A FEW BRST BICOMPLEXES

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Abstract

There are many double complexes in the mathematics and physics literature which are related to BRST transformations and anomalies, e.g. Variational-, BRST-, Faddeev-, Koszul-Tate-, Weil-, Gelfand-Fuks-, semi infinite-, Cech-DeRham-, foliation-, Lie group/algebra-bicomplex. The goal is to identify them and to establish relations between them; to compute their cohomologies and the corresponding anomalies. Here we list and identify only few of them and refer to [11] for more details.

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1 Variational bicomplex

Of all the bicomplexes mentioned above, the variational bicomplex is the "smallest" in the following sense. Let $\pi : P \to M^u$ be a fiber bundle and consider the jet bundle $J^\infty(\pi)$. Its exterior forms are bigraded by horizontal (r) and vertical (s) degrees

$$\Omega^p(J^\infty(\pi)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(\pi)),$$

$$d = d_H + d_V : \Omega^{r,s} \to \Omega^{r+1,s} \oplus \Omega^{r,s+1}.$$  

Horizontal and vertical derivatives satisfy Poincare lemmas and $d_H^2 = 0$, $d_V^2 = 0$, $d_H d_V = - d_V d_H$, hence $d^2 = 0$. With $\mathcal{F}^s = \{ \omega \in \Omega^{n,s} | I(\omega) = \omega \}$ (functional forms), $I$ the Euler operator we have the augmented variational bicomplex [1]:

$$\begin{array}{cccc}
0 & \to & \Omega^0,3 & \to & \Omega^1,3 & \to & \cdots & \to & \Omega^n,3 & \to & \mathcal{F}^3 & \to & 0 \\
0 & \to & \Omega^0,2 & \to & \Omega^1,2 & \to & \cdots & \to & \Omega^n,2 & \to & \mathcal{F}^2 & \to & 0 \\
0 & \to & \Omega^0,1 & \to & \Omega^1,1 & \to & \cdots & \to & \Omega^n,1 & \to & \mathcal{F}^1 & \to & 0 \\
\mathbb{R} & \to & \Omega^0,0 & \to & \Omega^1,0 & \to & \cdots & \to & \Omega^n,0 & & & \\
\end{array}$$

The local cohomologies $H^p(\pi) = \ker d / \text{im} d$ can be computed explicitly using spectral sequences [1]. These are local cohomologies because for any $\omega \in \Omega^{r,s}(J^\infty(\pi))$ we have $\omega(j, X_1, \cdots, X_s) \in \Omega^r(M)$, for any jet $j \in J^\infty(\pi)$ and vector fields $X_i$ on $J^\infty(\pi)$.

2 BRST bicomplex

The BRST bicomplex described in [2],[9],[10] is related to the variational bicomplex as follows: Let $\pi : P \to M$ be a principal $G$-bundle and $\pi^p : \Omega^p(P, \text{Lie}G) \to M$ Lie algebra valued equivariant p-forms. Let $\mathcal{G}$ denote the Lie group of gauge transformations and $\text{Lie}\mathcal{G}$ its Lie algebra. Set $C_{loc}^{q,p} =$
\[ \Lambda^q(\text{Lie} G, \Omega^p, (J^\infty(\pi^p))) \] 
(local \( q \) - cochains) and define \( \delta_{\text{loc}} : C^q_{\text{loc}} \rightarrow C^{q+1}_{\text{loc}} \) to be the Chevalley-Eilenberg coboundary operator with respect to a representation \( \rho \):

\[
(\delta_{\text{loc}} \phi)(\xi_0, \cdots, \xi_q) = \sum_{i=0}^{q} (-1)^i \rho'(\xi_i) \phi(\xi_0, \cdots, \hat{\xi}_i, \cdots, \xi_q) \
+ \sum_{i<j} (-1)^{i+j} \phi(\rho'(\xi_i) \xi_j, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_q),
\]

where \( \rho' \) is the derived representation of \( \text{Lie} G \) on \( \Omega(\mathbb{P}, \text{Lie} G) \) induced by a representation \( \rho \) of \( G \). We have \( \delta_{\text{loc}}^2 = 0 \). Then we define the BRST operator \( s : C^q_{\text{loc}} \rightarrow C^{q+1}_{\text{loc}} \) as

\[
s \equiv \frac{(-1)^{p+1}}{q+1} \delta_{\text{loc}}.
\]

It is clear that \( s \) is nilpotent, \( s^2 = 0 \). We call the associated local cohomology of this bicomplex \( \{ C^q_{\text{loc}}, s \} \) the \textit{BRST cohomology} of the gauge algebra \( \text{Lie} G \), denoted by \( H^*_{\text{BRST}}(\text{Lie} G) \). In [9],[10] we derived the classical BRST transformations using the Chevalley-Eilenberg constriction for the adjoint representation:

\[
sA = d\eta + [A, \eta], \quad s\eta = -\frac{1}{2}[\eta, \eta], \quad s\bar{\eta} = b, \quad sb = 0,
\]

where the vector potential \( A \in C^0_{\text{loc}} \) and the ghost field \( \eta \in C^1_{\text{loc}} \) is the Maurer Cartan form on \( G \). We derive a homotopy formula on this bicomplex and with the introduction of Chern-Simons type forms \( \omega_{2q-i} = \alpha_p(A, [A, A]^{i-1}, F_A^{q-1}) \) we obtain the associated descent equations \( \delta \omega_{2q-1} = -d \omega_{2q-2}, \delta \omega_{2q-2} = -d \omega_{2q-3}, \ldots, \delta \omega_0 = 0 \). We identify the non-Abelian anomaly, which automatically satisfies the Wess-Zumino consistency condition, as a cohomology class in \( H^1_{\text{loc}}(\text{Lie} G) \) represented by \( \omega_{2q-2} \) in \( n = 2q-2 \) dimensions.

For example, for \( q = 2, q = 3 \) we get the 2- and 4-dimensional non-Abelian anomaly respectively, represented by \( \omega_2 = Tr(\eta \delta_{\text{loc}} A) \) and \( \omega_4 = Tr(\eta \delta_{\text{loc}} (A \delta_{\text{loc}} A + \frac{2}{3} A^3)) \) resp., where \( A = A + \eta \).
3 Faddeev’s bicomplex

Let $\pi : (P, G) \to M$ be a principal bundle and consider $G^p = \{ f : S^p \to G | \infty \to 1 \}$, the space of $p$-loops. We have the exterior derivative $d : \Omega^q(P \times G^p) \to \Omega^{q+1}(P \times G^p)$ and the simplicial group coboundary operator $\Delta : \Omega^{q-p}(P \times G^p) \to \Omega^{q-p}(P \times G^{p+1})$ induced by $\Delta : P \times G^{p+1} \to P \times G^p : (x, g_1, \cdots, g_{p+1}) \mapsto (x, g_1, \cdots, g_{i+1}, \cdots, g_{p+1})$.

For example for $S^3$ the Chern-Simon form is $\omega^0 = Tp(A)$, where $dT p(A) = p(F) = Trace \, F^3$ ($p=$ invar. polynomial, $T=$ transgression). We get the stair case equations [12]:

$$
\begin{array}{c|ccc}
q=3 & \text{Tr} & F^3 \\
q=2 & d & \omega^0 & \Delta \\
q=1 & d & \omega^1 & \Delta \\
q=0 & d & \omega^2 & \Delta & 0 \\
p=0 & & & & \\
p=1 & & & & \\
p=2 & & & & \\
p=3 & & & & \\
\end{array}
$$

$\omega^2$ represents the anomaly.

4 Koszul-Tate complex

Let $M$ be a Poisson manifold with a Hamiltonian $G$ action. Extend the momentum map $J : \text{Lie}G \to C^\infty(M)$ to a super derivative $\delta$ and extend the Lie algebra $d$, $d : \Lambda \text{Lie}G \otimes C^\infty(M) \to \Lambda \text{Lie}G \otimes (\Lambda \text{Lie}G \otimes C^\infty(M))$ defined by $dk(\xi) = \xi \cdot k$, $\cdot =$ repres. of $\text{Lie}G$ on $\Lambda \text{Lie}G \otimes C^\infty(M)$ to $d$ such that we have
\[ \Lambda^p \text{Lie}^* G \otimes \Lambda^q \text{Lie} G \otimes C^\infty(M) \xrightarrow{\delta} \Lambda^p \text{Lie}^* G \otimes \Lambda^{q-1} \text{Lie} G \otimes C^\infty(M) \]

\[ \Lambda^{p+1} \text{Lie}^* G \otimes \Lambda^q \text{Lie} G \otimes C^\infty(M) \]

\( \delta \) and \( \tilde{d} \) being defined by

\[ \delta(\omega \otimes \xi \otimes 1) = \omega \otimes 1 \otimes J(\xi) , \quad \delta(\omega \otimes 1 \otimes f) = \omega \otimes 0 , \quad \tilde{d}(\omega \otimes k) = d\omega \otimes k + (-1)^p \omega \otimes k \cdot d. \]

We have \( \delta^2 = 0, \tilde{d}^2 = 0 \) and \( \tilde{d} \delta = \delta \tilde{d} \). The total differential defines the BRST operator \( D = \tilde{d} + (-1)^p 2\delta : C^k \rightarrow C^{k+1} \), satisfying nilpotency \( D^2 = 0 \), where \( C^k = \sum_{p-q=k} \Lambda^p \text{Lie}^* G \otimes \Lambda^q \text{Lie} G \otimes C^\infty(M) \).

The functions on the reduced phase space are given by the cohomology \[ C^\infty(J^{-1}(0)/G) = H_D(\Lambda \text{Lie}^* G \otimes \Lambda \text{Lie} G \otimes C^\infty(M)) \]

which equals the space \( E^{0,0}_2 \) of the associated spectral sequence.

5 Weil complex

Let \( \Lambda(\text{Lie}^* \mathcal{G}) \) be the exterior algebra and \( S(\text{Lie}^* \mathcal{G}) \) the symmetric algebra of \( \text{Lie} \mathcal{G} \), the Lie algebra of infinitesimal gauge transformations. The Weil algebra \( W(\text{Lie} \mathcal{G}) = \Lambda(\text{Lie}^* \mathcal{G}) \otimes S(\text{Lie}^* \mathcal{G}) \) is a graded differential \( \mathcal{G} \) algebra

\[ W(\text{Lie} \mathcal{G}) = \sum_r W^r , \quad W^r = \sum_{p+2q=r} \Lambda^p (\text{Lie}^* \mathcal{G}) \otimes S^q(\text{Lie}^* \mathcal{G}) \].

Let \( \{e_a\} \) be a basis of \( \text{Lie} \mathcal{G} \) and \( \{\theta^a\} \) its dual basis of \( \text{Lie}^* \mathcal{G} \), and let \( \{u^a\} \) be a basis of \( S(\text{Lie}^* \mathcal{G}) \). The antiderivation \( \delta_W \) of degree 1 on \( W(\text{Lie} \mathcal{G}) \) is given by \( \delta_W = \delta_A + \delta_S + h \), where \( \delta_A : W^p \rightarrow W^{p+1} \) is given by: for \( \phi \in \Lambda^p(\text{Lie} \mathcal{G}) \), \( x_i \in \text{Lie} \mathcal{G} \)

\[ (\delta_A \phi)(x_0, \ldots, x_p) = \sum_{\nu < \mu} (-1)^{\nu + \mu} \phi([x_\nu, x_\mu], x_0, \ldots, \hat{x}_\nu, \ldots, \hat{x}_\mu, \ldots, x_p), \]

or \( \delta_A = \frac{1}{2} \sum_a \mu(\theta^a)L_A(e_a) \), where \( L_A \) is the Lie derivative and \( \mu(a)b \equiv a \wedge b \).

We have \( \delta_A^2 = 0 \). Moreover \( \delta_S = \sum_a \mu(\theta^a)L_S(e_a) : W^p \rightarrow W^{p+1} \). Note that
\( \delta^2 \neq 0 \) but \((\delta_\Lambda + \delta_S)^2 = 0 \), so \( \delta_S^2 = - (\delta_\Lambda \delta_S + \delta_S \delta_\Lambda) \). The operator \( h : W^p \to W^{p+1} \) is defined by \( h = \sum_{\alpha} \mu_S(\theta^\alpha) \otimes i_A(\epsilon_\alpha) \) and is an antiderivative of degree 1 \( (i_A = \text{interior product}) \). The BRST operator is the total differential \( \delta_W = \delta_\Lambda + \delta_S + h \). The associated anomalies in \( H^1(\text{Lie} G) \) can be computed explicitly [7].

**References**


