, ω) be a symplectic manifold. A map $j : P \to P$ is called *symplectic* if it preserves the symplectic form, i.e. $j^*\omega = \omega$.

4.3 Reduction using the momentum mapping

As we have seen, the functions $\hat{J}_i \equiv \hat{J}(\xi_i)$ form a basis for the constraints, so the constraint manifold Σ is defined by the relations

$$\hat{J}_i = 0, \quad i = 1, \ldots, \dim \mathfrak{g}.$$

Since

$$\hat{J}(\xi_i)(x) = 0 \Leftrightarrow J(x)(\xi_i) = 0, \quad \forall \xi_i \in \mathfrak{g},, \ \forall x \in \Sigma,$$

the constraint manifold is just the submanifold $P' \subset P$ on which the r.h.s. of the above equation holds. After factoring out the action of the group, we get a manifold

$$P_0 = J^{-1}(0)/G = \Sigma/G,$$

which, under certain conditions, is symplectic. This is the reduced phase space.[2]

We will now give the mathematical theorems and definitions.

Theorem 8 *Let (P, ω) be a symplectic manifold on which the Lie group G acts symplectically and let $J : P \to \mathfrak{g}^*$ be an Ad^*-equivariant momentum mapping for this action. Assume G acts freely and properly on $\Sigma = J^{-1}(0)$. Then $P_0 = \Sigma/G$ has a unique symplectic form ω_0 with the property*

$$\pi^* \omega_0 = i^* \omega,$$

where $\pi : \Sigma \to P_0$ is the canonical projection and $i : \Sigma \to P$ is the inclusion.

The assumption that G acts freely and properly is made to ensure that Σ/G is a manifold. So P_0 is the phase space of the physical degrees of freedom. For example in electrodynamics, $J^{-1}(0)$ would be the manifold on which Gauss' law would hold, while dividing out the group action amounts to identifying the gauge invariant field configurations (though in this case G would be infinite dimensional).

For the proof of this theorem we'll need the following lemma.

Lemma 9 *For $p \in \Sigma$ and $G \cdot p = \{\Phi(g, p) | \, g \in G\}$, $T_p(\Sigma)$ is the ω-orthogonal complement of $T_p(G \cdot p) = \{X^\xi(p) | \, \xi \in \mathfrak{g}\}$.*

[2](In general, the constraint manifold is given by $J^{-1}(\mu)$, where μ is a regular value of J. We will restrict ourselves to the case of Marsden-Weinstein reduction, which is the one described above. This actually isn't a restriction, since one can always adjust the system such that $\mu = 0$. For a discussion of these matters, see for example [1].)

Proof of the Lemma. If $\xi \in \mathfrak{g}$ and $Y \in T_pP$, we have

$$\omega\left(X^{\xi}(p), Y\right) = \left(d\hat{J}(\xi)\right)_p (Y) = T_pJ(Y) \cdot \xi,$$

since J is a momentum mapping. Since $T_p(\Sigma) = \ker T_pJ$, $Y \in T_p(\Sigma)$ iff $\omega\left(X^{\xi}(p), Y\right) = 0$ for all $\xi \in \mathfrak{g}$, that is, Y is in the ω-orthogonal complement of $T_p(G \cdot p)$. ■

Now we can prove the theorem.

Proof of Theorem 8. For $X \in T_p(\Sigma)$, let $[X] = T\pi(X)$ denote the corresponding equivalence class in $T_p(\Sigma)/T_p(G \cdot p)$. The assertion $\pi^*\omega_0 = i^*\omega$ is

$$\omega_0([X], [Y]) = \omega(X, Y), \quad X, Y \in T_p(\Sigma).$$

Since π and $T\pi$ are surjective, ω_0 is unique.

Now, from the lemma it follows that ω_0 is a well-defined two-form. It is smooth since $\pi^*\omega_0$ is smooth. It is closed, since $d\pi^*\omega_0 = di^*\omega = i^*d\omega = 0$, so $\pi^*(d\omega_0) = 0$, and thus $d\omega_0 = 0$ because of the surjectivity of π and $T_p\pi$. For non-degeneracy of ω_0, suppose $\omega_0([X], [Y]) = 0$ for all $Y \in T_p\Sigma$, hence $\omega(X, Y) = 0$ for all $Y \in T_p(\Sigma)$. Thus, using the lemma, $X \in T_p(G \cdot p)$, that is, $[X] = 0$ and so ω_0 is a (weak) symplectic form. ■

To complete the reduction process, we now include the Hamiltonian.

Theorem 10 *Let Φ be a symplectic action of G on (P, ω) with a momentum mapping J. Suppose $H : P \to \mathbb{R}$ is invariant under the action, that is,*

$$H(p) = H(\Phi_g(p)) \quad \textit{for all } p \in P,\ g \in G.$$

Then J is an integral *for X_H, i.e. if F_t is the flow of X_H, then*

$$J(F_t(p)) = J(p).$$

Proof. For each $\xi \in \mathfrak{g}$ we have $H(\Phi_{\exp t\xi}(p)) = H(p)$ since H is invariant. Differentiating at $t = 0$ gives

$$dH(p)\left(X^{\xi}\right) = 0,$$

that is,

$$\omega\left(X_H, X^{\xi}\right) = 0,$$

that is,

$$\left\{H, \hat{J}(\xi)\right\} = 0$$

which proves that

$$\hat{J}(\xi)(F_t(p)) = \hat{J}(\xi)(p).$$

■

This is the geometric Hamiltonian version of Noether's theorem, which states that for every symmetry there exists a conserved quantity.

Theorem 11 *Under the assumptions of theorem 8, let $H : P \to \mathbb{R}$ be invariant under the action of G. Then the flow F_t of X_H leaves Σ invariant and commutes with the action of G on Σ, so it induces canonically a flow H_t on P_0 satisfying $\pi \circ F_t = H_t \circ \pi$. This flow is a Hamiltonian flow on P_0 with a Hamiltonian H_0 satisfying $H_0 \circ \pi = H \circ i$. H_0 is called the* reduced Hamiltonian.

Proof. From theorem 10 we know that J is an integral for X_H, i.e.

$$J(F_t(p)) = J(p).$$

It follows that Σ is invariant under the flow and that we get a well-defined flow H_t induced on P_0. We clearly have $\pi \circ F_t = H_t \circ \pi$, so $\pi^* H_t^* \omega_0 = F_t^* \pi^* \omega_0 = F_t^* i^* \omega = i^* \omega = \pi^* \omega$. But since π is a surjective submersion, we conclude $H_t^* \omega_0 = \omega_0$, so the flow H_t on P_0 is Hamiltonian. The relation $H_0 \circ \pi = H \circ i$ plus invariance of H under the action of G defines H_0 uniquely. Hence, if $[Y] = T\pi(Y) \in TP_0$, we have

$$dH_0[Y] = i^* dH(Y) = i^* \omega(X_H, Y).$$

But from the construction of H_t, its generator Z satisfies $T\pi \circ X_H = Z \circ \pi$, so $dH_0[Y] = i^* \omega(X_H, Y) = \omega_0(Z, [Y])$, that is, Z has energy H_0. ■

So here we finally have constructed the reduced phase space of the original system, consisting of the physical Hamiltonian H_0 and the physical state space P_0. This is the culmination of geometric theory of constrained systems.

Chapter 5

BRST Reduction in the Group Case

5.1 Introduction

The BRST formalism focuses on the space of functions on phase space rather than on phase space itself. We have seen earlier that the physically interesting functions (i.e. observables) are functions on the constraint manifold that are gauge invariant. In obtaining the set of these functions two steps have to be taken. First, functions on phase space that coincide on shell are to be identified. This gives us the set of functions on the constraint manifold. From that set, we should only take those functions that are invariant under gauge transformations.

In contrast to the methods of reduction described earlier, in the BRST formalism one first *enlarges* the space of phase space functions by adding so-called ghosts and their conjugate momenta. These are "extra" degrees of freedom. Their purpose is to allow the definition of two graded (super)derivations, δ and d. Through their (co)homology we can obtain the set of functions on the constraint manifold and gauge invariant functions, respectively.

Under certain condition these two differential operators can then be combined into one operator called the (classical) BRST operator D. In that case, gauge invariant functions on shell are then given by the cohomology of D.

When quantizing the system, the complete set of functions (including ghosts, ghost momenta, and non-physical functions) are quantized. At the quantum level the system is then reduced using the quantum BRST operator.

5.2 Restriction to the Constraint Surface

We will now treat Marsden-Weinstein reduction in a BRST context, explaining [11].

As we have seen in section 2.8, the space of functions $C^\infty(\Sigma)$ on the constraint surface Σ is given by

$$C^\infty(\Sigma) = \frac{C^\infty(P)}{\mathcal{N}},$$

where $\mathcal{N}$ denotes the ideal of functions that vanish on the constraint surface. The ideal $\mathcal{N}$ is generated by the functions $\hat{J}_i$ (which form a basis for the constraints), and we will denote it by $C^\infty(P)\cdot\hat{J}_i$.

Consider the algebra

$$\Lambda\mathfrak{g}\otimes C^\infty(P),$$

where $\Lambda\mathfrak{g}$ is the exterior algebra of the Lie algebra viewed as a vector space. Elements of $\Lambda\mathfrak{g}$ are called *ghost momenta* (because they will become conjugate to the ghosts defined below) and their degree q is called the *antighost number*.

Define the derivation

$$\delta : \Lambda^q\mathfrak{g}\otimes C^\infty(P) \to \Lambda^{q-1}\mathfrak{g}\otimes C^\infty(P)$$

by its action on the "generators" $\xi\otimes 1$, $\xi\in\mathfrak{g}$ and $1\otimes F$, $F\in C^\infty(P)$ and the fact that it's a (super)derivation. We define

$$\begin{array}{ll} \delta(\xi\otimes 1) = 1\otimes\hat{J}(\xi) & \forall\xi\in\mathfrak{g}; \\ \delta(1\otimes F) = 0 & \forall F\in C^\infty(P), \end{array}$$

which makes sense since $\hat{J}(\xi)\in C^\infty(P)$. Note that $1\in\mathbb{R}\simeq\Lambda^0\mathfrak{g}$. It is immediate to check that δ is a nilpotent derivation, i.e. $\delta^2=0$. The homology of δ, denoted by $H^q_\delta(\Lambda\mathfrak{g}\otimes C^\infty(P))$ is given (as usual) by

$$H^q_\delta(\Lambda\mathfrak{g}\otimes C^\infty(P)) = \frac{\ker\delta}{\operatorname{Im}\delta}.$$

It inherits the grading of $\Lambda\mathfrak{g}\otimes C^\infty(P)$. The algebra we're interested in is H^0_δ, since (using $\Lambda^0\mathfrak{g}=\mathbb{C}$ and $\otimes=\otimes_\mathbb{C}$),

$$\begin{array}{rcl} (\ker\delta)^0 &=& \Lambda^0\mathfrak{g}\otimes C^\infty(P)\sim C^\infty(P); \\ (\operatorname{Im}\delta)^0 &=& \Lambda^0\mathfrak{g}\otimes C^\infty(P)\cdot\hat{J}_i\sim C^\infty(P)\cdot\hat{J}_i, \end{array}$$

and so

$$H^0_\delta = \frac{(\ker\delta)^0}{(\operatorname{Im}\delta)^0} = \frac{C^\infty(P)}{C^\infty(P)\cdot\hat{J}_i} = C^\infty(\Sigma).$$

5.3 Factoring out the Gauge Group

The second part in the reduction process is factoring out the gauge transformations. In the BRST formalism this is achieved by looking for functions which are constant under the action of the gauge group. These are the gauge invariant (and hence physically relevant) functions.

First, we will assume we have a representation of $\mathfrak{g}$ on some vector space V. (In the end we will take V to be $\Lambda^q \mathfrak{g} \otimes C^\infty(P)$ or one of its homology spaces.) Define a map

$$d : V \to \mathfrak{g}^* \otimes V \simeq \mathrm{Hom}(\mathfrak{g},\ V)$$

by

$$dv(\xi) = \xi v, \quad \xi \in \mathfrak{g},\ v \in V, \tag{5.1}$$

where ξv means the representation of ξ acting on v.[1] Extend this map to

$$d : \Lambda^p \mathfrak{g}^* \otimes V \to \Lambda^{p+1} \mathfrak{g}^* \otimes V$$

by

$$d(\omega \otimes v) = d\omega \otimes v + (-1)^p \omega \wedge dv, \quad \omega \in \Lambda^p \mathfrak{g}^*$$

where $d\omega$ is the usual exterior derivative (when $\mathfrak{g}^*$ is viewed as the space of p-forms on the Lie group G with Lie algebra $\mathfrak{g}$). Elements of $\Lambda \mathfrak{g}^*$ are called *ghosts*, and their degree p is called the *pure ghost number*. Since $d^2 = 0$, we can look at its cohomology

$$H_d^p(\Lambda \mathfrak{g}^* \otimes V) = \frac{\ker d}{\mathrm{Im}\, d}.$$

We have

$$\begin{aligned} (\mathrm{Im}\, d)^0 &= 0; \\ (\ker d)^0 &= \{v \mid \xi v = 0\}, \end{aligned}$$

and since $\ker d$ at $p = 0$ are just those v that are infinitesimally invariant under $\mathfrak{g}$, i.e., are gauge invariant, we get

$$H_d^0(\Lambda \mathfrak{g}^* \otimes V) = \{\text{gauge invariant } v\}.$$

[1] This actually leads to the standard Lie algebra cohomology, which was known long before its application in the BRST method. See for example [2].

5.4 Bringing d and δ Together

To conclude the reduction process, form the algebra

$$\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P),$$

and redefine δ so it acts trivially on $\Lambda\mathfrak{g}^*$,

$$\delta : \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P) \to \Lambda^p\mathfrak{g}^* \otimes \Lambda^{q-1}\mathfrak{g} \otimes C^\infty(P).$$

We still have

$$d : \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P) \to \Lambda^{p+1}\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P),$$

where we have taken V in section 5.3 to be $\Lambda\mathfrak{g} \otimes C^\infty(P)$. The action of $\mathfrak{g}$ on this space is given by

$$\begin{aligned} \xi_i(\xi_j) &= [\xi_i, \xi_j] = C_{ij}{}^k\xi_k; \\ \xi_i(F) &= X^{\xi_i}F. \end{aligned}$$

Combining the results of the previous two sections, we immediately see that

$$H_d^0\left(H_\delta^0(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P))\right) = \{\text{gauge invariant functions on } \Sigma\}. \quad (5.2)$$

It is perhaps not immediately clear whether (5.2) is well defined, since the action of d could in principle mix up the equivalence classes in H_δ^0. It can be shown however [8, 11], that the relation

$$d\delta = \delta d$$

holds, which makes (5.2) well defined.

5.5 The BRST Operator

Consider the "total differential"

$$D = d + (-1)^p\,\delta.$$

It is a derivation

$$D : \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P) \to \Lambda^{p+1}\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P) + \Lambda^p\mathfrak{g}^* \otimes \Lambda^{q-1}\mathfrak{g} \otimes C^\infty(P).$$

We now combine the p and q grading into a $p-q$ grading (called the *ghost number*) and define the space C^k by

$$C^k(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)) = \sum_{k=p-q} \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P),$$

so that

$$D : C^k\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)\right) \to C^{k+1}\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)\right),$$

and

$$D^2 = 0.$$

This operator D is the classical BRST operator. Under certain conditions (to be specified below), the cohomology of D at $k = 0$ is given by

$$\begin{aligned} H^0_D\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)\right) &= H^0_d\left(H^0_\delta\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)\right)\right) \\ &= \{\text{gauge invariant functions on } \Sigma\}. \end{aligned} \tag{5.3}$$

On its generators it acts as follows

$$\begin{aligned} D\left(\eta^i\right) &= d\eta^i; \\ D\left(\xi_i\right) &= \hat{J}_i - C_{ib}{}^c\xi_c\eta^b; \\ D(F) &= dF. \end{aligned} \tag{5.4}$$

Now the conditions which must be fulfilled for (5.3) to hold are the following [4, 8]:

- The constraint surface $\Sigma = J^{-1}(0)$ should be a submanifold of P. This has of course been assumed all through this paper.
- The complex $\left(\Lambda\mathfrak{g} \otimes C^\infty(P), \delta\right)$ should be acyclic, i.e.

$$H^q_\delta\left(\Lambda\mathfrak{g} \otimes C^\infty(P)\right) = 0, \quad q \neq 0.$$

- The space $C^\infty(\Sigma)$ is given by $H^0_\delta\left(\Lambda\mathfrak{g} \otimes C^\infty(P)\right)$. That this is the case was shown in section (5.2).

In [4] it is shown that all three conditions are fulfilled if and only if $0 \in g^*$ is a regular value of the momentum mapping J, which in its turn is the case if and only if G acts (almost) freely on $J^{-1}(0)$.

5.6 Super-Poisson Bracket

Since we will want to quantize the whole algebra $\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)$, we need to have a Poisson bracket defined on it. This is a non-trivial matter. Kostant and Sternberg [11] achieve this by using the fact that $\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} = \Lambda(\mathfrak{g}^* \oplus \mathfrak{g})$ has a scalar product by evaluation of $\mathfrak{g}^*$ on $\mathfrak{g}$. They use this scalar product to construct the Clifford algebra $Cl(\mathfrak{g}^* \oplus \mathfrak{g})$, which induces a Poisson bracket on $\Lambda(\mathfrak{g}^* \oplus \mathfrak{g})$. Then, by taking the tensor product, $\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(P)$

becomes a super-Poisson algebra. Some facts about super-Poisson algebra are gathered in appendix B.

We will not give the derivation here, but only give its results. Let $\{\eta^i\}$ be a basis for $\mathfrak{g}^*$ and $\{\xi_i\}$ a basis for $\mathfrak{g}$. Furthermore, let F be any element of $C^\infty(P)$. Their super-Poisson brackets are given by

$$\begin{aligned} \{\eta^i,\eta^j\} = \{\xi_i,\xi_j\} &= 0; \\ \{\eta^i,\xi_j\} = \{\xi_j,\eta^i\} &= \delta^i_j; \\ \{\eta^i,F\} = \{\xi_j,F\} &= 0, \end{aligned}$$

and $\{F,G\}$ is given by the usual Poisson bracket on $C^\infty(P)$.[2] These relations then define a Poisson bracket on the whole phase space by means of the graded Leibniz rule.

5.7 The BRST Generator

We now want find an element Ω of $\Lambda(\mathfrak{g}^*\oplus\mathfrak{g})\otimes C^\infty(P)$ which is such that Poisson bracket with Ω is just the operator D of section 5.5, i.e.

$$\{\Omega, .\} = D.$$

As we will check, it is given by

$$\Omega = \eta^a \hat{J}_a - \frac{1}{2} C_{ab}{}^c \eta^a \eta^b \xi_c.$$

First, we will calculate the Poisson bracket with a function $F\in\Lambda(\mathfrak{g}^*\oplus\mathfrak{g})\otimes C^\infty(P)$,

$$\left\{\eta^a \hat{J}_a - \frac{1}{2} C_{ab}{}^c \eta^a \eta^b \xi_c, F\right\} = \eta^a\left\{\hat{J}_a, F\right\} = \eta^a X^{\xi_a} F.$$

But according to (5.1) this is just dF,

$$\left(\eta^a X^{\xi_a} F\right)\cdot\chi = \left(\eta^a X^{\xi_a} F\right)\cdot \chi^i\xi_i = \chi^a X^{\xi_a} F = X^\chi F.$$

For the basis vectors ξ_i of $\mathfrak{g}$ we find

$$\begin{aligned} \left\{\eta^a \hat{J}_a - \frac{1}{2} C_{ab}{}^c \eta^a \eta^b \xi_c, \xi_i\right\} &= \{\eta^a,\xi_i\}\hat{J}_a - \frac{1}{2} C_{ab}{}^c\left\{\eta^a\eta^b\xi_c,\xi_i\right\} \\ &= \hat{J}_i - \frac{1}{2} C_{ab}{}^c \xi_c\left(\delta^a_i\eta^b - \delta^b_i\eta^a\right) \\ &= \hat{J}_i - C_{ib}{}^c\xi_c\eta^b. \end{aligned}$$

[2]Actually, Kostant and Sternberg set $\{\eta^a,\xi_b\} = 2\delta^a_b$. We drop the factor 2 here.

And finally, for the basis η^i of $\mathfrak{g}^*$ we get

$$\begin{aligned}\left\{\eta^a \hat{J}_a - \frac{1}{2}C_{ab}{}^c\eta^a\eta^b\xi_c, \eta^i\right\} &= -\frac{1}{2}C_{ab}{}^c\eta^a\eta^b\delta^i_c \\ &= -\frac{1}{2}C_{ab}{}^i\eta^a\eta^b.\end{aligned}$$

To check that this is just $d\eta^i$ we evaluate

$$\begin{aligned}\left(-\frac{1}{2}C_{ab}{}^i\eta^a\eta^b\right)\cdot(\xi_k,\xi_l) &= -\frac{1}{2}C_{ab}{}^i\left(\delta^a_k\delta^b_l - \delta^a_l\delta^b_l\right) \\ &= -C_{kl}{}^i,\end{aligned}$$

which is the same result we get when using the Eilenberg-Cartan formula:

$$\begin{aligned}d\eta^i(\xi_k,\xi_l) &= \xi_k\eta^i(\xi_l) - \xi_l\eta^i(\xi_k) - \eta^i([\xi_k,\xi_l]) \\ &= -C_{kl}{}^m\eta^i(\xi_m) \\ &= -C_{kl}{}^i.\end{aligned}$$

Summarizing, we get

$$\begin{aligned}\{\Omega,\eta^i\} &= d\eta^i; \\ \{\Omega,\xi\} &= \hat{J}_i - C_{ib}{}^c\xi_c\eta^b; \\ \{\Omega,F\} &= dF,\end{aligned}$$

which is in complete accordance with (5.4). So

$$\{\Omega,.\} = D.$$

Since $D^2 = 0$, the relation $\{\Omega,\Omega\} = 0$ should hold. Indeed, we have

$$\begin{aligned}\{\Omega,\Omega\} &= \left\{\eta^a\hat{J}_a - \frac{1}{2}C_{ab}{}^c\eta^a\eta^b\xi_c, \eta^i\hat{J}_i - \frac{1}{2}C_{ij}{}^k\eta^i\eta^j\xi_k\right\} \\ &= \left\{\eta^a\hat{J}_a, \eta^i\hat{J}_i\right\} + 2\left\{\eta^a\hat{J}_a, -\frac{1}{2}C_{ij}{}^k\eta^i\eta^j\xi_k\right\} \\ &\quad + \left\{\frac{1}{2}C_{ab}{}^c\eta^a\eta^b\xi_c, \frac{1}{2}C_{ij}{}^k\eta^i\eta^j\xi_k\right\}.\end{aligned}$$

The first two terms cancel each other:

$$\begin{aligned}&\left\{\eta^a\hat{J}_a, \eta^b\hat{J}_b\right\} + 2\left\{\eta^d\hat{J}_d, -\frac{1}{2}C_{ab}{}^c\eta^a\eta^b\xi_c\right\} \\ &= \eta^a\eta^b\left\{\hat{J}_a,\hat{J}_b\right\} - C_{ab}{}^c\hat{J}_d\left\{\eta^d,\eta^a\eta^b\xi_c\right\} \\ &= \eta^a\eta^b C_{ab}{}^c\hat{J}_c - C_{ab}{}^c\hat{J}_d\eta^a\eta^b\delta^d_c \\ &= C_{ab}{}^c\eta^a\eta^b\hat{J}_c - C_{ab}{}^c\eta^a\eta^b\hat{J}_c \\ &= 0,\end{aligned}$$

while the third term is identically zero:

$$\begin{aligned}
&\left\{\frac{1}{2}C_{ab}{}^{c}\eta^a\eta^b\xi_c, \frac{1}{2}C_{ij}{}^{k}\eta^i\eta^j\xi_k\right\} = \frac{1}{4}C_{ab}{}^{c}C_{ij}{}^{k}\left\{\eta^a\eta^b\xi_c, \eta^i\eta^j\xi_k\right\}\\
&= \frac{1}{4}C_{ab}{}^{c}C_{ij}{}^{k}\left(\left\{\eta^a\eta^b\xi_c, \eta^i\eta^j\right\}\xi_k + \eta^i\eta^j\left\{\eta^a\eta^b\xi_c, \xi_k\right\}\right)\\
&= \frac{1}{4}C_{ab}{}^{c}C_{ij}{}^{k}\left(\eta^a\eta^b\left\{\xi_c, \eta^i\eta^j\right\}\xi_k + \eta^i\eta^j\left\{\xi_k, \eta^a\eta^b\right\}\xi_c\right)\\
&= \frac{1}{2}C_{ab}{}^{c}C_{ij}{}^{k}\left(\eta^a\eta^b\left(\delta^i_c\eta^j - \eta^i\delta^j_c\right)\xi_k - \eta^i\eta^j\left(\delta^a_k\eta^b - \eta^a\delta^b_k\right)\xi_c\right)\\
&= \frac{1}{2}C_{ab}{}^{c}C_{cj}{}^{k}\eta^a\eta^b\eta^j\xi_k - \frac{1}{2}C_{ij}{}^{k}C_{kb}{}^{c}\eta^i\eta^j\eta^b\xi_c\\
&+\frac{1}{2}C_{ij}{}^{k}C_{ak}{}^{c}\eta^i\eta^j\eta^a\xi_c - \frac{1}{2}C_{ab}{}^{c}C_{ic}{}^{k}\eta^a\eta^b\eta^i\xi_k\\
&= 0.
\end{aligned} \tag{5.5}$$

(That this is zero can be seen as follows. For any anticommuting function P, the following equation holds:

$$\{\{P, P\}, P\} = 0.$$

Taking $P = \eta^a \hat{J}_a$ we get

$$\begin{aligned}
\left\{\left\{\eta^a\hat{J}_a, \eta^b\hat{J}_b\right\}, \eta^c\hat{J}_c\right\} &= \left\{\eta^a\eta^b C_{ab}{}^{d}\hat{J}_d, \eta^c\hat{J}_c\right\}\\
&= \eta^a\eta^b C_{ab}{}^{d}\eta^c\left\{\hat{J}_d, \hat{J}_c\right\}\\
&= C_{ab}{}^{d}C_{dc}{}^{e}\eta^a\eta^b\eta^c\hat{J}_e = 0.
\end{aligned}$$

This result and the fact that the $\hat{J}_e$ are independent proves (5.5)).

5.8 The Ghost Number Generator

The grading k of the algebra

$$C^k\left(\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(P)\right) = \sum_{k=p-q} \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(P),$$

defined in section 5.5, usually called the ghost number, has the nice property that for any element of $c \in C^k$, k can be evaluated by taking the Poisson bracket of c with the *ghost number generator* $\mathcal{G}$ given by

$$\mathcal{G} = \eta^a\xi_a.$$

By checking on generators,

$$\begin{aligned}
\{\mathcal{G}, \xi_a\} &= -\xi_a;\\
\{\mathcal{G}, \eta^a\} &= \eta^a;\\
\{\mathcal{G}, F\} &= 0,
\end{aligned}$$

we see that for any $c \in C^k$,

$$\{\mathcal{G}, c\} = kc.$$

Part II

Quantization of Constrained Systems

Chapter 6

Standard Operator Methods

6.1 Introduction

In general, quantizing a system means that each function F on phase space is represented by an operator $\hat{F}$, acting on some (pseudo)Hilbert space, which contains the state vectors. The Poisson bracket is then replaced by the commutator, under the identification[1]

$$[.,.] \longleftrightarrow i\hbar\,\{.,.\}\,.$$

When dealing with gauge theories, there exist unphysical degrees of freedom, which under the above prescription would be quantized as well. There are in general two ways of approaching this problem. One could first try to reduce the classical theory and then quantize the reduced system. This is called reduced phase space quantization. On the other hand, one could first quantize the complete theory, and then try to reduce the quantum system. This is the approach of the Dirac and BRST quantization methods. We will throughout assume we know how to quantize the original system (though not including ghosts and ghost momenta).

For notational convenience and compatibility with most literature on this subject, the constraints $\hat{J}_a$ will from now on be denoted by G_a in the classical theory. At the quantum level, operators which are the quantization of some phase space function F will be denoted by $\hat{F}$. (So the quantization of G_a will be denoted by $\hat{G}_a$.)

[1] Since the η^a and ξ_a are anticommuting, their Poisson bracket becomes an anticommutator after quantization. To avoid notational clutter, both the commutator and the anticommutator will be denoted by $[.,.]$. It is an anticommutator in the case where both arguments are odd.

6.2 Reduced Phase Space Quantization

When quantizing gauge systems, it would appear to be logical to quantize only the reduced system. This way, only the physically meaningful observables would be realized as operators an some Hilbert space, which would contain only physical states. There are, however, some serious drawbacks to this approach.

First of all, it may be very hard or even impossible to explicitly construct the set of gauge invariant functions. Also, it could be that the reduced phase space doesn't have some sort of canonical coordinates $\tilde{q}, \tilde{p}$ which obey

$$\{\tilde{q}_i, \tilde{p}_j\} = \delta_{ij}.$$

Finally, going to the reduced system may spoil some manifest symmetry, like Lorentz invariance.

Because of this, methods have been invented to reduce the system at the quantum level, i.e. to first quantize the complete system, after which some sort of physical condition is imposed upon the operators and states.

6.3 Dirac quantization

In the Dirac quantization method, one first quantizes the complete gauge theory. As said before, there exists then states which are not physical. To remove this unphysical states, Dirac proposed the following "physical state condition": a state $|\psi\rangle$ is a physical state if it obeys

$$\hat{G}_a |\psi\rangle = 0 \tag{6.1}$$

for all a. This would seem reasonable, since then physical states would be left unchanged by the finite transformation

$$e^{i\varepsilon^a \hat{G}_a} |\psi\rangle = |\psi\rangle .$$

There are, however, some complications. The classical condition for fist class constraints,

$$\{G_a, G_b\} = C_{ab}{}^c G_c,$$

could be spoiled by quantum corrections of order $\hbar^2$,

$$\left[\hat{G}_a, \hat{G}_b\right] = i\hbar C_{ab}{}^c \hat{G}_c + \hbar^2 \hat{D}_{ab}.$$

In this case (6.1) would imply

$$\hat{D}_{ab} |\psi\rangle = 0.$$

This could however restrict the physical subspace too much, and in extreme cases reduce it to an empty subspace. Also, the classical relation

$$\{H, G_a\} = V_a{}^b G_b$$

may pick up quantum corrections,

$$\left[\hat{H}, \hat{G}_a\right] = i\hbar V_a{}^b \hat{G}_b + \hbar^2 \hat{C}_a.$$

Therefore, to be able to apply the Dirac quantization method, one has to assume $\hat{D}_{ab} = \hat{C}_a = 0$.

Another problem arises from the inner product. In the Schrödinger representation, the scalar product on the Dirac Hilbert space is given by

$$\langle\psi|\,\phi\rangle = \int dq^1 dq^2 \ldots dq^N \;\psi^*\left(q^1, \ldots, q^N\right)\phi\left(q^1, \ldots, q^N\right). \tag{6.2}$$

Since the integration is over all degrees of freedom, including non-physical ones, this integral could diverge. For example, in the simple case of just one constraint given by $p_1 = 0$, the physical state condition reads

$$i\hbar \frac{\partial}{\partial q^1}\,|\psi\rangle = 0,$$

so physical states do not depend on q^1. So if $\langle\psi|$ and $|\phi\rangle$ are both physical states, the inner product given by (6.2) would be infinite. In other words, (6.1) may have no solutions in the Hilbert space to which the $|\psi\rangle$'s belong.

Chapter 7

BRST Quantization

7.1 Introduction

As said before, in the BRST quantization procedure, one quantizes the complete system (including ghosts and ghost momenta) and then imposes some physical state condition to retrieve physical states and operators (as in the Dirac method). We will throughout assume we know how to quantize the original system without ghosts and ghost momenta.

7.2 Quantum BRST Operator

To include the ghosts and their momenta in the quantization procedure, the state space should also carry a representation of the anticommutation relations[1] between the $2m$ hermitian ghost and ghost momenta operators

$$\left[\hat{\eta}^a, \hat{\xi}_b\right] = \delta^a{}_b.$$

The BRST generator is then represented by a hermitian operator $\hat{\Omega}$, given by

$$\hat{\Omega} = \hat{G}_a \hat{\eta}^a - \frac{i}{2} C_{ab}{}^c \hat{\eta}^a \hat{\eta}^b \hat{\xi}_c + \frac{i}{2} C_{ab}{}^b \hat{\eta}^a.$$

The classical condition $D^2 = 0$ now reads

$$\left[\hat{\Omega}, \hat{\Omega}\right] = 2\hat{\Omega}^2 = 0.$$

The ghost number operator is given by

$$\hat{\mathcal{G}} = \hat{\eta}^a \hat{\xi}_a + \text{constant}.$$

[1]From now on we'll set $\hbar = 1$.

The constant is real and reflects the ordering ambiguity. We will use a form of the ghost number operator which is antihermitian,

$$\begin{aligned}\hat{\mathcal{G}} &= \frac{1}{2}\left(\hat{\eta}^a\hat{\xi}_a - \hat{\xi}_a\hat{\eta}^a\right);\\ \hat{\mathcal{G}}^* &= -\hat{\mathcal{G}}.\end{aligned}$$

The ghost number operator obeys

$$\begin{aligned}\left[\hat{\mathcal{G}},\hat{\xi}_a\right] &= -\hat{\xi}_a,\\ \left[\hat{\mathcal{G}},\hat{\eta}^a\right] &= \hat{\eta}^a,\\ \left[\hat{\mathcal{G}},\hat{q}\right] = \left[\hat{\mathcal{G}},\hat{p}\right] &= 0.\end{aligned}$$

We shall assume $\hat{\mathcal{G}}$ to be a conserved quantity. This allows us to classify the physical states according to their ghost number,

$$\hat{\mathcal{G}}\,|\text{phys},n\rangle = n\,|\text{phys},n\rangle\,.$$

Since $\hat{\mathcal{G}}$ is anti-hermitian, these states satisfy

$$\langle\text{phys},m|\,\text{phys},n\rangle = C_n\delta_{n,-m}. \tag{7.1}$$

7.3 Quantum Ghost Algebra

We'll assume we can write the linear space that quantizes the classical BRST extended phase space as a tensor product of a Hilbert space (containing the so-called matter states) and a space with indefinite inner product (containing the ghost states).

Consider states of the form [9]

$$|\psi\rangle = |M\rangle\,|G\rangle\,,$$

where $|M\rangle$ and $|G\rangle$ denote matter and ghost states respectively. The ghost algebra may be specified exactly.

A vacuum ghost state is defined by

$$\hat{\eta}^a\left|\frac{m}{2}\right\rangle = 0, \quad a = 1,\ldots,m,$$

which has ghost number $m/2$,

$$\begin{aligned}\hat{\mathcal{G}}\left|\frac{m}{2}\right\rangle &= \frac{1}{2}\hat{\eta}^a\hat{\xi}_a\left|\frac{m}{2}\right\rangle\\ &= \frac{1}{2}\left[\hat{\eta}^a,\hat{\xi}_a\right]\left|\frac{m}{2}\right\rangle\\ &= \frac{m}{2}\left|\frac{m}{2}\right\rangle.\end{aligned}$$

The conjugate ghost vacuum is defined by

$$\hat{\xi}_a \left| -\frac{m}{2} \right\rangle = 0, \quad a = 1, \ldots, m,$$

which has ghost number $-m/2$. The inner product between these two states is normalized by defining

$$\left\langle -\frac{m}{2} \right| \left. \frac{m}{2} \right\rangle = \left\langle \frac{m}{2} \right| \left. -\frac{m}{2} \right\rangle = 1, \tag{7.2}$$

obeying (7.1).

The states with ghost number n in the range $-m/2 < n < m/2$ are not uniquely determined by their ghost number. This requires an extra index on these states:

$$\begin{aligned} \left| -\frac{m}{2} + k \right\rangle^{a_1 \ldots a_k} &\equiv \hat{\eta}^{a_1} \ldots \hat{\eta}^{a_k} \left| -\frac{m}{2} \right\rangle ; \\ \left| \frac{m}{2} - k \right\rangle_{a_1 \ldots a_k} &\equiv \hat{\xi}_{a_1} \ldots \hat{\xi}_{a_k} \left| \frac{m}{2} \right\rangle . \end{aligned}$$

The states with lower and upper indices are not independent, since

$$\left| \frac{m}{2} \right\rangle = (-1)^{m(m-1)/2} \, \hat{\eta}^1 \ldots \hat{\eta}^m \left| -\frac{m}{2} \right\rangle ,$$

which satisfies (7.2). This leads to the following relation (up to a sign):

$$\begin{aligned} \left| \frac{m}{2} - k \right\rangle_{a_1 \ldots a_k} &= \hat{\xi}_{a_1} \ldots \hat{\xi}_{a_k} \left| \frac{m}{2} \right\rangle \\ &= (-1)^{m(m-1)/2} \, \hat{\xi}_{a_1} \ldots \hat{\xi}_{a_k} \hat{\eta}^1 \ldots \hat{\eta}^m \left| -\frac{m}{2} \right\rangle \\ &= (-1)^{m(m-1)/2} \, \varepsilon_{a_1 \ldots a_m} \left| -\frac{m}{2} + k \right\rangle^{a_{k+1} \ldots a_m} , \end{aligned}$$

(with no summation over the indices), where ε is the totally antisymmetric symbol in m dimensions.

A general state $|\psi\rangle$ may be written as

$$|\psi\rangle = \sum_{k=0}^{m} |M\rangle_{a_1 \ldots a_k} |G\rangle^{a_1 \ldots a_k} .$$

To avoid double counting the indices are restricted to be in increasing order. This general state may be decomposed according to ghost number:

$$\begin{aligned} |\psi\rangle &= \sum_{k=0}^{m} \left| \psi, -\frac{m}{2} + k \right\rangle ; \\ \left| \psi, -\frac{m}{2} + k \right\rangle &= |M\rangle_{a_1 \ldots a_k} \left| -\frac{m}{2} + k \right\rangle^{a_1 \ldots a_k} . \end{aligned}$$

Using the material from appendix C we can represent the states $|-m/2+k\rangle$ as products of Grassmann variables θ^i,

$$\begin{aligned} \left|-\frac{m}{2}\right\rangle &\rightarrow 1; \\ \left|-\frac{m}{2}+k\right\rangle^{a_1\dots a_k} &\rightarrow \theta^{a_1}\theta^{a_2}\dots\theta^{a_k}, \end{aligned}$$

and the ghost operators as

$$\begin{aligned} \hat{\eta}^a &\rightarrow \theta^i; \\ \hat{\xi}_a &\rightarrow \frac{\partial}{\partial\theta^i}. \end{aligned}$$

Using these identifications, we find

$$\left|\psi, -\frac{m}{2}+k\right\rangle = |M\rangle_{a_1\dots a_k}\theta^{a_1}\theta^{a_2}\dots\theta^{a_k}.$$

7.4 Quantum BRST Cohomology

By analog with (6.1), the physical state condition in the BRST formalism is given by

$$\hat{\Omega}\,|\psi\rangle = 0.$$

Since $\hat{\Omega}$ is nilpotent, we can define BRST closed and exact operators in the same way as in the classical case. We get:

$$\begin{aligned} \left[\hat{\Omega}, \hat{A}\right] = 0 &\quad\Leftrightarrow\quad \hat{A} \text{ is BRST-closed}, \\ \hat{A} = \left[\hat{\Omega}, \hat{B}\right] &\quad\Leftrightarrow\quad \hat{A} \text{ is BRST-exact}. \end{aligned}$$

BRST-exact operators are BRST-closed, so one can define the "quantum operator cohomology" $H_{op}\left(\hat{\Omega}\right)$ as the set of equivalence classes of BRST-closed operators modulo the exact ones.

Nilpotency of $\hat{\Omega}$ also allows the definition of closed and exact states:

$$\begin{aligned} \hat{\Omega}\,|\psi\rangle = 0 &\quad\Leftrightarrow\quad |\psi\rangle \text{ is BRST-closed}; \\ |\psi\rangle = \hat{\Omega}\,|\phi\rangle &\quad\Leftrightarrow\quad |\psi\rangle \text{ is BRST-exact}. \end{aligned}$$

So we define "quantum state cohomology" $H_{st}\left(\hat{\Omega}\right)$ as the equivalence classes of BRST-closed states modulo exact ones,

$$H_{st}\left(\hat{\Omega}\right) = \frac{\ker\hat{\Omega}}{\operatorname{Im}\hat{\Omega}}.$$

The claim of quantum BRST reduction is that $H_{st}\left(\hat{\Omega}\right)$ will contain the physical states.

To examine the quantum BRST condition we will first look at the conceptually simple system in which some of the variables (q^a, p_a), $(a = 1, \ldots m)$, are unphysical[2]. In that case the BRST operator is given by

$$\hat{\Omega} = \hat{p}_a \hat{\eta}^a.$$

The obvious outcome of the BRST procedure should be that the phase space should reduce to a space with no ghosts and ghost momenta, and with no $(\hat{q}_a, \hat{p}^a)$ degrees of freedom.

Consider a general state with ghost number $-m/2 + k$,

$$\left|\psi, -\frac{m}{2} + k\right\rangle = |M\rangle_{a_1 \ldots a_k} \left|-\frac{m}{2} + k\right\rangle^{a_1 \ldots a_k}, \tag{7.3}$$

where repeated indices denote a summation over all different ghost states. BRST invariance implies

$$\hat{p}_{[a_1} |M\rangle_{a_2 \ldots a_k]} = 0, \tag{7.4}$$

where the brackets denotes anti-symmetric summation. These are $\binom{m}{k+1}$ equations on $\binom{m}{k}$ matter states.

The dual state to (7.3) is given by

$$\left\langle \phi, \frac{m}{2} - k\right| = {}^{b_1 \ldots b_{m-k}} \left\langle \frac{m}{2} - k\right|_{b_1 \ldots b_{m-k}} \langle M| \tag{7.5}$$

and satisfies

$$\left\langle \phi, \frac{m}{2} - k \right| \left. \psi, -\frac{m}{2} + k \right\rangle = {}_{b_1 \ldots b_{m-k}} \langle M| \, M\rangle_{a_1 \ldots a_k}.$$

BRST invariance of 7.5 implies

$${}_{b_1 \ldots b_{m-k}} \langle M| \, \hat{p}_a = 0, \quad a \neq b_1, \ldots, b_{m-k}. \tag{7.6}$$

So for BRST invariant states we have

$${}_{b_1 \ldots b_{m-k}} \langle M| \, \hat{p}_c \, |M\rangle_{a_1 \ldots a_k} = 0.$$

In [9] it is shown that even though we have restricted our analysis to states of the form $|M\rangle\,|G\rangle$, the resulting space of physical states is still too big. A way out proposed in [9] is to require that all inner products in

[2] This is a typical example of a system which runs into trouble when trying to quantize it in the Dirac quantization scheme, see 6.3.

the original state space are finite[3]. But in [9] this is enforced through the definition of vacuum states $|0\rangle_p$ and $|0\rangle_q$, obeying

$$\begin{aligned} \hat{q}^a |0\rangle_q &= 0; \\ \hat{p}_a |0\rangle_p &= 0; \\ {}_q\langle 0|\, 0\rangle_p &= 1. \end{aligned} \tag{7.8}$$

But this involves finding the solution to Dirac's physical state condition given by (7.8), which is exactly what BRST was trying to avoid. (A similar construction is found in [8].) This, in our opinion, defeats the purpose of BRST as a quantization procedure.

Let's however finish the argument. After having found the vacua defined in (7.8), construct states of the form

$$\langle A| = {}_q\langle 0|\, A(\hat{p}), \quad |B\rangle = B(\hat{q})\, |0\rangle_p . \tag{7.9}$$

From the requirement that $\langle A|\, B\rangle$ should be finite follows that $|B\rangle$ cannot be an eigenstate of p_a, for else

$$\begin{aligned} \langle A|\, B\rangle &= {}_q\langle 0|\, A(\hat{p})\, |B\rangle \\ &= const \times {}_q\langle 0|\, B(\hat{q})\, |0\rangle_p = 0. \end{aligned}$$

Let's also assume that $A(\hat{p})$ and $B(\hat{q})$ may be given in terms of power series,

$$A(\hat{p}) = \sum_{n=0}^{\infty} A^n (\hat{p})^n , \quad B(\hat{q}) = \sum_{n=0}^{\infty} B_n (\hat{q})^n .$$

Now consider general states with ghost number $-m/2+k$, where the matter

[3]To see what happens without this condition, suppose we impose another condition

$$p_{a_k} |M\rangle_{a_1 \ldots a_k} = 0. \tag{7.7}$$

This can be interpreted as the BRST condition on the matter state $|M\rangle_{a_1 \ldots a_{k-1}}$, which implies

$$\begin{aligned} \left|phys, -\frac{m}{2} + k\right\rangle = \eta^{a_k} \left|phys, -\frac{m}{2} + k - 1\right\rangle \\ = \Omega\, |\; \rangle , \end{aligned}$$

since

$$\eta^a = i\, [\Omega, q^a] .$$

So after imposing the condition (7.7), the original BRST invariant physical state turns into a null state.

On the other hand, (7.7) implies that the state $|M\rangle_{a_1 \ldots a_k}$ must contain the state $|0\rangle_p$ satisfying $p_{a_k} |0\rangle_p = 0$. But this makes the inner product infinite, since ${}_p\langle 0|\, 0\rangle_p = \infty$. This is contradictory to the previous statement that $|M\rangle_{a_1 \ldots a_k}$ is a null state.

parts are constructed from states of the form (7.9),

$$\left|\psi, N, -\frac{m}{2}+k\right\rangle =$$
$$|M^*\rangle_{a_1\ldots a_k}\left(\sum_{\lambda_1+\lambda_2+\ldots+\lambda_m=N-k}\Phi_{N-k}[\lambda]\prod_{b=1}^{m}\left(\hat{q}^b\right)^{\lambda_b}\right)|0\rangle_p\left|-\frac{m}{2}+k\right\rangle,$$

where $|M^*\rangle_{a_1\ldots a_k}$ is a matter state independent of the unphysical q^a, p_a, and N is the eigenvalue of the so-called BRST scaling operator $\hat{L}$, defined by

$$\hat{L} = i\hat{q}^a\hat{p}_a + \hat{\eta}^a\hat{\xi}_a = i\left[\hat{\Omega}, \hat{q}^a\hat{\xi}_a\right].$$

Suppose the state $\left|\psi, N, -\frac{m}{2}+k\right\rangle$ is BRST invariant,

$$\hat{\Omega}\left|\psi, N, -\frac{m}{2}+k\right\rangle = 0.$$

Then using

$$\hat{L}\left|\psi, N, -\frac{m}{2}+k\right\rangle = N\left|\psi, N, -\frac{m}{2}+k\right\rangle$$

we get

$$\left|\psi, N, -\frac{m}{2}+k\right\rangle = \hat{\Omega}\left(i\frac{\hat{q}^a\hat{\xi}_a}{N}\left|\psi, N, -\frac{m}{2}+k\right\rangle\right).$$

So all BRST invariant states are BRST exact, unless $N=0$. This leads to the conclusion that the physical states are given by

$$\left|phys, -\frac{m}{2}\right\rangle = |M^*\rangle\,|0\rangle_p\left|-\frac{m}{2}\right\rangle.$$

7.5 The Non-Abelian Case

We will now look at the case where the structure constants $C_{ab}^{\ c}$ are nonzero, and try to follow the treatment given in the previous section.

Consider once again a state of the form

$$\left|\psi, -\frac{m}{2}+k\right\rangle = |M\rangle_{a_1\ldots a_k}\left|-\frac{m}{2}+k\right\rangle^{a_1\ldots a_k}.$$

Since the BRST operator is now given by

$$\hat{\Omega} = \hat{G}_a\hat{\eta}^a - \frac{i}{2}C_{ab}{}^{c}\hat{\eta}^a\hat{\eta}^b\hat{\xi}_c + \frac{i}{2}C_{ab}{}^{b}\hat{\eta}^a,$$

BRST invariance implies

$$\left(\hat{G}_{[a_1} + \frac{i}{2} C_{[a_1 b}{}^{b}\right) |M\rangle_{a_2 \ldots a_{k+1}]} - \frac{i}{2} \sum_{j=2}^{k+1} C_{[a_j a_1}{}^{b} \, |M\rangle_{a_2 \ldots a_{j-1} b a_{j+1} \ldots a_{k+1}]} = 0.$$

Just as in the simple case described before, these are again $\binom{m}{k+1}$ equations on $\binom{m}{k}$ matter states, but this time the structure constants make the analysis much more difficult.

Once again, let's first consider simpler cases. Suppose the states are of the form (7.3),

$$\left|\psi, -\frac{m}{2} + k\right\rangle = |M\rangle_{a_1 \ldots a_k} \left|-\frac{m}{2} + k\right\rangle^{a_1 \ldots a_k},$$

with only one matter state, say $|M\rangle_{12 \ldots k}$, different from zero. Then the BRST condition implies

$$\left(\hat{G}_a + \frac{i}{2} C_{ab}{}^{b} - i \sum_{b=1}^{k} C_{ab}{}^{b}\right) |M\rangle_{12 \ldots k} = 0, \quad a \neq 1, \ldots, k; \tag{7.10}$$

$$C_{ab}{}^{c} = 0, \quad c = 1, \ldots k, \ a, b \neq 1, \ldots k. \tag{7.11}$$

On the dual state of the form (7.5), with ${}_{k+1 \ldots m}\langle M|$ nonzero, the BRST condition results in

$${}_{k+1 \ldots m}\langle M| \left(\hat{G}_a + \frac{i}{2} C_{ab}{}^{b} - i \sum_{b=1}^{k} C_{ab}{}^{b}\right) = 0, \quad a = 1, \ldots k; \tag{7.12}$$

$$C_{ab}{}^{c} = 0, \quad c \neq 1, \ldots k; \ a, b = 1, \ldots k. \tag{7.13}$$

So we see that in the general case the structure constants must obey consistency conditions, namely (7.11) and (7.13). Because of this, not all ghost states are formally allowed. The states at minimum and maximum ghost number, however, trivially obey (7.11) and (7.13) and are therefore always allowed.

In the previous section, it was shown to be necessary to demand finite inner products in the original state space. Since the case at hand is more general, this has to be demanded now as well. It may however (depending on the particular model at hand) be necessary to impose further restrictions.

At a formal level, it is however possible to say something about the resulting BRST cohomology. In the previous section it was shown that in that simple case, all BRST invariant states not of the form $|M\rangle\,|G\rangle$ were BRST exact. Now, in the classical theory there always exists a canonical transformation which puts the BRST operator into the form $\Omega' = G_a \eta^a$ [8]. Therefore in the quantum theory one would expect the existence of a unitary operator U satisfying $\hat{\Omega}' = U^\dagger \hat{\Omega} U$. The states in the two representations would then be related according to $|phys\rangle = U^\dagger \, |phys\rangle'$. Then we could just follow the treatment in the previous section. However, to explicitly specify U is not doable, so all this remains a formal point.

Conclusion

As should now be clear, a mathematically rigorous treatment of classical BRST theory is not only possible, it also addresses certain conceptual problems (such as the origin of the ghosts and ghost momenta). BRST quantization however is a totally different matter. While claiming to solve certain problems arising in for example Dirac's method, it turns out that at some point the BRST formalism still requires the solution to Dirac's physical state condition. This in our view defeats the purpose of BRST quantization.

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Amsterdam, December 15, 1998.

Appendix A

Lie Groups and Lie Algebras

In this appendix we have collected some useful definitions from the theory of Lie groups and Lie algebras [1].

Definition 12 *A Lie group is a finite-dimensional smooth manifold G that is also a group and for which the group operations of multiplication, $G \times G \to G : (g,h) \to gh$, and inversion, $^{-1} : G \to G : g \to g^{-1}$ are smooth. The identity is denoted by e.*

Let for every $g \in G$ *left translation by* g be defined by the map $L_g : G \to G : h \to gh$. A vector field X on G is said to be *left invariant* if, for every $g \in G$,

$$(L_g)_* X = X.$$

Let $\mathfrak{X}_L(G)$ be the set of left invariant vector fields on G. It can be shown that $\mathfrak{X}_L(G)$ and T_eG are isomorphic as vector spaces. Let $\xi \in T_eG$. Then $X_\xi(g) \equiv T_eL_g\xi$ is a left invariant vector field. A Lie bracket in T_eG is defined by

$$[\xi, \zeta] = [X_\xi, X_\zeta](e),$$

for $\xi, \zeta \in T_eG$. The vector space T_eG with this Lie algebra structure is called the *Lie algebra* of G and is denoted by $\mathfrak{g}$. For every $\xi \in T_eG$, $\phi_\xi : \mathbb{R} \to G : t \to \exp t\xi$ denotes the integral curve of X_ξ passing through e at $t = 0$.

Let $R_g : G \to G : h \to hg$, called *right translation by* g. Then we can define the map $I_g : G \to G : h \to ghg^{-1}$ as $I_g = R_{g^{-1}}L_g$. Associated with it is the so called adjoint mapping Ad_g defined by

$$Ad_g = T_eI_g = T_e\left(R_{g^{-1}}L_g\right) : T_eG \to T_eG.$$

Definition 13 *An* action *of a Lie group G on a manifold M is a smooth mapping $\Phi : G \times M \to M$ such that:*

1. *For all* $m \in M$, $\Phi(e, m) = m$;
2. *For every* $g, h \in G$, $\Phi(g, \Phi(h, m)) = \Phi(gh, m)\ \forall m \in M$.

Suppose $\Phi : G \times M \to M$ is a smooth action. If $\xi \in T_eG$, then $\Phi^\xi : \mathbb{R} \times M \to M : (t, m) \to \Phi(\exp t\xi, m)$ is an $\mathbb{R}$-action on M, that is, Φ^ξ is a flow on M. The corresponding vector field on M given by

$$X^\xi(m) = \left.\frac{d}{dt}\Phi(\exp t\xi, m)\right|_{t=0}$$

is called the *infinitesimal generator* of the action corresponding to ξ.

Appendix B

Super-Poisson Algebras

From [11] we have collected some definitions regarding superalgebras and super-Poisson algebras.

An associative *superalgebra* A is a $\mathbb{Z}_2$-graded vector space,

$$A = A_0 \oplus A_1,$$

together with a multiplication $A \times A \to A$, which is $\mathbb{Z}_2$-graded, i.e.

$$\begin{aligned} A_0 \times A_0 &\to A_0; \\ A_1 \times A_1 &\to A_0; \\ A_1 \times A_0 &\to A_1; \\ A_0 \times A_1 &\to A_1, \end{aligned}$$

which is associative. A is called *supercommutative* if , for $i, j = 0, 1$,

$$a_i b_j = (-1)^{ij} b_j a_i, \quad a_i \in A_i,\ b_j \in A_j.$$

A *superderivation* is a map from A to itself which can be either *even* or *odd*. An even derivation is a map $d : A_0 \to A_0$ and $d : A_1 \to A_1$ and obeys the usual Leibniz rule

$$d(ab) = (da)\, b + a\,(db).$$

An odd derivation is a map $d : A_0 \to A_1$ and $d : A_1 \to A_0$ such that

$$d(a_i b) = (da_i)\, b + (-)^i a_i\,(db), \quad a_i \in A_i,\ b \in A.$$

A superalgebra L is called a *Lie superalgebra* if the multiplication (called the Lie bracket and denoted by $[,]$) is superanticommutative:

$$[l_i, l_j] = -(-)^{ij} [l_j, l_i],$$

and left multiplication is a superderivation, i.e.

$$[l_i, [l_j, l_k]] = [[l_i, l_j], l_k] + (-)^{ij} [l_j, [l_i, l_k]] .$$

A *Poisson algebra* P is an algebra which is both a commutative algebra and a Lie algebra, so there are two multiplications defined on it:

1. An associative commutative multiplication $P \times P \to P$.
2. A Lie bracket multiplication $P \times P \to P$, called a Poisson bracket, and denoted by $\{\ ,\ \}$.

The Poisson bracket is a derivation for the multiplication:

$$\{a, bc\} = \{a, b\} c + \{a, c\} b.$$

A *super-Poisson algebra* is a Poisson algebra for which both the ordinary multiplication and the Poisson bracket are now supercommutative. So it is a superalgebra with:

1. An associative supercommutative multiplication $B_i \times B_j \to B_{i+j \mod 2}$.
2. A Lie bracket multiplication $B_i \times B_j \to B_{i+j \mod 2}$.

The Lie bracket multiplication is still called a Poisson bracket, denoted by $\{\ ,\ \}$. Furthermore, the Poisson bracket is a superderivation for the multiplication, i.e.

$$\{b_i, b_j b_k\} = \{b_i, b_j\} b_k + (-)^{ij} b_j \{b_i, b_k\} .$$

Appendix C

Grassmann Variables

Following the treatment in [10] and [7], this appendix deals with the principles and applications of anticommuting variables. Anticommuting variables, like the η's and ξ's appearing in the classical BRST formalism, are usually called Grassmann variables. They satisfy the basic anticommutation relations

$$\theta^i \theta^j + \theta^j \theta^i = 0.$$

So for any i,

$$\left(\theta^i\right)^2 = 0.$$

The algebra generated by these symbols contains all expressions of the form

$$f\left(\theta\right) = f_0 + \sum_i f_i \theta^i + \sum_{i<j} f_{ij} \theta^i \theta^j + \sum_{i<j<k} f_{ijk} \theta^i \theta^j \theta^k + \cdots . \tag{C.1}$$

This allows the definition of an associative product

$$\begin{aligned} f\left(\theta\right) g\left(\theta\right) &= f_0 g_0 + \sum_i \left(f_0 g_i + f_i g_0\right) \theta^i \\ &+ \frac{1}{2} \sum_{i,j} \left(f_{ij} g_0 + f_i g_j - f_j g_i + f_0 g_{ij}\right) \theta^i \theta^j + \cdots . \end{aligned}$$

A left derivative $\partial / \partial \theta^i$ is defined as follows. It gives zero on a product of θ's which does not contain θ^i. If the product does contain the θ^i, it is moved to the left and then dropped, i.e.

$$\frac{\partial}{\partial \theta^2}\left(\theta^1 \theta^2\right) = \frac{\partial}{\partial \theta^2}\left(-\theta^2 \theta^1\right) = -\theta^1.$$

The left derivative obeys

$$\begin{aligned} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} + \frac{\partial}{\partial \theta^j} \frac{\partial}{\partial \theta^i} &= 0; \\ \frac{\partial}{\partial \theta^i} \theta^j + \theta^j \frac{\partial}{\partial \theta^i} &= \delta_{ij}. \end{aligned}$$

Now, in the BRST quantization, the following anticommutation relations for the ghost and ghost momenta emerge:

$$\begin{aligned} \left[\eta^i, \eta^j\right] &= \left[\xi_i, \xi_j\right] = 0; \\ \left[\eta^i, \xi_j\right] &= \delta_{ij}. \end{aligned}$$

Under the identification

$$\begin{aligned} \eta^i &\to \theta^i; \\ \xi_i &\to \frac{\partial}{\partial \theta^i}, \end{aligned}$$

we can represent the ghost algebra as a Hilbert space of "functions" $f(\theta)$, with the inner product

$$\langle g|\, f\rangle = \sum_{k=1}^{N} \frac{1}{k!} \sum_{i_1,\ldots i_k} \bar{g}^{i_1\ldots i_k} f^{i_1\ldots i_k}, \tag{C.2}$$

where the bar denotes complex conjugation.

Integration over anticommuting variables (called Berezin integration) is defined by

$$\begin{aligned} \int d\theta \; f(\theta) &= \frac{\partial f}{\partial \theta}; \\ \int d\theta^1 d\theta^2 \; f\left(\theta^1, \theta^2\right) &= \frac{\partial}{\partial \theta^1} \frac{\partial}{\partial \theta^2} f\left(\theta^1, \theta^2\right), \end{aligned}$$

and so on. This yields for the scalar product defined by equation (C.2)

$$\langle g|\, f\rangle = \int d\theta^N d\theta^H \ldots d\theta^1 d\theta^1 \;\exp\left(-\sum_{i=1}^{N} \theta^i \theta^i\right) \bar{g}(\theta)\, f(\theta),$$

which can be verified by expanding the exponential and using the definition of Berezin integration.

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