GOSSET'S FIGURE IN A CLIFFORD ALGEBRA

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Abstract. This note describes a way to realize a "projective" version of Gosset's 240-vertex semiregular polytope 4_{21} using the Clifford algebra Cl(8) generated by an 8-dimensional vector space equipped with a non-degenerate quadratic form. The 120 vertices of this projective Gosset figure are also seen to coincide with a particular basis for the Lie algebra $\mathfrak{so}(16)$.

1. Introduction

Gosset's figure 4_{21} is perhaps most concisely described as the boundary of the convex hull of the 240 roots of the E_8 lattice. Correspondingly, the symmetry group of Gosset's figure is the Weyl group E_8 for the exceptional Lie algebra $\mathfrak{e}_8$. Note that the order of this group is

$$|E_8| = 3! \cdot 4! \cdot 5! \cdot 8! = 696729600,$$

so Gosset's figure is a highly symmetrical object, considering it exists in only 8 dimensions. It is named after Thorold Gosset because he was apparently the first to describe it in print, [2, 6]. A sketch of the vertices and most of the edges of Gosset's figure appears as the frontispiece to [3]. Despite the naturality of using coordinates in Euclidean space, there are quite a few different ways to conceive Gosset's figure. In particular, this note is intended to serve as a tutorial on another way to "visualize" Gosset's 8-dimensional semiregular polytope, namely, using the 256-dimensional Clifford algebra Cl(8).

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The result in this note first sprouted as a numerological observation. Suppose $\{e_1, e_2, ..., e_8\}$ is a generating set for Cl(8), satisfying the relations

$$e_i^2 = -1,\ e_i e_j + e_j e_i = 0,\ (i \neq j).$$

For each $k \in \{0, 1, 2, ..., 8\}$, let Σ_k be the set of all products of k distinct generators, and let $\mathrm{Cl}(8)_k$ denote their span. Then clearly

$$|\Sigma_k| = \dim(\mathrm{Cl}(8)_k) = \frac{8!}{k!(8-k)!},$$

and there is a decomposition

$$\mathrm{Cl}(8) = \bigoplus_{k=0}^{8} \mathrm{Cl}(8)_k.$$

Let $\Sigma = \bigcup_{k=0}^{8} \Sigma_k$ and $\Sigma^- = \{x \in \Sigma : x^2 = -1\}$. Then one can check that

$$|\Sigma^-| = |\Sigma_1 \cup \Sigma_2 \cup \Sigma_5 \cup \Sigma_6| = 8 + 28 + 56 + 28 = 120.$$

The first part of the observation is that 120 is half the number of vertices of Gosset's figure. Next, each vertex in Gosset's figure is joined by an edge to 56 other vertices, while given an element $x \in \Sigma^-$, one can check that there are precisely 56 elements $y \in \Sigma^-$ for which $yx = -xy$. Such a pair is said to "anticommute". Based on these meager data, one might hope for a correspondence,

$$\begin{array}{rcc} \text{vertices} & \leftrightarrow & \text{elements } x \in \Sigma^-, \\ \text{edges} & \leftrightarrow & \text{anticommuting pairs } x, y \in \Sigma^-. \end{array}$$

This note aims to show that this is not merely a coincidence. Moreover, it will be seen that Σ^- represents a basis for the Lie algebra $\mathfrak{so}(16)$, if Cl(8) is represented by 16×16 matrices.

Based on the data given so far, one may guess that the Clifford-algebra model described in this note yields a "projectivized" version of Gosset's figure. Note that Gosset's figure bounds a convex cell in 8 dimensions, so, by central projection, it yields a cellular decomposition of the sphere S^7. This decomposition is invariant under the antipodal map

$$\nu : \mathbf{v} \mapsto -\mathbf{v},$$

mapping each point to its negative. In other words, if c is a cell of any dimension appearing in Gosset's figure, then $-c$ is another cell appearing in the Gosset figure. One may therefore quotient by this action to obtain $P4_{21}$, a combinatorial polytope with exactly half the number of cells of 4_{21}. One can also see this by noting that the center of its symmetry group E_8 has two elements, the involutive element of this subgroup corresponding to the antipodal map. The notation $P4_{21}$ is intended to suggest a cellular decomposition of the projective space $\mathbb{R}P^7$.

The 240 vertices of Gosset's figure 4_{21} generate the E_8 lattice. This lattice is famous for several reasons. First, it corresponds to the most complicated of the 5 exceptional simple complex Lie algebras, [8]. Second, it is the most accessible non-trivial example of an even unimodular lattice in existence, as all other such lattices exist in dimensions of the form $d = 8n$ where $n \geq 2$, [4]. Third, no other lattice in 8 dimensions has higher density, so this lattice is an important example in coding theory and the general sphere-packing problem, [4]. Finally, this lattice has made an appearance in the theory of quasicrystals, as it appears that many quasicrystalline phenomena may be placed within the framework of certain natural projections of the lattice down to 2, 3, and 4 dimensions, [5, 9].

2. A General Fact about Clifford Algebras

Clifford algebras have a lot of structure in general, (see [7]), but here we are mainly concerned with when two elements of a canonical basis commute or anticommute. The purpose of this section is thus to describe these conditions.

Start with a set S. Then the power set $\mathcal{P}(\mathcal{S})$ is a vector space over the 2-element field $\mathbb{F}_2$. Addition on $\mathcal{P}(\mathcal{S})$ is given by the symmetric difference operation

$$I + J = I - J = (I \cup J)\backslash(I \cap J),$$

for all $I, J \subset S$, and scalar multiplication is given by

$$1 \cdot I = I \text{ and } 0 \cdot I = \emptyset,$$

for all $I \subset S$.

Define the function

$$|\ | : \mathcal{P}(S) \to \mathbb{F}_2$$

as the cardinality of S mod 2. Thus, $|\emptyset| = 0$, $|\{1\}| = 1$, $|\{1,2\}| = 0$, $|\{1,2,3\}| = 1$, and so on. Then there is a symmetric bilinear form $\langle\ ,\ \rangle : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathbb{F}_2$ defined according to the prescription

$$\langle I, J\rangle = |I| \cdot |J| - |I \cap J|.$$

Since $\mathrm{char}(\mathbb{F}_2) = 2$, this form is also skew-symmetric.

Suppose now that $S = \{1, 2, 3, ..., r\}$, a finite set with r elements. Let $\mathrm{Cl}(r)$ denote the Clifford algebra with generators $\{e_i : i \in S\}$. In particular, assume e_i^2 is non-zero for all $i \in S$ and

$$e_j e_i = -e_i e_j$$

for all $i, j \in S$. Define a map $e : \mathcal{P}(S) \to \mathrm{Cl}(r)$ by

$$e(I) = e_I = \prod_{i \in I} e_i,$$

where, for definiteness, the indices are written in increasing order. As in the introduction, a basis for $\mathrm{Cl}(r)$ is given by

$$\Sigma = \{e_I : I \subset S\},$$

This shall be called the "canonical basis" for $\mathrm{Cl}(r)$.

Proposition 2.1 *Suppose $I, J \subset S$. Then*

$$e_J e_I = (-1)^{\langle I,J\rangle} e_I e_J.$$

Remark. If $k \in \mathbb{F}_2$, then the meaning of $(-1)^k$ should be clear: One has $(-1)^0 = 1$, and $(-1)^1 = -1$, and in fact this yields a homomorphism from the additive part part of $\mathbb{F}_2$ to the group $\{\pm 1\}$.

Proof. Use induction on $m = |I|$, $n = |J|$, and $k = |I \cap J|$. If either $m = 0$, or $n = 0$, the formula holds because then either e_I or e_J is the identity element of $\mathrm{Cl}(r)$. Next, suppose $k = 0$. Then one transforms the expression $e_I e_J$ to the expression $e_J e_I$ by interchanging precisely mn distinct pairs of the form $e_i e_j$ into the form $e_j e_i$. For each of these mn pairs, one has $i \neq j$, and thus $e_j e_i = -e_i e_j$ for every pair. Therefore, if $k = 0$, then $e_J e_I = (-1)^{mn} e_I e_J$. Now one must handle the induction step. Suppose $e_J e_I = (-1)^{mn-k} e_I e_J$ for some particular I and J. Choose i such that $i \in \{1, 2, 3, ..., r\}$ but $i \notin I$. Form a new set by uniting $I' = \{i\} \cup I$. There are two cases to consider, one where $i \in J$ and the other where $i \notin J$. First suppose $i \in J$. Using the induction hypothesis and the relations among the generators, one has

$$\begin{aligned} e_J e_{I'} &= e_J e_i e_I \\ &= (-1)^{n-1} e_i e_J e_I \\ &= (-1)^{n-1}(-1)^{mn-k} e_i e_I e_J \\ &= (-1)^{mn+n-k-1} e_{I'} e_J. \end{aligned}$$

Since, however, $i \in J$, one then has $|I'| = m + 1$ and $|I' \cap J| = k + 1$. Thus,

$$|I'| \cdot |J| - |I' \cap J| = (m + 1)n - (k + 1) = mn + n - k - 1,$$

exactly the exponent appearing above. Next suppose $i \notin I$. Using the same reasons as above, one has

$$\begin{aligned} e_J e_{I'} &= e_J e_i e_I \\ &= (-1)^{n} e_i e_J e_i \\ &= (-1)^{n}(-1)^{mn-k} e_i e_I e_J \\ &= (-1)^{mn+n-k} e_{I'} e_J. \end{aligned}$$

In this case one has $i \notin J$, so $|I'| = m + 1$ while $|I' \cap J| = k$. Thus

$$|I'| \cdot |J| - |I' \cap J| = (m + 1)n - k = mn + n - k,$$

which is the exponent appearing above. The arguments for cases obtained by adjoining new elements to J are identical. This completes the induction. □

It is worth noting here that the set $\{e_i : i \in S\}$ generates a finite group given by the union

$$\mathbb{Q}(r) = \Sigma \cup (-\Sigma).$$

The notation $\mathbb{Q}(r)$ is intended to reflect the fact that this is a generalization of the 8-element group of quaternions. Also note that $\mathbb{Q}(r)$ is a non-split extension of the two-element group by the additive part of the vector space $\mathcal{P}(S)$.

3. Gosset's 8-Dimensional Figure and Cl(8)

An easy way to identify Gosset's figure is to recognize it as a particular G-set for the Coxeter group $G = E_8$. For reference, the Coxeter diagram is given.

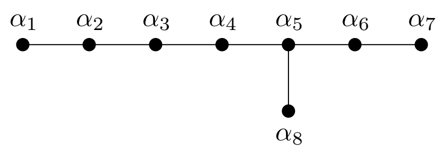

Figure 1. Coxeter Diagram for E_8.

Recall that the diagram contains information on how E_8 is presented by generators and relations: Each node corresponds to an involutive generator, each edge corresponds to a pair generating the 6-element nonabelian group, and each non-edge corresponds to a commuting pair of generators. If one can find an action by a suitable set of operations obeying these relations, then one has identified an action of E_8. We shall employ this principle in order to discover the projectivized Gosset's figure inside Cl(8).

With that, first notice that each $J \subset S$ induces a map r_J of $\mathcal{P}(S)$ according to the prescription

$$r_J(I) = I - \langle I, J \rangle \, J.$$

Proposition 3.1 *(a) Each r_J is an involutive $\mathbb{F}_2$-linear map of $\mathcal{P}(S)$. (b) If $\langle I, J \rangle = 0$, then $(r_I r_J)^2 = Id$. (c) If $\langle I, J \rangle = 1$, then $(r_I r_J)^3 = Id$.*

Proof. Straightforward computation. □

Throughout the rest of this note, assume that $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then $\mathcal{P}(S)$ carries an action by E_8, which will be described here. Denote

$$\begin{array}{llll} \alpha_1 = \{1,2\}, & \alpha_2 = \{2,3\}, \\ \alpha_3 = \{3,4\}, & \alpha_4 = \{4,5\}, \\ \alpha_5 = \{5,6\}, & \alpha_6 = \{6,7\}, \\ \alpha_7 = \{7,8\}, & \alpha_8 = \{1,2,3,4,5\}, \end{array}$$

and for each $k \in \{1, 2, ..., 8\}$, let r_k be the corresponding involution of $\mathcal{P}(S)$. One quickly notices that $\langle \alpha_i, \alpha_j \rangle = 1$ if and only if α_i and α_j are joined by an edge in the Coxeter diagram given. The corresponding involutions r_k of $\mathcal{P}(S)$ clearly generate a group acting on $\mathcal{P}(S)$. However, using the preceding proposition and the fact that these involutions obey the relations of the generators of E_8, it is clear that these involutions yield an action of E_8 on $\mathcal{P}(S)$. Given that E_8 acts on $\mathcal{P}(S)$, one must now determine the orbits. To facilitate this, for each $k \in \{0, 1, 2, ..., 8\}$, let $\mathcal{P}(S)_k$ denote the set of subsets of S with cardinality k. Then we have the following:

Proposition 3.2 *The action of E_8 on $\mathcal{P}(S)$ has three orbits, specifically*

$$O_1 = \mathcal{P}(S)_0,$$

$$O_{120} = \mathcal{P}(S)_1 \cup \mathcal{P}(S)_2 \cup \mathcal{P}(S)_5 \cup \mathcal{P}(S)_6,$$

and

$$O_{135} = \mathcal{P}(S)_3 \cup \mathcal{P}(S)_4 \cup \mathcal{P}(S)_7 \cup \mathcal{P}(S)_8.$$

Proof. First, it is clear that O_1 is an orbit. Next, notice that the involutions $\{r_1, r_2, ..., r_7\}$ generate the full symmetric group S_8, and that each $\mathcal{P}(S)_k$ is an orbit under the action of this subgroup. The involution r_8 corresponding to $\alpha_8 = \{1, 2, 3, 4, 5\}$, however, is the only generator which can transform a set of cardinality k to a set which does not have cardinality k. After some calculations, one observes that O_{120} and O_{135} are each closed with respect to the action of r_8. □

There is another approach to proving the preceding proposition, provided one knows some facts about the subgroups of E_8. In particular, E_8 has maximal subgroups with indicies 120 and 135, and aside from the index-2 subgroup of "even" elements, no larger maximal subgroups, [10]. Thus, given that the action of E_8 on $\mathcal{P}(S)$ has non-trivial orbits with more than two elements, one is led to the conclusion that the only other orbits besides O_1 must be O_{120} and O_{135}.

Recall the map e which maps a set $I \subset S$ to the canonical basis element $e(I) = e_I \in \Sigma$. Since E_8 acts on $\mathcal{P}(S)$, the bijection e induces a corresponding action on Σ.

Theorem 3.3 *(a) The action of E_8 on Σ preserves commutative and anticommutative pairs. (b) The action of E_8 on Σ has three orbits, specifically $\Sigma_1 = e(O_1)$, $\Sigma_{120} = e(O_{120})$, and $\Sigma_{135} = e(O_{135})$. (c) $\Sigma_{120} = \Sigma^- = \{e_I : e_I^2 = -1\}$.*

Proof. Part (a) follows from the proposition in the preceding section, giving the conditions under which a pair $x, y \in \Sigma$ commutes or anticommutes. Part (b) is immediate from the preceding proposition. Part (c) is a straightforward computation. □

If it is not clear by now, this is how the projectivized Gosset figure $P4_{21}$ lies inside of Cl(8): The vertices correspond to elements of Σ^- and the edges correspond to anticommuting pairs of elements of Σ^-. According to the preceding theorem, this configuration of vertices and edges carries an action by E_8, and therefore corresponds to the projective Gosset figure.

Next we have an interesting fact about the orbit Σ^-:

Proposition 3.4 *If* Cl(8) *is represented by* 16×16 *matrices, then* Σ^- *is represented by a basis for* $\mathfrak{so}(16)$.

Proof. One may construct the representation explicitly. First recall that Cl(8) is isomorphic to the 4-fold tensor product of the algebra Cl(2), and that Cl(2) is isomorphic to the algebra of 2×2 matrices, [7]. Denote

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $J = HD$. Then $\{I, H, D, J\}$ represents a basis for Cl(2), and the 4-fold tensor products of these matrices represents a basis for Cl(8). The representation is determined uniquely by the values on the generators of Cl(8). With that, make the assignments

$$\begin{array}{llll} e_1 \mapsto I \otimes I \otimes H \otimes J, & e_2 \mapsto I \otimes H \otimes J \otimes I, \\ e_3 \mapsto I \otimes I \otimes D \otimes J, & e_4 \mapsto I \otimes J \otimes I \otimes H, \\ e_5 \mapsto I \otimes J \otimes I \otimes D, & e_6 \mapsto I \otimes D \otimes J \otimes I, \\ e_7 \mapsto H \otimes J \otimes J \otimes J, & e_8 \mapsto D \otimes J \otimes J \otimes J. \end{array}$$

One quickly checks that these matrices satisfy the same relations as the generators. Moreover, for each element of Σ^-, one can check that its image under this mapping is an orthogonal matrix. Finally, $\mathfrak{so}(16)$ has dimension 120, coinciding with the cardinality of Σ^-. □

4. Conclusion

It has been demonstrated that the projectivized Gosset figure $P4_{21}$ lies imbedded inside the Clifford algebra Cl(8) and the Lie algebra $\mathfrak{so}(16)$. Moreover, there is a basis for $\mathfrak{so}(16)$ for which anti-commuting pairs of basis elements correspond to the edges of $P4_{21}$.

It is appropriate at this point to mention that Clifford algebras may contain other exceptional configurations as well. For example, consider the semiregular polytope 2_{21}, whose 27 vertices correspond to the lines on the general cubic surface, [1]. This polytope is imbedded in the Clifford algebra Cl(6) as follows. Suppose $\{a_1, a_2, ..., a_6\}$ is a generating set for Cl(6) which satisfies

$$a_i^2 = 1, \ a_j a_i + a_i a_j = 0, \ (i \neq j).$$

As before, set Σ_k to be the set of products of k distinct generators. Then $z = a_1 a_2 a_3 a_4 a_5 a_6$ is the only basis element for Σ_6. For each $k \in \{1, 2, 3, 4, 5, 6\}$, let

$$b_k = a_k z,$$

and for each $i, j \in \{1, 2, 3, 4, 5, 6\}$ with $i < j$, let

$$c_{ij} = a_i a_j z.$$

Evidently $\{a_k\}$ is a basis for Σ_1, $\{b_k\}$ is a basis for Σ_5, and $\{c_{ij}\}$ is a basis for Σ_4. Moreover, all of these basis elements square to unity in Cl(6). One can check that the array

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array}$$

is an example of Schläfli's double-six. This means that the two rows and six columns each constitute a mutually anti-commuting set, while every other pair in the array commutes. The other double-sixes appearing in 2_{21} either have the form

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{array}$$

or

$$\begin{array}{cccccc} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{array}.$$

Again, notice that two entries in these arrays commute if and only if they are not in the same row and not in the same column.

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