SUPERGEOMETRY

1. Supercalculus

We begin by considering the analog of domains in $\mathbb{R}^p$. A superdomain $\mathcal{U}$ of dimension $p|q$ is a pair a domain $U \subset \mathbb{R}^p$ and the algebra of functions on it

$$C^\infty(\mathcal{U}) = C^\infty(U) \otimes_{\mathbb{R}} \Lambda^*_{\mathbb{R}}(\xi^1, \ldots, \xi^q)$$

Notice that functions on $\mathcal{U}$ are not determined by their values at points in U. The central object in this defintion is not the domain U but the ring of functions on it (this is of course a very useful viewpoint). Each element F in $C^\infty(\mathcal{U})$ can be decomposed as

$$F = \sum_\alpha f_\alpha(u)\xi^\alpha$$

with f_α smooth functions on U. $C^\infty(U)$ sits inside $C^\infty(\mathcal{U})$ as the 'constants' with respect to the grassmann variables.

For each point $x \in U$, there is an evaluation map $\mathrm{ev}_x : C^\infty(\mathcal{U}) \to \mathbb{R}$. $\mathrm{ev}_x(f \otimes 1) = f(x)$ and $ev_x(1 \otimes \xi) = 0$. The kernel of the evaluation map at x gives the maximal ideal m_x. This is no longer just the ideal of functions vanishing at x. It is the ideal generated by the functions vanishing at x and the grassmann variables.

For two superdomains $\mathcal{U}, \mathcal{V}$, a morphism $\phi : \mathcal{U} \to \mathcal{V}$ is a pair (ϕ_0, ϕ^*) where $\phi_0 : U \to V$ is a smooth map and a $\mathbb{R}$-superalgebra homomorphism $\phi^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ with the compatiblity condition that $\phi^*(m_{\phi_0(x)}) \subset m_x$. Notice that this is equivalent to $\mathrm{ev}_{\phi_0(x)}(f) = \mathrm{ev}_x(\phi^* f)$. Those in the know will recognize this as a morphism of locally ringed spaces.

The main computational tool in differential geometry is the notion of coordinates. To specify a mapping between domains in Euclidean spaces one only needs to determine the map in coordinates. Fortunately for us, we have not strayed too far from ordinary differential geometry and this result continues to hold. For a superdomain $\mathcal{U}$, we have a natural choice for coordinates, namely the set (u^i, ξ^i) where (u^i) are the coordinates on $\mathbb{R}^p$.

Lemma

(1) A morphism $\phi : \mathcal{V} \to \mathcal{U}$ is uniquely determined by the image of the coordinate functions (u^i, ξ^i) of $\mathcal{U}$

(2) Take any p even functions $(v^1, \ldots, v^p)$ and q odd functions $(\theta^1, \ldots, \theta^q)$ in $C^\infty(\mathcal{V})$ so that $(\mathrm{ev}_x(v^1), \ldots, ev_x(v^p)) \in U$ for every $x \in V$. Then there exists a morphism $\phi : \mathcal{V} \to \mathcal{U}$ with $\phi^* u^i = v^i$ and $\phi^* \xi^i = \theta^i$.

The key point in the proof of part one is that a morphism that vanishes on the coordinate functions, also vanishes on all polynomials. The image of any function must then lie in $\bigcap_x (m_x)^{q+1} = \{0\}$. For the second part, we generalize the ordinary definition of the pullback of a function in coordinates. We restrict ourselves to the case $1|q$ for ease of exposition. Ordinarily, the pullback of a functions of one real variable $F(y)$ under a map ϕ is given by $\phi^* F(x) = F(\phi(x))$. To extend this definition, we must determine how to 'evaluate' F on terms that have a even combinations of grassmann variables. First we decompose $\phi^* x = y$ into the piece y_{red} lying in $C^\infty(V)$ and a nilpotent piece y_{nil}. We define

$$\phi^* F(y_{\text{red}} + y_{\text{nil}}) = \sum_r \frac{1}{r!} \partial_r F(y_{red}) y_{\text{nil}}^r$$

This enables one to pull back Taylor grassmann series and then one checks that this gives a homomorphism. Extension to larger even dimensions is straightforward.

When we have an isomorphism of superdomains $\phi : \mathcal{V} \to \mathcal{U}$ we will also label as coordinates the image of the standard coordinates on $\mathcal{U}$. The above lemma holds with definition since we can simply reduce to the above case.

The basic object of calculus is the tangent space $T_{\mathcal{U}}$ to the superdomain $\mathcal{U}$. We will view locally free sheaves as the central notion for vector bundles over supermanifolds. Consequently, we declare that the tangent space to $\mathcal{U}$ is the $C^\infty(\mathcal{U})$-module of derivation of the algebra $C^\infty(\mathcal{U})$. Just as in the usual calculus setting, we have coordinate vector fields.

$$\frac{\partial}{\partial u^i} f_\alpha(u) \xi^\alpha = \frac{\partial f_\alpha}{\partial u^i} \xi^\alpha$$

$$\frac{\partial}{\partial \xi^i} f_\alpha(u) \xi^\alpha = \alpha_i (-1)^{\tilde{\alpha_1} + \cdots + \tilde{\alpha_{i-1}}} f_\alpha (\xi^1)^{\alpha_1} \cdots (\xi^{i-1})^{\alpha_{i-1}} (\xi^i)^{\alpha_i - 1} (\xi^{i+1})^{\alpha_{i+1}} \cdots (\xi^q)^{\alpha_q}$$

The parity of the coordinate vector is determined by the parity of the coordinate function.

Lemma $T_{\mathcal{U}}$ is a free $C^\infty(\mathcal{U})$-module with basis $(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial \xi^i}$.

Derivations outside of the span of $(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial \xi^i}$ must annihilate all polynomials; so the result follows as in part one of the previous result.

The next object one tackles after the tangent space is its dual, the cotangent space. The cotangent space of a superdomain is free $C^\infty(\mathcal{U})$-module given by

$$\Omega^1(\mathcal{U}) = \text{Hom}(T_{\mathcal{U}}, C^\infty(\mathcal{U}))$$

It has the standard dual basis $(du^i, d\xi^i)$ with du^i even and $d\xi^i$ odd. From here we move to the exterior algebra of $\mathcal{U}$. We have a canonical even derivation $d : C^\infty(\mathcal{U}) \to \Omega^1(\mathcal{U})$ given by $df(X) = Xf$. The exterior algebra of a manifold is commutative as a superalgebra if we declare all one-forms to be odd. Thanks to our definition of the exterior algebra of a super vector space, in $\Lambda^*(\Omega^1(\mathcal{U})) = \Omega^*((\mathcal{U})$ the parity of the $(du^i, d\xi^i)$ are reversed. Consequently, d extends to an odd derivation of $\Lambda^*((\mathcal{U})$ with $d^2 = 0$.

Lemma The complex $(\Omega^*(\mathcal{U}), d)$ is a resolution of $\mathbb{R}$.

This follows from the usual Poincare lemma and an examination of this complex for $\mathbb{R}^{0|1}$, see [2].

For a morphism $\phi : \mathcal{U} \to \mathcal{V}$, and coordinates (y^i) on $\mathcal{V}$ and (x^i) on $\mathcal{U}$ the Jacobian of ϕ is defined as in a familiar manner

$$(J_{xy})_{ij} \frac{\partial}{\partial x^i} \phi^* y^j$$

The standard statement of the implicit function theorem holds with this definition; from this, the multitude of corollaries follow, e.g. the inverse function theorem, the rank theorem, etc. See [4].

Another result whose validity extends into the superrealm is the Frobenius theorem. See [2]. This allows us to define flows of nonvanishing vector fields.

The Lie derivative for functions and vector fields is defined in the same way as in the ordinary case. It extends to sections of the tensor products of powers of the tangent and cotangent bundles as a derivation that respects the usual contractions.

This covers the majority of differential calculus. The next object of interest is integral calculus on supermanifolds, or more precisely, just integration. So what do we integrate? The answer is sections of the Berezinian of the cotangent bundle, $B(\Omega^1(\mathcal{U})$. In the superrealm, there is usually no top power exterior power of a free module, hence no top degree differential forms. What we need for integration is a proper transformation law under change of coordinates. Sections of $B(\Omega^1(\mathcal{U})$ have this property. Given coordinates (u^i, ξ^i) on $\mathcal{U}$ we get a section $[du^1 \dots d\xi^q]$ of $B(\Omega^1(\mathcal{U})$. For $F = \sum_\alpha f_\alpha(u)\xi^\alpha$ we set

$$\int_{\mathcal{U}} F[du^1 \dots d\xi^q] = (-1)^n m \int_U f_{1,\dots,1} du^1 \wedge \cdots \wedge du^p$$

Of course, this only makes sense for compactly supported sections. We leave it as an exercise to check that the integral is invariant under oriented change of coordinates, see [4].

Example On $\mathbb{R}^{0|1}$ we have $\int_{\mathbb{R}^{0|1}} \xi d\xi = 1$ whereas $\int_{\mathbb{R}^{0|1}} d\xi = 0$.

One often views integration against the grasmmann components as being akin to differentiation since only the term with all nonzero constant grassmann variables.

2. Supermanifolds

This section is quite short because after the definition, most everything is a straightforward globalization of the local notions defined (and presented in an invariant manner) above. So first the definition.

A supermanifold of dimension $p|q$ is a pair $(M, \mathcal{O}_M)$ with M a smooth manifold and a supercommutative sheaf of rings $\mathcal{O}_M$ so that $(M, \mathcal{O}_M$ is locally isomorphic (as a locally ringed manifold) to a superdomain $\mathcal{U}$ of $\mathbb{R}^{p|q}$. Morphisms of supermanifolds are morphisms as locally ringed manifolds. Some examples are below.

Example

(1) Superdomains are supermanifolds.
(2) Manifolds are supermanifolds. The sheaf is simply the sheaf of smooth functions with even parity. Also, if $(M, \mathcal{O}_M)$ (or just $\mathcal{M}$) is a supermanifold, there is a natural quotient $\mathcal{O}_M \to \mathcal{O}_M/\mathcal{N}$ where $\mathcal{N}$ is the ideal sheaf of nilpotents. This induces a morphism of supermanifolds $M \hookrightarrow \mathcal{M}$ which embeds the underlying manifold M into the supermanifold $\mathcal{M}$.
(3) A super Lie group is a supermanifold $\mathcal{G}$ with multiplication $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and inversion $i : \mathcal{G} \to \mathcal{G}$ morphisms and an identity element $e : \mathbb{R}^{0|0} \to \mathcal{G}$ so that they satisfy the usual relations.
(4) An oft useful way of viewing supermanifolds is by the collection of morphism sets into $\mathcal{M}$. More precisely, instead of focusing on the space itself we look at the set of morphisms $\mathrm{Hom}(\mathcal{B}, \mathcal{M})$ for each supermanifold $\mathcal{B}$. A super Lie group then can be defined as a functorial assignment of group structures to the sets $\mathrm{Hom}(\mathcal{B}, \mathcal{G})$.

References

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