A FEW BRST BICOMPLEXES

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Abstract

There are many double complexes in the mathematics and physics literature which are related to BRST transformations and anomalies, e.g. Variational-, BRST-, Faddeev-, Koszul-Tate-, Weil-, Gelfand-Fuks-, semi infinite-, Cech-DeRham-, foliation-, Lie group/algebra-bicomplex. The goal is to identify them and to establish relations between them; to compute their cohomologies and the corresponding anomalies. Here we list and identify only few of them and refer to [11] for more details.

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1 Variational bicomplex

Of all the bicomplexes mentioned above, the variational bicomplex is the "smallest" in the following sense. Let $\pi : P \to M^n$ be a fiber bundle and consider the jet bundle $J^\infty(\pi)$. Its exterior forms are bigraded by horizontal (r) and vertical (s) degrees

$$\Omega^p(J^\infty(\pi)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(\pi)) , \qquad d = d_H + d_V : \Omega^{r,s} \to \Omega^{r+1,s} \oplus \Omega^{r,s+1}.$$

Horizontal and vertical derivatives satisfy Poincare lemmas and $d_H^2 = 0$, $d_V^2 = 0$, $d_H d_V = -d_V d_H$, hence $d^2 = 0$. With $\mathcal{F}^s = \{\omega \in \Omega^{n,s} | I(\omega) = \omega\}$ (functional forms), I the Euler operator we have the augmentad variational bicomplex [1]:

$$\begin{array}{ccccccccccccc}
 & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V & & \\
0 & \longrightarrow & \Omega^{0,3} & & & & \cdots & & \Omega^{n,3} & \xrightarrow{I} & \mathcal{F}^3 & \longrightarrow & 0 \\
 & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V & & \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} \cdots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{I} & \mathcal{F}^2 & \longrightarrow & 0 \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V & & \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} \cdots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{I} & \mathcal{F}^1 & \longrightarrow & 0 \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & & & \\
\mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} \cdots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} & & & &
\end{array}$$

The *local* cohomologies $H^p(\pi) = ker\ d\ /im\ d$ can be computed explicitly using spectral sequences [1]. These are *local* cohomologies because for any $\omega \in \Omega^{r,s}(J^\infty(\pi))$ we have $\omega(j, X_1, \cdots, X_s) \in \Omega^r(M)$, for any jet $j \in J^\infty(\pi)$ and vector fields X_i on $J^\infty(\pi)$.

2 BRST bicomplex

The BRST bicomplex described in [2],[9],[10] is related to the variational bicomplex as follows: Let $\pi : P \to$ M be a principal $G-$bundle and π^p : $\Omega^p(P, LieG) \to M$ Lie algebra valued equivariant p-forms. Let $\mathcal{G}$ denote the Lie group of gauge transformations and $Lie\mathcal{G}$ its Lie algebra. Set $\mathbf{C}^{q,p}_{loc} =$

$\Lambda^q(Lie\mathcal{G}, \Omega^{p,0}(J^\infty(\pi^p))$ (local q - cochains) and define $\delta_{loc} : \mathbf{C}^{q,p}_{loc} \to \mathbf{C}^{q+1,p}_{loc}$ to be the Chevalley-Eilenberg coboundary operator with respect to a representation ρ:

$$\begin{aligned}(\delta_{loc}\phi)(\xi_0, \cdots, \xi_q) &= \sum_{i=0}^{q}(-1)^i \rho'(\xi_i)\phi(\xi_0, \cdots, \hat{\xi}_i, \cdots, \xi_q) \\ &+ \sum_{i<j}(-i)^{i+j}\phi(\rho'(\xi_i)\xi_j, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_q),\end{aligned}$$

where ρ' is the derived representation of $Lie\mathcal{G}$ on $\Omega(P, LieG)$ induced by a representation ρ of G . We have $\delta^2_{loc} = 0$. Then we define the BRST operator $\mathbf{s} : \mathbf{C}^{q,p}_{loc} \to \mathbf{C}^{q+1,p}_{loc}$ as

$$\mathbf{s} \equiv \frac{(-1)^{p+1}}{q+1}\delta_{loc} \ .$$

It is clear that $\mathbf{s}$ is nilpotent, $\mathbf{s}^2 = 0$. We call the associated *local* cohomology of this bicomplex $\{\mathbf{C}^{q,p}_{loc}, \mathbf{s}\}$ the *BRST cohomology* of the gauge algebra $Lie\mathcal{G}$, denoted by $H^*_{BRST}(Lie\mathcal{G})$. In [9],[10] we derived the classical BRST transformations using the Chevalley-Eilenberg constriction for the *adjoint* representation:

$$\mathbf{s}A = d\eta + [A, \eta] \ , \qquad \mathbf{s}\eta = -\frac{1}{2}[\eta, \eta] \ , \qquad \mathbf{s}\bar{\eta} = b \ , \qquad \mathbf{s}b = 0 \ ,$$

where the vector potential $A \in \mathbf{C}^{0,1}_{loc}$ and the ghost field $\eta \in \mathbf{C}^{1,0}_{loc}$ is the Maurer Cartan form on $\mathcal{G}$. We derive a homotopy formula on this bicomplex and with the introduction of Chern-Simons type forms $\omega^{i-1}_{2q-i} = a_i p(A, [A, A]^{i-1}, F^{q-1}_A)$ we obtain the associated descent equations $\delta\omega^0_{2q-1} = -d\omega^1_{2q-2}$, $\delta\omega^1_{2q-2} = -d\omega^2_{2q-3}$, ... , $\delta\omega^{2q-1}_0 = 0$. We identify the non-Abelian anomaly, which automatically satisfies the Wess-Zumino consistency condition, as a cohomology class in $H^1_{loc}(Lie\mathcal{G})$ represented by ω^1_{2q-2} in $n = 2q-2$ dimensions.

For example, for $q = 2, q = 3$ we get the 2- and 4-dimensional non-Abelian anomaly respectively, represented by $\omega^1_2 = Tr(\eta\delta_{loc}\tilde{A})$, and $\omega^1_4 = Tr(\eta\delta_{loc}(\tilde{A}\delta_{loc}\tilde{A} + \frac{2}{3}\tilde{A}^3))$ resp., where $\tilde{A} = A + \eta$.

3 Faddeev's bicomplex

Let $\pi : (P, G) \to M$ be a principal bundle and consider $G^p = \{f : S^p \to G | \infty \to 1\}$, the space of p-loops. We have the exterior derivative $d : \Omega^q(P \times G^p) \to \Omega^{q+1}(P \times G^p)$ and the simplicial group coboundary operator $\Delta : \Omega^{q-p}(P \times G^p) \to \Omega^{q-p}(P \times G^{p+1})$ induced by $\Delta_i : P \times G^{p+1} \to P \times G^p : (x, g_1, \cdots, g_{p+1}) \mapsto (x, g_1, \cdots, g_i g_{i+1}, \cdots, g_{p+1})$.

For example for S^3 the Chern-Simon form is $\omega_5^0 = Tp(A)$, where $dTp(A) = p(F) = Trace\ F^3$ (p=invar. polynomial, T= transgression). We get the stair case equations [12]:

$$
\begin{array}{c|ccccccc}
\text{q=3} & \text{Tr } F^3 & & & & & & \\
 & \uparrow \text{d} & & & & & & \\
\text{q=2} & \omega_5^0 & \xrightarrow{\Delta} & \bullet & & & & \\
 & & & \uparrow \text{d} & & & & \\
\text{q=1} & & & \omega_4^1 & \xrightarrow{\Delta} & \bullet & & \\
 & & & & & \uparrow \text{d} & & \\
\text{q=0} & & & & & \omega_3^2 & \xrightarrow{\Delta} & 0 \\
\hline
 & \text{p=0} & & \text{p=1} & & \text{p=2} & & \text{p=3}
\end{array}
$$

ω_3^2 represents the anomaly.

4 Koszul-Tate complex

Let M be a Poisson manifold with a Hamiltonian G action. Extend the momentum map $J : LieG \to C^\infty(M)$ to a super derivative δ and extend the Lie algebra d, $d : \Lambda LieG \otimes C^\infty(M) \to Lie^*G \otimes (\Lambda LieG \otimes C^\infty(M))$ defined by $dk(\xi) = \xi \cdot k$, ($\cdot$ = repres. of $LieG$ on $\Lambda LieG \otimes C^\infty(M)$) to $\tilde{d}$ such that we have

$$\begin{array}{ccc} \Lambda^p Lie^*G \otimes \Lambda^q LieG \otimes C^\infty(M) & \xrightarrow{\delta} & \Lambda^p Lie^*G \otimes \Lambda^{q-1} LieG \otimes C^\infty(M) \\ \tilde{d}\big\downarrow & & \\ \Lambda^{p+1} Lie^*G \otimes \Lambda^q LieG \otimes C^\infty(M) & & \end{array}$$

δ and $\tilde{d}$ being defined by

$$\delta(\omega \otimes \xi \otimes 1) = \omega \otimes 1 \otimes J(\xi) \ , \ \delta(\omega \otimes 1 \otimes f) = \omega \otimes 0 \ , \ \tilde{d}(\omega \otimes k) = d\omega \otimes k + (-1)^p \wedge dk.$$

We have $\delta^2 = 0, \tilde{d}^2 = 0$ and $\tilde{d}\delta = \delta\tilde{d}$. The total differential defines the BRST operator $D = \tilde{d} + (-1)^p 2\delta : \mathcal{C}^k \to \mathcal{C}^{k+1}$, satisfying nilpotency $D^2 = 0$, where $\mathcal{C}^k = \sum_{p-q=k} \Lambda^p Lie^*G \otimes \Lambda^q LieG \otimes C^\infty(M)$.

The functions on the reduced phace space are given by the cohomology [8]

$$C^\infty(J^{-1}(0)/G) = H_D(\Lambda Lie^*G \otimes \Lambda LieG \otimes C^\infty(M)) \ .$$

which equals the space $E_2^{0,0}$ of the associated spectral sequence.

5 Weil complex

Let $\Lambda(Lie^*\mathcal{G})$ be the exterior algebra and $\mathbf{S}(Lie^*\mathcal{G})$ the symmetric algebra of $Lie\mathcal{G}$, the Lie algebra of infinitesimal gauge transformations. The Weil algebra $W(Lie\mathcal{G}) \equiv \Lambda(Lie^*\mathcal{G}) \otimes \mathbf{S}(Lie^*\mathcal{G})$ is a graded differential $\mathcal{G}$ algebra

$$W(Lie\mathcal{G}) = \sum_r W^r \ , \ W^r = \sum_{p+2q=r} \Lambda^p(Lie^*\mathcal{G}) \otimes S^q(Lie^*\mathcal{G}).$$

Let $\{e_\alpha\}$ be a basis of $Lie\mathcal{G}$ and $\{\theta^\alpha\}$ its dual basis of $Lie^*\mathcal{G}$, and let $\{u^\alpha\}$ be a basis of $\mathbf{S}(Lie^*\mathcal{G})$. The antiderivation δ_W of degree 1 on $W(Lie\mathcal{G})$ is given by $\delta_W = \delta_\Lambda + \delta_S + h$, where $\delta_\Lambda : W^p \to W^{p+1}$ is given by: for $\phi \in \Lambda^p(Lie\mathcal{G}), \quad x_i \in Lie\mathcal{G}$

$$(\delta_\Lambda \phi)(x_0, \cdots, x_p) = \sum_{\nu<\mu} (-1)^{\nu+\mu} \phi([x_\nu, x_\mu], x_0, \cdots, \hat{x}_\nu, \cdots, \hat{x}_\mu, \cdots, x_p),$$

or $\delta_\Lambda = \frac{1}{2} \sum_\alpha \mu(\theta^\alpha) L_\Lambda(e_\alpha)$, where L_Λ is the Lie derivative and $\mu(a)b \equiv a \wedge b$. We have $\delta_\Lambda^2 = 0$. Moreover $\delta_S = \sum_\alpha \mu(\theta^\alpha) L_S(e_\alpha) : W^p \to W^{p+1}$. Note that

$\delta_s^2 \neq 0$ but $(\delta_\Lambda + \delta_S)^2 = 0$, so $\delta_S^2 = -(\delta_\Lambda\delta_S + \delta_S\delta_\Lambda)$. The operator $h : W^p \to W^{p+1}$ is defined by $h = \sum_\alpha \mu_S(\theta^\alpha) \otimes i_A(e_\alpha)$ and is an antiderivative of degree 1 ($i_A =$ interior product). The BRST operator is the total differential $\delta_W = \delta_\Lambda + \delta_S + h$. The associated anomalies in $H^1(Lie\mathcal{G})$ can be computed explicitly [7].

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