More Examples of Path Integrals

Now let's consider a quantum particle of mass m on the real line, but in a potential

$$V : \mathbb{R} \longrightarrow \mathbb{R}$$

Let's compute its time evolution using a path integral, using the fact that we already found the answer for the free particle. Our particle can trace out any path

$$\gamma : [0, T] \longrightarrow \mathbb{R}$$

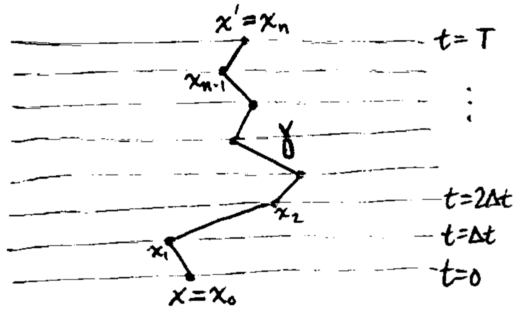
(where w.l.o.g. we start the clock ticking at 0) and its action is

$$S(\gamma) = \int_0^T \frac{m}{2} \dot{\gamma}(t)^2 - V(\gamma(t))\, dt$$

so the path integral philosophy tells us given a wavefunction $\psi_0 \in L^2(\mathbb{R})$ at time 0, it will evolve to $\psi_T \in L^2(\mathbb{R})$ at time T where

$$\psi_T(x') = \int_{\substack{\gamma : [0,T] \to \mathbb{R} \\ \text{with } \gamma(0) = x \\ \gamma(T) = x'}} e^{iS(\gamma)/\hbar}\, \psi_0(x)\, \mathcal{D}\gamma\, dx$$

To do this, we first integrate over piecewise linear paths, like this:



$\Delta t = T/n \qquad \Delta x_k = x_k - x_{k-1}$

$\gamma(k\Delta t) = x_k$

and then take the limit as $n \to \infty$, if it exists. So we hope:

$$\psi_T(x') = \lim_{n\to\infty} \int e^{iS(\gamma)/\hbar}\, \psi_0(x_0)\, \frac{dx_0}{c(\Delta t)} \cdots \frac{dx_{n-1}}{c(\Delta t)}$$

where

$$\begin{aligned} S(\gamma) &= \int_0^T \frac{m}{2}\dot\gamma(t)^2 - V(\gamma(t))\; dt \\ &\approx \sum_{k=1}^{n} \left(\frac{m}{2}\left(\frac{\Delta x_k}{\Delta t}\right)^2 - V(x_{k-1})\right)\Delta t \end{aligned}$$

So we hope:

$$\psi_T(x') = \lim_{n\to\infty} \int \prod_{k=1}^{n} e^{\frac{i}{\hbar}\frac{m}{2}\frac{\Delta x_k^2}{\Delta t}}\, e^{-\frac{i}{\hbar}V(x_{k-1})\Delta t}\, \psi(x_0)\frac{dx_0}{c(\Delta t)} \cdots \frac{dx_{n-1}}{c(\Delta t)}$$

In this formula, we're repeatedly multiplying our wavefunction by $e^{-\frac{i}{\hbar}V\Delta t}$ and then evolving it for a time Δt as if it were a free particle. The latter type of evolution, we've seen, amounts to

$$\psi \longmapsto e^{-iH_0\Delta t/\hbar}\,\psi$$

where H_0 is the Hamiltonian for a free particle:

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

So

$$\psi_T = \lim_{n\to\infty}\left(e^{-\frac{i}{\hbar}H_0\Delta t}\, e^{-\frac{i}{\hbar}V\Delta t}\right)^n \psi_0 .$$

The <u>Lie-Trotter theorem</u> says: if A & B are self-adjoint (unbounded!) operators such that $A+B$ is essentially self adjoint on $D(A)\cap D(B)$, then

$$e^{i(A+B)t}\psi = \lim_{n\to\infty}\left(e^{iAt/n}\, e^{iBt/n}\right)^n \psi$$

for all ψ in our Hilbert space (and $\forall t$, $e^{i(A+B)t}$ is unitary). For details*, see vol. 1 of Reed & Simon's Methods of Modern Mathematical Physics.

In fact H & V are self-adjoint on $L^2(\mathbb{R})$ & $H_0 + V$ is essentially self-adjoint on $D(H_0)\cap D(V)$ if V is reasonably nice (e.g. continuous and bounded below). So in this case

$$\lim_{n\to\infty}\left(e^{-\frac{i}{\hbar}H_0\Delta t}\, e^{-\frac{i}{\hbar}V\Delta t}\right)^n \psi$$

exists and equals

$$e^{-\frac{i}{\hbar}(H_0+V)T}\psi .$$

So we get

$$\psi_T = e^{-\frac{i}{\hbar}(H_0+V)T}\psi_0 = e^{-\frac{i}{\hbar}HT}\psi_0$$

where

$$H = H_0 + V$$

is the Hamiltonian. ~~So we get~~ If $\psi_0 \in D(H_0+V)$ we can differentiate this and get

$$\frac{d}{dt}\psi_t = -\frac{i}{\hbar}(H_0+V)\psi_t$$

which is Schrödinger's equation.

We can also handle the case of a particle on a complete Riemannian manifold, Q. Here again

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2$$

&

$$\text{"}V\text{"} = \text{multiplication by } V: Q \to \mathbb{R}$$

are self adjoint operators on $L^2(Q)$ & if V is continuous & bded below, H_0+V is ess. s.a. on $D(H)\cap D(V)$. So again, skipping lots of steps, we obtain this formula for time evolution of a wavefunction for a particle on Q with potential V:

$$\psi_T = \lim_{n\to\infty} \left(e^{-\frac{i}{\hbar}H_0\frac{T}{n}} e^{-\frac{i}{\hbar}V\frac{T}{n}}\right)^n \psi_0$$

Lie-Trotter formula

$$= e^{-\frac{i}{\hbar}HT}\psi_0$$

where $H = H_0 + V$ is the Hamiltonian for our particle. But, we can't compute $e^{-\frac{i}{\hbar}H_0T}$ exactly as an integral over piecewise geodesic paths with only a few pieces — need to take a limit as the number of pieces goes to infinity.

Now let's return to our general story where we have a category C whose objects are thought of as "configurations" and whose morphisms are thought of as "paths". In the example we just saw, objects were points in $\mathbb{R}\times Q$ (spacetime) & a morphism

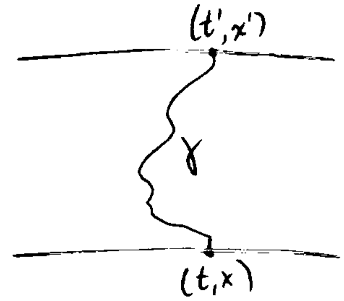
$$\gamma : (t,x) \longrightarrow (t',x')$$

is a path

$$\gamma : [t,t'] \longrightarrow Q$$

s.t.

$$\gamma(t) = x \quad \& \quad \gamma(t') = x'$$



We also have a functor

$$S : C \longrightarrow \mathbb{R}$$

(where $\mathbb{R}$ is viewed as a category with one object with addition as composition) serving as our action, giving

$$e^{iS/\hbar} : C \longrightarrow U(1) \subseteq \mathbb{C}$$

How do we get a Hilbert space in this more general framework? In the examples above, we could use $L^2(Q)$ — but there's no "Q" in general: Q came from our ability to "slice" the set of objects $\mathbb{R} \times Q$ into slices $\{t = \text{const}\}$ called "Cauchy surfaces" — surfaces on which we can freely specify "initial data" (= "Cauchy data") for our wavefunction.