LOCAL COHOMOLOGY IN GAUGE THEORIES
BRST TRANSFORMATIONS AND ANOMALIES

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Abstract

We introduce a geometric framework needed for a mathematical understanding of the BRST symmetries and chiral anomalies in gauge field theories. We define the BRST bicomplex in terms of local cohomology using differential forms on the infinite jet bundle and consider variational aspects of the problem in this cohomological context. The adjoint representation of the structure group induces a representation of the infinite dimensional Lie algebra $\mathbf{g}$ of infinitesimal gauge transformations on the space of local differential forms, with respect to which the BRST bicomplex is defined using the Chevalley-Eilenberg construction. The induced coboundary operator of the associated cohomology $H^*_{loc}(\mathbf{g})$ is the BRST operator $\mathbf{s}$. With this we derive the classical BRST transformations of the vector potential A and the ghost field η as $\mathbf{s}A = d\eta + [A, \eta]$, and $\mathbf{s}\eta = -1/2[\eta, \eta]$. Moreover the ghost field η is identified with the canonical Maurer- Cartan form of the infinite dimensional Lie group $\mathbf{G}$ of gauge transformations. We give a homotopy formula on the BRST bicomplex and with the introduction of Chern-Simon type forms we derive the associated descent equations and show that the non-Abelian anomalies, which satisfy the Wess-Zumino consistency condition, represent cohomology classes in $H^1_{loc}(\mathbf{g})$.

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1 Introduction

In recent years various cohomological ideas have been introduced in quantum field theories and string theories to explain BRST transformations and anomalies from a purely algebraic or differential geometric point of view [3], [6], [8], [11], [12], [17]. It was first noticed by Becchi, Rouet and Stora [2] and Tyutin [22] (unpublished) that in gauge field theories the effective action, which is no longer gauge invariant, has a new global symmetry , now called BRST symmetry. This BRST transformation $\mathbf{s}$ mixes the ghost fields with the other fields and has proved to be a very important tool in the quantization of gauge theories. The classical BRST transformations of a vector potential A and a ghost field η are given by [2],[13]:

$$\mathbf{s}A = d\eta + [A, \eta] \ , \quad \mathbf{s}\eta = -\frac{1}{2}[\eta, \eta] \ \ .$$

The main property of the BRST transformation is its nilpotency $\mathbf{s}^2 = 0$ which is the key for unitarity of the S-matrix. Anomalies were first discovered by Adler, Bardeen, Bell, Jackiw and Schwinger as quantum effects of conservation laws; e.g. in QED the Noether current associated to the chiral symmetry is conserved at the classical level but is not conserved after quantizing the theory. There are many different descriptions to find these anomalies, the original one was by perturbation theory using Feynman diagrams. Later one noticed that the Adler - Bardeen anomaly is related to the index of the Dirac operator and that it has a topological interpretation; namely, the anomaly is a reflection of the non vanishing of a certain cohomology of the gauge group. A representation of the anomaly can be obtained by applying the BRST operator $\mathbf{s}$ to the vacuum functional. If the difference of two anomalies is the variation of a *local* functional then the two anomalies have to be considered as physically equivalent. The anomaly ω must satisfy certain properties which follow from the structure relations of the gauge group, called the Wess - Zumino consistency condition $\mathbf{s}(\eta \cdot \omega) = 0$. This and the nilpotency of the BRST operator $\mathbf{s}^2 = 0$ lead to the consideration of local cohomologies of the group of gauge transformations.

Our construction is more general than the more algebraic ones given by Bonora and Cotta-Ramusino [5], Kastler and Stora [11], Dubois-Violette [6] in the sense that in our local cohomology the functionals $\Phi(\xi_1, \cdots, \xi_q)$ need not be polynomials in derivatives of the fields ξ_i but they can be differential or pseudodifferential operators. They are *local* in the physical sense [13],[20],[24], i.e. they can be any functional of the $\xi_i s$ and of finitely many derivatives of them, which is the way they occur in physical examples. This notion of locality is defined in terms of infinite jet bundles. Moreover, the more general definition of the BRST cohomology with respect to arbitrary representations (not just the adjoint) of the gauge algebra allows new, different kinds of anomalies which might be of physical interest in the future. These are currently under investigation.

2 Local Cohomology

Let $\pi : B \to M$ be a fixed smooth fiber bundle and let $\Gamma^\infty(\pi)$ denote the manifold of smooth sections of π . The spaces $J^k(\pi)$ of k - *jets* , $0 \le k \le \infty$, of local sections of π are smooth manifolds and we have the canonical projections $\pi^l_k : J^k(\pi) \to J^l(\pi)$, $0 \le l \le k$, and $\pi_k : J^k(\pi) \to M$, as well as the k-jet extension maps $j^k : M \times \Gamma^\infty(\pi) \to J^k(\pi); j^k(x,s) = [x,s]_k$ the k-jet equivalence class of (x,s). Note that $J^0(\pi) = B$ and $\pi_0 = \pi$.

There is a natural splitting of the tangent space $T_s J^\infty(\pi) = H_s \oplus V_s$ at each $s \in J^\infty(\pi)$ and hence of the space $\mathbf{X}(J^\infty(\pi))$ of vector fields on $J^\infty(\pi)$: $\mathbf{X}(J^\infty(\pi)) = \mathbf{H} \oplus \mathbf{V}$ as follows: $\mathbf{H}$ is the space of *horizontal* vector fields, i.e. lifts of vector fields $\bar{X}$ on M ; $\bar{X} \in \mathbf{X}(M) \mapsto X \in \mathbf{X}(J^\infty(\pi))$ defined by $X(f)(s) = \bar{X}(f \circ S)(\pi_\infty(s))$ where $f \in C^\infty(J^\infty(\pi))$, $s \in J^\infty(\pi)$ and S is a local section at $\pi_\infty(s)$. The subspace $\mathbf{V}$ is the space of *vertical* vector fields on $J^\infty(\pi)$; i.e. $Y \in \mathbf{V}$ if and only if $Y(f \circ \pi_\infty) = 0$ for all $f \in C^\infty(M)$. It should be remarked that such a canonical splitting of $\mathbf{X}(\mathbf{J}^\infty(\pi)) = \mathbf{H} \oplus \mathbf{V}$ cannot be constructed for $J^k(\pi)$ if $k < \infty$.

We denote by $\Omega^q_p(\pi)$ the vector space of those $(q+p)$-forms ω on $J^\infty(\pi)$ with $\omega(X_1, \cdots, X_{q+p}) = 0$ if more than q of the vector fields X_i , $1 \le i \le q+p$, are vertical or more than p of them are horizontal. Elements of $\Omega^q_p(\pi)$ are called *local forms* on $J^\infty(\pi)$. If $\omega \in \Omega^q_p(\pi)$ then $d\omega \in \Omega^{q+1}_p(\pi) \oplus \Omega^q_{p+1}(\pi)$, i.e. $d : \Omega^q_p(\pi) \to \Omega^{q+1}_p(\pi) \oplus \Omega^q_{p+1}(\pi)$ and we can define the *vertical* exterior derivative $\partial : \Omega^q_p(\pi) \to \Omega^{q+1}_p(\pi)$ and the *horizontal* exterior derivative D : $\Omega^q_p(\pi) \to \Omega^q_{p+1}(\pi)$ by $d = \partial + D$. Then $d^2 = D^2 = \partial^2 = D\partial + \partial D = 0$. This bicomplex of local forms is often called the *variational bicomplex,* (see eg. Anderson [1], Saunders [16]).

There is another characterization of local forms, which justifies their names. Consider the de Rham complex $\Omega(M \times \Gamma^\infty(\pi))$ of smooth differential forms on $M \times \Gamma^\infty(\pi)$ with exterior derivative d . From the product structure of $M \times \Gamma^\infty(\pi)$ the space $\Omega(M \times \Gamma^\infty(\pi)$ inherits a bigradation and we can write

$$\Omega(M \times \Gamma^\infty(\pi)) = \coprod_{p,q} \Omega^{p,q}(M \times \Gamma^\infty(\pi)).$$

Corresponding to this bigradation the exterior derivative d on $M \times \Gamma^\infty(\pi)$ breaks into two operators; D of type $(1,0)$, $D : \Omega^{p,q}(M \times \Gamma^\infty(\pi)) \rightarrow \Omega^{p+1,q}(M \times \Gamma^\infty(\pi))$, and ∂ of type $(0,1)$, $\partial : \Omega^{p,q}(M \times \Gamma^\infty(\pi)) \rightarrow \Omega^{p,q+1}(M \times \Gamma^\infty(\pi))$. With these we have $d = D + \partial$ and $d^2 = D^2 = \partial^2 = D\partial + \partial D = 0$.

If $A \in \Omega^{p,0}(M \times \Gamma^\infty(\pi))$ and $s \in \Gamma^\infty(\pi)$, define a p-form $A(s)$ on M by $A(s)(x) = A(x,s)$, $x \in M$. Then $DA \in \Omega^{p+1,0}(M \times \Gamma^\infty(\pi))$ and we have $(DA)(s) = d_M(A(s))$ where d_M is the exterior derivative on M . More generally, if $A \in \Omega^{p,q}(M \times \Gamma^\infty(\pi))$, $s \in \Gamma^\infty(\pi)$ and $X_1, \cdots, X_q \in \mathbf{X}(J^\infty(\pi))$ we can define a p-form $A(s, X_1, \cdots, X_q)$ on M by $A(s, X_1, \cdots, X_q)(x) = (i_{X_1(s)} \cdots i_{X_q(s)} A)(x,s)$. Again $DA \in \Omega^{p+1,q}(M \times \Gamma^\infty(\pi))$ is given by $(DA)(s, X_1(s), \cdots, X_q(s)) = d_M(A(s, X_1, \cdots, X_q))$.

The bicomplex $\Omega(M \times \Gamma^\infty(\pi))$ has a canonical sub-bicomplex $\Omega_{loc}(M \times \Gamma^\infty(\pi))$ defined as follows: The ∞ - jet extension map $j_\infty : M \times \Gamma^\infty(\pi) \rightarrow J^\infty(\pi)$ induces a map of the de Rham complexes $j_\infty^* : \Omega(J^\infty(\pi)) \rightarrow \Omega(M \times \Gamma^\infty(\pi))$. The image $j_\infty^* \Omega(J^\infty(\pi))$ in $\Omega(M \times \Gamma^\infty(\pi))$ is stable under both D and ∂ , and hence is a sub-bicomplex which we denote by $\Omega_{loc}(M \times \Gamma^\infty(\pi))$. We write

$$\Omega_{loc}(M \times \Gamma^\infty(\pi)) = \coprod_{p,q} \Omega_{loc}^{p,q}(M \times \Gamma^\infty(\pi)).$$

The map j_∞^* induces an isomorphism of bicomplexes between local forms in $\Omega(J^\infty(\pi))$ and $\Omega_{loc}(M \times \Gamma^\infty(\pi))$, i.e. $\Omega_p^q(\pi) \cong \Omega_{loc}^{p,q}(M \times \Gamma^\infty(\pi))$.

We call a form A on $M \times \Gamma^\infty(\pi)$ *local* if A lies in $\Omega_{loc}(M \times \Gamma^\infty(\pi))$. Thus if $A \in \Omega_{loc}^{p,q}(M \times \Gamma^\infty(\pi))$, then for $s \in \Gamma^\infty(\pi)$ and $X_1, \cdots, X_q \in \mathbf{X}(J^\infty(\pi))$ the p-form $A(s, X_1, \cdots, X_q)$ on M depends on $s, X_1(s), \cdots, X_q(s)$ in a local fashion, that means $A(s, X_1, \cdots, X_q)(x)$ depends only on finite jets (i.e. finitely many derivatives) of $s, X_1(s), \cdots, X_q(s)$ at x . In local coordinates of M , a local form A can be written as follows:

$$A = \sum_{i,j} A_{i_1 \cdots i_p, j_1 \cdots j_q} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge \partial u_{j_1} \wedge \cdots \wedge \partial u_{j_q}$$

where the coordinates $A_{i_1,\cdots,i_p,j_1,\cdots,j_q}$ are local $(0,0)$-forms, the dx_is are local $(1,0)$-forms and the u_js are local $(0,1)$-forms. This justifies the terminology of local forms.

For the bicomplex of local forms $\Omega^{p,q}_{loc}(M\times\Gamma^\infty(\pi))\cong\Omega^q_p(\pi)$ we have the following exactness theorems [20]: We write in short $\Omega^{p,q}_{loc}$ for $\Omega^{p,q}_{loc}(M\times\Gamma^\infty(\pi))\cong\Omega^q_p(\pi)$, and let $n=dimM$. One sets $\Omega^{p,q}_{loc}=0$ whenever $p>n, p<0$ or $q<0$. For each $\Omega^{p,q}_{loc}$ we denote by $\tilde\Omega^{p,q}_{loc}$ its sheaf of germs of sections.

Poincare Lemma: *If* $\alpha_0\in\tilde\Omega^{m,0}_{loc}$, $\alpha_1\in\tilde\Omega^{m-1,1}_{loc},\cdots,$ $\alpha_m\in\tilde\Omega^{0,m}_{loc}$, $m\geq 1$ *, and* $D\alpha_0=0$, $\partial\alpha_0+D\alpha_1=0,\cdots,\partial\alpha_{m-1}+D\alpha_m=0$ *and* $\partial\alpha_m=0$ *; then there exist* $\beta_0\in\tilde\Omega^{m-1,0}_{loc}$, $\beta_1\in\tilde\Omega^{m-2,1}_{loc},\cdots,\beta_{m-1}\in\tilde\Omega^{0,m-1}_{loc}$ *such that* $D\beta_0=\alpha_0$, $\partial\beta_0+D\beta_1=\alpha_1,\cdots,\partial\beta_{m-2}+D\beta_{m-1}=\alpha_{m-1}$ *, and* $\partial\beta_{m-1}=\alpha_m$.

D - Exactness Theorem: *Let* $\omega\in\tilde\Omega^{p,q}_{loc}$, $0<p<n=dimM$ *, and* $D\omega=0\in\tilde\Omega^{p+1,q}_{loc}$ *. Then there exists an* $\eta\in\tilde\Omega^{p-1,q}_{loc}$ *such that* $D\eta=\omega$ *. However D-closed* $(0,q)$*-forms need not be D-exact ; moreover,* (n,q)*- forms, which are always D-closed , need not be D-exact.* $D:\Omega^{0,q}_{loc}\to\Omega^{1,q}_{loc}$ *is injective for* $q>0$ *, the kernel of* $D:\Omega^{0,0}_{loc}\to\Omega^{1,0}_{loc}$ *consists of locally constant functions in* $\Omega^{0,0}_{loc}$.

∂ - Exactness Theorem: *Let* $\omega\in\tilde\Omega^{p,q}_{loc}$, $q>0$ *, and* $\partial\omega=0$ *. Then there exists an* $\eta\in\tilde\Omega^{p,q-1}_{loc}$ *such that* $\partial\eta=\omega$ *. Moreover* $\partial:\tilde\Omega^{n,0}_{loc}/D\tilde\Omega^{n-1,0}_{loc}\to\tilde\Omega^{n,1}_{loc}/D\tilde\Omega^{n-1,1}_{loc}$ *is injective.*

We have the following useful cohomology result for the (p,q)-th D-cohomology groups [20]:

$$H^{p,q}_D=\frac{Ker(D:\Omega^{p,q}_{loc}\to\Omega^{p+1,q}_{loc})}{Im(D:\Omega^{p-1,q}_{loc}\to\Omega^{p,q}_{loc})}\cong\begin{cases}0 & ,\quad \text{if}\quad q\neq 0,\quad 0<p\leq n\\ H^p(B,\mathbf{R}) & ,\quad \text{if}\quad q=0,\quad 0\leq p\leq n\ .\end{cases}$$

Remark 1: Classical field theories can be formulated in terms of local forms as follows [20]: A *variational problem* or *Lagrangian* on the fiber bundle $\pi:B\to M$ is an operator L which assigns to each smooth local section $s:M\to B$ an n-form $L(s)$ on the domain of s , such that $L(s)(x)$ only depends smoothly on the value of $s(x)$ and on a finite number of derivatives $D^js(x)$, $0\leq j\leq k<\infty$. In our formulation a Lagrangian L on π therefore is an element of $\Omega^{n,0}_{loc}$. Indeed, any $L\in\Omega^{n,0}_{loc}$ defines an n-form $L(s)$ on M by

$L(s)(x) = L(s,x)$ which is local in the sense above. Interpreting this n-form $L(s)$ as Lagrangian density (we fix a volume form on M) the *action* $\mathbf{L}(s)$ in any domain $\mathbf{U} \subset M$ is defined as the integral

$$\mathbf{L}(s) = \int_{\mathbf{U}} L(s).$$

The space $\Omega^{n,1}_{loc}$ has a distinguished subspace $\Omega^{n,1}_{source}$: we call $A \in \Omega^{n,1}_{loc}$ a *source form* if for any $s \in \Gamma^\infty(\pi)$ and $X \in \mathbf{X}(J^\infty(\pi))$ the n-form $A(s, X(s))(x)$ depends only on a finite jet of s and the zero-jet of $X(s)$ at x . We have a direct sum of vector spaces

$$\Omega^{n,1}_{loc} = \Omega^{n,1}_{source} \oplus D\Omega^{n-1,1}_{loc}.$$

If $L \in \Omega^{n,0}_{loc}$ then $\partial L \in \Omega^{n,1}_{loc}$ and we can write

$$\partial L = E + DH.$$

A section $s \in \Gamma^\infty(\pi)$ is an *extremal* for the Lagrangian field theory determined by L if the variation of the action $\delta \int_{\mathbf{U}} L(s) = 0$ for all domains $\mathbf{U}$ in M and all variations $X(s) \in T_s\Gamma^\infty(\pi)$ of s which vanish on the boundary of $\mathbf{U}$; hence

$$\delta \int_{\mathbf{U}} L(s) = \int_{\mathbf{U}} \partial L(s, X(s)) = \int_{\mathbf{U}} E(s, X(s)) = 0$$

which is satisfied if and only if s satisfies the Euler-Lagrange equations

$$E(s, X(s)) = 0$$

for all variations $X(s)$, [20], [24]. (In local coordinates the system $E(s, X(s)) = 0$ is equivalent to the standard Euler-Lagrange equations).

It follows from the ∂-cohomology theorem that each locally variational source equation E is globally variational provided that $H^{n+1}(B, \mathbf{R}) = 0$.

Remark 2: A new universal conserved current for Lagrangian field theories has been defined by Zuckerman [25]: Let $L \in \Omega^{n,0}_{loc}$ be a Lagrangian and write $\partial L = E + DH$. Then the local form $U = \partial H \in \Omega^{n-1,2}_{loc}$ is a conserved current for L (called the universal conserved current [25]). We have $\partial U = 0$ and $DU = D\partial H = -\partial DH = -\partial(\partial L - E) = \partial E$. So if $s \in \Gamma^\infty(\pi)$ is an extremal of L (i.e. s satisfies the Euler-Lagrange equations $E(s, X) = 0$ for all $X \in \mathbf{X}(J^\infty(\pi))$ and hence the Jacobi equations $\partial E(s, X_1, X_2) = 0$ are satisfied for all Jacobi fields X_1, X_2) then $U(s)$ defines a closed $(n-1)$- form on M by $U(s, X_1, X_2)(x) = (i_{X_1(s)}, i_{X_2(s)}U)(x, s) = (i_{X_1(s)}, i_{X_2(s)}\partial H)(x, s)$, and $dU(s, X_1, X_2) = (\partial U + DU)(s, X_1, X_2) = \partial E(s, X_1, X_2) = 0$.

Example: For the Yang-Mills action on any space-time the universal current U is given by $U(A, X_1(A), X_2(A)) = Tr(X_1(A) \wedge *X_2(F_A) - X_2(A) \wedge *X_1(F_A))$, [24].

3 BRST Transformations and Anomalies

The BRST transformation $\mathbf{s}$ on a vector potential A and a ghost field η are given by [2], [13] :

$$\mathbf{s}A = d\eta + [A, \eta] \quad , \qquad \mathbf{s}\eta = -\frac{1}{2}[\eta, \eta] \quad .$$

Moreover $\mathbf{s}$ satisfies the nilpotency condition $\mathbf{s}^2 = 0$.

In [18] we derived these equations as the coboundary operator of the Chevalley-Eilenberg cohomology of the Lie algebra of infinitesimal gauge transformations with respect to the adjoint representation. The ghost field η was identified with the canonical Maurer-Cartan form on the infinite dimensional Lie group of gauge transformations.

We summarize these results: The construction is more general than presented here, i.e. the Chevalley-Eilenberg cohomology can be defined with respect to any representation of the Lie algebra [18], but we restrict ourselves to the adjoint representation because that's the one relevant for the BRST cohomology and the anomalies. In a future paper we will investigate to what physical interpretations the corresponding cohomologies with respect to other representations of the gauge algebra will lead to.

We consider a principal fiber bundle $\pi : P \to M$ with structure group G . Denote by $\mathbf{G}$ the infinite dimensional Lie group of gauge transformations on P called the *gauge group*

$$\mathbf{G} = \{\varphi : P \to G \mid \varphi(p \cdot a) = a^{-1}\varphi(p)a, \quad p \in P, a \in G\} \ .$$

Its Lie algebra $\mathbf{g}$ called the *gauge algebra* is the infinite dimensional Lie algebra of infinitesimal gauge transformations on P

$$\mathbf{g} = \{\xi : P \to g \mid \xi(p \cdot a) = Ad_{a^{-1}}\xi(p), \quad p \in P, a \in G\}$$

where g is the Lie algebra of the structure group G . Denote by $\mathcal{A}$ the space of connection one-forms (or gauge potentials) on P , and let $\Lambda^k(P,g)$ be the space of g-valued, Ad-equivariant k-forms Φ on P , and $\Lambda(P,g) = \sum_k \Lambda^k(P,g)$. We complete these spaces with respect to suitable Sobolev H_s-topologies so that $\Lambda^k(P,g)$ and $\Lambda(P,g)$ are Hilbert spaces and $\mathbf{G}$ becomes a smooth infinite dimensional Hilbert Lie group with Lie algebra $\mathbf{g}$, [15], [18].

We define a representation ρ of $\mathbf{G}$ on $\Lambda(P,g)$ by

$$\rho(\varphi)\Phi = (\varphi^{-1})^*\Phi, \qquad \varphi \in \mathbf{G}, \Phi \in \Lambda(P,g).$$

The induced action of $\mathbf{G}$ on $\Lambda(P,g)$ is smooth since inversion $\varphi \mapsto \varphi^{-1}$ and pull back $\varphi \mapsto \varphi^*$ are both smooth mappings. The derived representation ρ' of $\mathbf{g}$ on the subspace $\Lambda^0(P,g) \cong \mathbf{g}$ is the adjoint representation of $\mathbf{g}$

$$\rho'(\xi)\eta = ad_\xi(\eta) = [\xi,\eta], \quad \xi,\eta \in \mathbf{g}.$$

The induced action of ρ' on $\mathcal{A} \subset \Lambda^1(P,g)$ is given by

$$\rho'(\xi)A = D_A\xi$$

where ξ is identified with the fundamental vector field Z_ξ generated by $\xi \in \mathbf{g}$ and D_A denotes the exterior covariant derivative with respect to $A \in \mathcal{A}$.

Recall the Chevalley-Eilenberg cohomology of a Lie algebra with respect to a representation [4]; in our case of the gauge algebra $\mathbf{g}$ with respect to the representation ρ' : Let $\mathbf{C}^{q,p} = C^q(\mathbf{g}, \Lambda^p(P,g))$ be the space of $\Lambda^p(P,g)$-valued q-cochains on $\mathbf{g}$, let $\mathbf{C}^{0,p} = \Lambda^p(P,g)$ and note that $\mathbf{C}^{1,0} = C^1(\mathbf{g}, \Lambda^0(P,g)) \cong \mathcal{L}(\mathbf{g},\mathbf{g})$. The Chevalley-Eilenberg coboundary operator $\delta : \mathbf{C}^{q,p} \to \mathbf{C}^{q+1,p}$ is given by

$$\begin{aligned}(\delta\Phi)(\xi_0,\cdots,\xi_q) &= \textstyle\sum_{i=0}^q (-1)^i \rho'(\xi_i)\Phi(\xi_0,\cdots,\hat{\xi}_i,\cdots,\xi_q) \\ &+ \textstyle\sum_{i<j} (-1)^{i+j}\Phi([\xi_i,\xi_j],\cdots,\hat{\xi}_i,\cdots,\hat{\xi}_j,\cdots,\xi_q) \ .\end{aligned}$$

For $q = 0$ and $\Phi \in \mathbf{C}^{0,p}$, $\delta\Phi$ is defined by $(\delta\Phi)(\xi) = \rho'(\xi)\Phi$. The coboundary operator δ satisfies $\delta^2 = 0$. We define the BRST *transformation* $\mathbf{s} : \mathbf{C}^{q,p} \to \mathbf{C}^{q+1,p}$ by

$$\mathbf{s} = \frac{(-1)^{p+1}}{q+1}\delta \ .$$

Again $\mathbf{s}$ satisfies $\mathbf{s}^2 = 0$ and we call the cohomology of the complex $\{\mathbf{C}^{q,p}, \mathbf{s}\}$ the *BRST cohomology* of the gauge algebra $\mathbf{g}$, denoted by $H^*_{BRST}(\mathbf{g})$.

Theorem: *Let A be a vector potential and η a ghost field, that means $A \in \mathcal{A} \subset \Lambda^1(P,g) \cong C^0(\mathbf{g}, \Lambda^1(P,g)) = \mathbf{C}^{0,1}$ and $\eta \in \mathcal{L}(\mathbf{g},\mathbf{g}) \cong C^1(\mathbf{g}, \Lambda^0(P,g)) = \mathbf{C}^{1,0}$ is the Maurer-Cartan form on $\mathbf{G}$; i.e such that $\eta(\xi) = \xi$ for all $\xi \in \mathbf{g}$. Then the BRST transformations for A and η are*

$$\mathbf{s}A = d\eta + [A, \eta], \quad \mathbf{s}\eta = -\frac{1}{2}[\eta, \eta].$$

For the proof we refer to Schmid [18]. These transformations are the classical BRST transformations for the vector potential A and the ghost field η [2], [13].

Next we describe the cohomology which accommodates the Adler-Bardeen anomalies as elements of its first cohomology group. This is an analogue construction as before but using a different representation of the gauge algebra $\mathbf{g}$ on the space of *local forms* .

Consider $\Omega_{loc}(M \times \Gamma^\infty(\pi)$ with $\Gamma^\infty(\pi) = \Lambda^k(P,g)$. We restrict ourselves to the suspace $\mathcal{A} \subset \Lambda^1(P,g)$. Let C be a smooth q-chain on M and $\omega \in \Omega^{q,0}_{loc}$. Consider the functional $\mathbf{L}$ on $\mathcal{A}$ given by

$$\mathbf{L}(A) = \int_C \omega(A), \quad A \in \mathcal{A},$$

and denote the space of all such functionals by

$$\Gamma_{loc} = \{\mathbf{L} : \mathcal{A} \to \mathbf{R} \mid \mathbf{L}(A) = \int_C \omega(A), \ \omega \in \Omega^{q,0}_{loc}\}.$$

We define the representation ρ_{loc} of the gauge group $\mathbf{G}$ on the space Γ_{loc} by

$$(\rho_{loc}(\varphi)\mathbf{L})(A) = \mathbf{L}(\rho(\varphi^{-1})A), \quad \varphi \in \mathbf{G}, A \in \mathcal{A}.$$

Then the derived representation ρ'_{loc} of the gauge algebra $\mathbf{g}$ on Γ_{loc} is given by

$$(\rho'_{loc}(\xi)\mathbf{L})(A) = \frac{d}{dt}_{|t=0}\mathbf{L}(\rho(e^{-t\xi})A) = \mathbf{L}(\rho(Z_\xi)A), \quad \xi \in \mathbf{g}, A \in \mathcal{A}.$$

Now we consider the Chevalley-Eilenberg complex of $\mathbf{g}$ with respect to the representation ρ'_{loc} on Γ_{loc} . That means that the coboundary operator δ_{loc} : $C^q(\mathbf{g}, \Gamma_{loc}) \to C^{q+1}(\mathbf{g}, \Gamma_{loc})$ is given by

$$
\begin{aligned}
(\delta_{loc}\omega)(\xi_0,\cdots,\xi_q) &= \textstyle\sum_{i=0}^{q}(-1)^i\rho'_{loc}(\xi_i)\omega(\xi_0,\cdots,\hat{\xi}_i,\cdots,\xi_q) \\
&+ \textstyle\sum_{i<j}(-1)^{i+j}\omega([\xi_i,\xi_j],\xi_0,\cdots,\hat{\xi}_i,\cdots,\hat{\xi}_j,\cdots,\xi_q).
\end{aligned}
$$

We have $\delta_{loc}^2 = 0$ and a straightforward computation shows that

$$\delta_{loc}\mathbf{s} + \mathbf{s}\delta_{loc} = 0.$$

We define the total differential as

$$\Delta = \delta_{loc} + \mathbf{s}.$$

Then we have $\delta_{loc}^2 = \mathbf{s}^2 = \delta_{loc}\mathbf{s} + \mathbf{s}\delta_{loc} = 0$ which implies $\Delta^2 = 0$. We denote the induced *local* cohomology by $H^*_{BRST}(\mathbf{g},\Gamma_{loc})$, called the *local BRST cohomology* of $\mathbf{g}$.

Next we show that the Wess-Zumino consistency condition implies that the anomalies are elements of the first local cohomology group $H^1_{BRST}(\mathbf{g},\Gamma_{loc})$. We define the total differential as

$$\Delta = \delta_{loc} + \mathbf{s}.$$

Then we have $\delta_{loc}^2 = \mathbf{s}^2 = \delta_{loc}\mathbf{s} + \mathbf{s}\delta_{loc} = 0$ which implies $\Delta^2 = 0$. We derive the Chern-Weil homotopy formula as follows: Let $\tilde{A} = A + \eta \in \mathbf{C}^{0,1} \oplus \mathbf{C}^{1,0}$ and $\tilde{F} = \Delta\tilde{A} + \tilde{A}^2$. It follows from the BRST transformation theorem that $\tilde{F} = (\delta_{loc} + \mathbf{s})(A+\eta) + (A+\eta)^2 = \delta_{loc}A + A^2 = F_{\tilde{A}}$. For $t \in [0,1]$ let $\tilde{F}_t = t\tilde{F} + (t^2 - t)\tilde{A}^2$ and define the Chern-Simons form

$$\omega_{2q-1} = q\int_0^1 Tr(\tilde{A}\tilde{F}_t^{q-1})dt$$

and we get

$$\Delta\omega_{2q-1} = Tr\tilde{F}^q \quad . \qquad (*)$$

We write ω_{2q-1} as sum of homogeneous terms in the ghost number (upper index) and the degree (lower index)

$$\omega_{2q-1} = \omega^0_{2q-1} + \omega^1_{2q-2} + \omega^2_{2q-3} + \cdots + \omega_0^{2q-1}.$$

Then the relation $(*)$ yields the *descent equations*

$$\begin{aligned} \delta_{loc}\omega^0_{2q-1} &= 0 \\ \mathbf{s}\omega^0_{2q-1} + \delta_{loc}\omega^1_{2q-2} &= 0 \\ \mathbf{s}\omega^1_{2q-2} + \delta_{loc}\omega^2_{2q-3} &= 0 \\ &\vdots \\ \mathbf{s}\omega^{2q-2}_1 + \delta_{loc}\omega^{2q-1}_0 &= 0 \\ \mathbf{s}\omega^{2q-1}_0 &= 0\ . \end{aligned}$$

We are particularly interested in the third relation which will be used to identify the anomaly. Let q be such that $2q-2 = n = dimM$. Then we get the n-form on M

$$(\mathbf{s}\omega^1_{2q-2})(\xi_1,\xi_2) + (\delta_{loc}\omega^2_{2q-3})(\xi_1,\xi_2) = 0, \quad \xi_1,\xi_2 \in \mathbf{g}.$$

Notice that $(\delta_{loc}\omega^2_{2q-3})(\xi_1,\xi_2) = d_M(\omega^2_{2q-3}(\xi_1,\xi_2))$ where d_M is the exterior derivative on M . Stoke's theorem now implies

$$\int_M \mathbf{s}\omega^1_{2q-2}(\xi_1,\xi_2) = 0, \quad \xi_1,\xi_2 \in \mathbf{g}\ ,$$

or equivalently

$$\int_M \int_0^1 Tr(\mathbf{s}\tilde{F}_t^{2q-2})dt = 0\ . \qquad (**)$$

Let $\omega(\xi,A) = \int_M \omega^1_{2q-2}(\xi)$, or with the Chern-Simon form $\omega^1_{2q-2} = \int_0^1 \tilde{A}\tilde{F}_t^{q-1}dt$ we get

$$\omega(\xi,A) = \int_M \int_0^1 \tilde{A}\tilde{F}_t^{q-1}(\xi)dt\ .$$

The relation $(**)$ implies the *Wess-Zumino consistency condition*

$$(\delta_{loc}\omega)(\xi_1,\xi_2,A) = 0, \quad \xi_1,\xi_2 \in \mathbf{g}, A \in \mathcal{A}$$

which implies that the cohomology class $[\omega]$ of ω is an element of the first local cohomology group:

$$[\omega] \in H^1_{BRST}(\mathbf{g},\Gamma_{loc}).$$

Summarizing we proved the following:

Theorem: *The form* $\omega(\xi, A) = \int_M \int_0^1 \tilde{A}\tilde{F}_t^{q-1}(\xi)dt$ *satisfies the Wess-Zumino consistency condition* $(\delta_{loc}\omega)(\xi_1, \xi_2, A) = 0$ *and represents the anomaly* $[\omega]$ *in the local BRST cohomology* $H^1_{BRST}(\mathbf{g}, \Gamma_{loc})$.

An explicit form for the anomaly in (2q - 2)-dimensions is given by

$$\omega^1_{2q-2} = q(q-1)\int_0^1 (1-t)Tr(\eta\delta_{loc}(\tilde{A}\tilde{F}_t^{q-2}))dt.$$

We obtain for
$q = 2:\ \ \omega^1_2 = Tr(\eta\delta_{loc}\tilde{A})$, which is the non-Abelian anomaly in 2-dimensions; or for
$q = 3:\ \ \omega^1_4 = Tr(\eta\delta_{loc}(\tilde{A}\delta_{loc}\tilde{A} + \frac{1}{2}\tilde{A}^3)$, which is the non-Abelian anomaly in 4-dimensions [26], [27].

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