LECTURE NOTES ON EQUIVARIANT DE RHAM THEORY, 2/22/06

BEN MARES

- References

<u>Supersymmetry and Equivariant de Rham Theory</u>, Guillemin+Sternberg

<u>Differential Forms in Algebraic Topology</u>, Bott+Tu

- Introduction

Rough overview of how one makes the transition from topological concepts to differential concepts. The strategy is to explain the functorial aspects of the theory. Then I will indicate how to transition from the topological perspective to the geometric perspective, reviewing the Chern-Weil theory of characteristic classes along the way. Will use poetic license to convey general idea. I roughly follow Guillemin, but try to give a complementary persepctive whenever possible.

Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. (Think $U(n)$.) We want to define "equivariant cohomology" for the category of G-manifolds. Should be a *functorial* construction.

- Review of ordinary (complex) deRham cohomology:

It's a composition of functors

$M \Rightarrow \Omega^\bullet(M;\mathbb{C}) \Rightarrow H^\bullet(\Omega^\bullet(M;\mathbb{C}))$

differential manifolds⇒differential $\mathbb{Z}$-graded superalgebras over $\mathbb{C}$⇒graded $\mathbb{C}$-algebras

Where does the $\mathbb{C}$ come from? In the category of differential manifolds, the point is the *terminal object*: $M \to \text{pt}$. This induces a map $H^\bullet_{dR}(\text{pt}) \to H^\bullet_{dR}(M)$. This gives M an $\mathbb{C}$-module structure.

Let M be a G-manifold, i.e. a manifold with a smooth left action $G \to \text{Diffeo}(M)$. In the category of G-manifolds, morphisms are smooth G-equivariant maps:

$$\begin{array}{ccc} M & \to & M \\ \downarrow & & \downarrow \\ M' & \to & M' \end{array}$$

$$\begin{array}{ccc} x & \to & gx \\ \downarrow & & \downarrow \\ f(x) & \to & gf(x) = f(gx) \end{array}$$

Now we can apply a bunch of functors that we define to be the equivariant cohomology:

$M \Rightarrow P_M \Rightarrow P_M/G \Rightarrow H^\bullet_{\text{sing}}(P_M/G;\mathbb{C})$

G-manifolds⇒Topological principal G-bundles⇒Topological spaces⇒Graded $\mathbb{C}$-algebras

- Explanation of topological background

By a topological principal G-bundle, I mean a G-space where the G-action is free. It's essentially (the total space of) a principal bundle, but it's not generally a finite-dimensional manifold.

Given a Lie group G, there's a construction due to topologists called the universal bundle EG, which is a contractible free G-space. (Can't ever be finite-dimensional manifold if G is non-contractible.) It follows from the theory that the base space $BG := EG/G$ is a classifying space for principal G-bundles.

Given a topological principal G-bundle P, there exists a unique homotopy class of maps $f : P/G \to BG$ such that P is the pullback of EG along f.

$$\begin{array}{ccc} P \cong f^*(EG) & \to & EG \\ \downarrow & & \downarrow \\ P/G & \to & BG \end{array}$$

Examples: If G is discrete, $BG = K(G, 1)$. If $G = U(n)$ (resp. $O(n)$), then $BG = \mathrm{Gr}_n(\mathbb{C}^\infty)$ (resp. $\mathrm{Gr}_n(\mathbb{R}^\infty)$). In the case $G = S^1$, $BG = \mathrm{Gr}_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$.

- Explanation of the functors

The first step is to take the G-bundle and turn it into a topological principal G-bundle. This is accomplished by multiplying by EG:

$M \Rightarrow M \times EG$. This is a G-space with the product action $g(m, e) := (gm, ge)$. Clearly G acts freely on the product since G acts freely on EG. One should view this process as "stabilizing" a G-space to a principal G-space.

Next we pass to the base space

$$\frac{M \times EG}{G}.$$

One might worry the homotopy type of this depends on the choice of EG. To see that it doesn't, we can use the following handy formula:

If P is principal and E is contractible, then

$$\frac{P \times E}{G} \cong \frac{P}{G}.$$

(One sees this as a weak homotopy equivalence by analyzing the long exact homotopy sequence of the fibration $E \to (P \times E)/G \to P/G$.) Thus if M is already principal,

$$\frac{M \times E}{G} \cong \frac{M}{G}.$$

Moreover, if $(EG)_1$ and $(EG)_2$ are two different classifying space representatives, then

$$\frac{M \times (EG)_1}{G} \cong \frac{M \times (EG)_1 \times (EG)_2}{G} \cong \frac{M \times (EG)_2}{G}.$$

Therefore, the resulting space is independent of choices.

Now we apply singular cohomology with $\mathbb{C}$ coefficients to the resulting space. (Topologically, it makes sense to use $\mathbb{Z}$ coefficients, or whatever else. Our choice of $\mathbb{C}$ allows us connect with deRham theory later on.) Thus we define the equivariant cohomology

$$H^\bullet_G(M) := H^\bullet_{\text{sing}}\left(\frac{M \times EG}{G}; \mathbb{C}\right).$$

Any equivariant map $f : M \to M'$ induces a map $f^* : H^*_G(M') \to H^*_G(M)$.

- Cohomology of a point

Just as in the case of the category of differentiable manifolds, the point is a terminal object in the category of G-manifolds. Thus we have an induced map

$$H^\bullet_G(\text{pt}) \to H^\bullet_G(M)$$

that turns $H^\bullet_G(M)$ into a $H^\bullet_G(\text{pt})$-module. But

$$H^\bullet_G(\text{pt}) = H^\bullet_{\text{sing}}\left(\frac{\text{pt} \times EG}{G}; \mathbb{C}\right) = H^\bullet_{\text{sing}}(BG; \mathbb{C}).$$

For example, when $G = S^1$, we have $H^\bullet(BG; \mathbb{C}) = H^\bullet(\mathbb{CP}^\infty; \mathbb{C}) = \mathbb{C}[x]$, $\deg x = 2$. More generally, if $G = U(n)$, then $H^\bullet(BG; \mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n]$, $\deg x_i = 2i$.

In the case $G = U(n)$, the image of the generator x_i in $H^\bullet_G(M)$ is the *i-th equivariant Chern class of M*.

For general G, an *equivariant characteristic class* is the image of a particular element of $H^\bullet_G(\text{pt})$.

This includes the traditional notion of a characteristic class for a fiber bundle. (e.g. vector bundle or principal bundle) A characteristic class is thus a cohomology class on the base coming from the twisting of the fiber. The recipe is as follows:

Take the associated principal bundle P, which is a G-space.

The associated equivariant characteristic classes live in $H^\bullet_G(P)$.

Since the action on P is free, we identify this with the ordinary cohomology of the base $H^\bullet(P/G)$.

- The geometric point of view: Cartan's model

The Cartan model is a completely different construction of a cohomology theory, this time geometric rather than topological. We will see why (at least in principle) these two different models produce the same result.

Suppose M is a manifold with a smooth G-action. Consider a polynomial ω on $\mathfrak{g}$ with coefficients in $\Omega^\bullet(M)$.

More concretely, elements of $\mathfrak{g}^*$ are linear functions on $\mathfrak{g}$. General polynomials on $\mathfrak{g}$ are given by elements of $\mathrm{Sym}(\mathfrak{g}^*)$, the symmetric algebra. For example, to evaluate a monomial $\nu^1 \cdots \nu^k$ at a point $\xi \in \mathfrak{g}$, we compute $\nu^1(\xi)\cdots\nu^k(\xi)$.

It follows that any polynomial ω on $\mathfrak{g}$ with coefficients in $\Omega^\bullet(M)$ can be viewed as an element of $S(\mathfrak{g}^*)\otimes\Omega^\bullet(M)$.

There is a natural G-action on $S(\mathfrak{g}^*)\otimes\Omega^\bullet(M)$. On the $\Omega^\bullet(M)$ component, one acts by "translation," or pulling back the forms via the multiplication map on M. On the $S(\mathfrak{g}^*)$ component, one acts by the "coadjoint action." When an element of $S(\mathfrak{g}^*)$ is viewed as a function, this simply means "translation" of the function via the adjoint action on its domain. I'll explain this total action in more detail later on.

Given the G-action we just defined, we can ask what are the G-invariant elements $(S(\mathfrak{g}^*)\otimes\Omega^\bullet(M))^G$? When viewed as polynomial functions, invariant elements correspond to equivariant polynomials, i.e. $\omega(g\cdot\xi) = g\cdot\omega(\xi)$.

We now have an algebra. We want a cohomology theory. Thus we seek to make this algebra graded with a differential.

Define a grading on equivariant polynomials by

$$\deg_{\mathrm{tot}}\omega = 2\deg_{\mathrm{poly}}\omega + \deg_{\mathrm{form}}\omega.$$

For example, if $\theta\in\mathfrak{g}^*$ and $\alpha\in\Omega^3(M)$, then $\deg_{\mathrm{tot}}(\theta^2\otimes\alpha) = 2\cdot 2+3 = 7$.

We can define a differential d_G on $(S(\mathfrak{g}^*)\otimes\Omega^\bullet(M))^G$ that increases the total degree by one. At the level of functions on $\mathfrak{g}$, it's defined as

$$(d_G\omega)(\xi) := d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

The first term leaves the polynomial degree unchanged, but increases the form degree by one. The second term decreases the form degree by one, but increases the polynomial degree by one. (If $\deg_{\mathrm{poly}}\omega = k$, then $\omega(\xi)$ is homogeneous of degree k in ξ. Ignoring what happens on the forms level, $\iota_\xi(\omega(\xi))$ is homogeneous of degree $k+1$ in ξ.)

In addition to the function-level description, it's also useful to know how d_G acts on expressions in $(S(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G$. It's easy to verify that

$$d_G = 1 \otimes d + \sum_a \mathrm{mult}_{\nu^a} \otimes \iota_{\xi_a},$$

for any basis $\{\xi_a\}$ of $\mathfrak{g}$ with corresponding dual basis $\{\nu^a\}$.

Now that we have a differential graded algebra, we define the equivariant de Rham cohomology

$$H^\bullet_{G,\mathrm{dR}}(M) := H^\bullet\left((S(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G, d_G\right).$$

This is the Cartan model.

- From topology to geometry

The goal is to convince everyone that it's plausible) that the Cartan model and the topological equivariant cohomology are equivalent. Thus we want to understand how the Cartan model expresses all the previous topological nonsense in differential geometric terms. The answer is provided by the Chern-Weil theory of connections and curvature.

- Connections

There are many different, but compatible, definitions of a connection.

(1) For a vector bundle $E \to B$, it's a map $\nabla : \Gamma(E) \to \Omega^1(B) \otimes \Gamma(E)$, satisfying $\nabla(fs) = df \otimes s + f\nabla s$ for all $f \in C^\infty(B)$, $s \in \Gamma(E)$.

(2) For a fiber bundle with fiber F, structure group G, and a collection of trivializations $\{U_\alpha\}$ with transition maps $\phi_{\beta\alpha}(x, f)_\alpha = (x, g_{\beta\alpha} f)_\beta$, it's a collection of "$\mathfrak{g}$-valued one-forms" $\{A_\alpha \in \Gamma(T^*M) \otimes \mathfrak{g}\}$ satisfying

$$(d + A_\beta)g_{\beta\alpha} = g_{\beta\alpha}(d + A_\alpha)$$

as operators on $C^\infty(U_\alpha) \otimes \mathfrak{g}$. This is equivalent to the gauge transformation formula

$$A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} - d(g_{\beta\alpha}) g_{\beta\alpha}^{-1}.$$

(3) For a principal bundle P, there is a distinguished vertical subbundle $VP \subset TP$. (It is the kernel of $d\pi$ where $\pi : P \to B := P/G$.) A connection on P is a smooth G-invariant projection $t : TP \to VP$. I claim that any such projection can be represented uniquely by a differential form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$. Given certain hypotheses, such a form also corresponds to a connection.

(In what follows, there may be a few minus signs missing. In large part, this is because I can't decide whether the action on a principal bundle should be on the right or left. See Guillemin's book for the more careful treatment.)

To realize this third construction, we must consider the action $G \to \mathrm{Diffeo}(P)$. The differential of this action gives a map $\mathfrak{g} \to \mathrm{Vect}(P)$. We denote this map by $\mathfrak{g} \ni \xi \mapsto \xi_P \in \mathrm{Vect}(P)$. Because the group action preserves fibers, the ξ_P are actually vertical, that is sections of VP. Because the action of G is free, the vector fields ξ_P are nonzero for every nonzero $\xi \in \mathfrak{g}$. Thus the map $P \times \mathfrak{g} \to VP$ determined by $(p, \xi) \mapsto \xi_P|_p$ is a global trivialization of VP.

Since the vertical bundle is trivial with fiber $\mathfrak{g}$, any fiberwise-linear map $TP \to VP$ is equivalent to a map $TP \to \mathfrak{g}$. Furthermore, any such map $TP \to \mathfrak{g}$ is equivalent to an element $\theta \in \Omega^1(P) \otimes \mathfrak{g}$.

This map/element is assumed to have two properties: projection, and G-invariance. Being a projection simply means that $\theta(\xi_P) = \xi$ for all $\xi \in \mathfrak{g}$. G-invariance is much more subtle. The group G acts on $\Omega^\bullet$ in the obvious way of "translation". (Pulling back by group multiplication). Explicitly, $\omega \cdot g = \rho^*_{g^{-1}}(\omega)$, where $\rho_g : P \to P$ is right-multiplication by $g \in G$. One might think that in our identification $VP \cong P \times \mathfrak{g}$, since we have a trivial $\mathfrak{g}$-bundle, the G-action on $\mathfrak{g}$ should be trivial. This is not the case.

To understand what's going on, observe that we have the following expression for the vector field ξ_P at a point:

$$\xi_P|_{p\in P} = \frac{d}{dt}|_{t=0} p \cdot \exp(t\xi).$$

Thus the vector field is *left*-invariant. However, when we act on the *right* by $g \in G$, we get

$$\begin{aligned}
\xi_P|_{p\in P} \cdot g &= \frac{d}{dt}|_{t=0} p \cdot \exp(t\xi) g \\
&= \frac{d}{dt}|_{t=0} pg \cdot g^{-1} \exp(t\xi) g \\
&= \frac{d}{dt}|_{t=0} pg \cdot \exp(t g^{-1} \xi g) \\
&= (\mathrm{Ad}_{g^{-1}} \xi)_{p\cdot g \in P}.
\end{aligned}$$

Therefore, whenever we act by G on the manifold, we need to act by the adjoint representation on the $\mathfrak{g}$ in our trivialization.

Our current example of $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ transforms as follows. Suppose $\mathfrak{g}$ is a matrix algebra, which is of course always a valid assumption. Then we may write θ as a "matrix of one-forms"

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \cdots \\ \theta_{21} & \ddots & \\ \vdots & & \end{pmatrix}.$$

Suppose $g = M \in G$ is a matrix, and we want to compute the action of M on θ. We have

$$\theta \cdot M = M^{-1} \begin{pmatrix} \rho^*_{M^{-1}}(\theta_{11}) & \rho^*_{M^{-1}}(\theta_{12}) & \cdots \\ \rho^*_{M^{-1}}(\theta_{21}) & \ddots & \\ \vdots & & \end{pmatrix} M.$$

Thus the components of the connection do not transform individually: they are mixed by the adjoint action. More precisely, we have the following natural-looking identity for infinitesimal invariance, where $\mathfrak{L}_\eta$ is the Lie derivative by the vector field η_P for $\eta \in \mathfrak{g}$:

$$\begin{aligned} 0 = \mathfrak{L}_\eta(\theta^a \otimes \xi_a) &= (\mathfrak{L}_\eta \theta^a) \otimes \xi_a + \theta^a \otimes [\eta, \xi_a] \\ \implies (\mathfrak{L}_\eta \theta^a) \otimes \xi_a &= -\theta^a \otimes [\eta, \xi_a]\,. \end{aligned}$$

- Basic forms

Since the projection $\pi : P \to B := P/G$ is a submersion, we have

$$\pi_* : TP \twoheadrightarrow TB \implies \pi^* : T^*B \hookrightarrow T^*P \implies \Omega^*(B) \hookrightarrow \Omega^*(P).$$

Thus, we may ask when a form ω on P comes from a form on B. The necessary and sufficient conditions are

$$\begin{aligned} i_\xi \omega := \omega(\xi_M, \cdot) = 0 \qquad & \text{(horizontal)}, \\ \omega \cdot g = \omega \qquad & \text{(invariant)}. \end{aligned}$$

Such forms are called *basic forms* since they come from the base.

Note that θ is not basic because it is not horizontal: $i_\xi \theta = \theta$. (Also it is not invariant unless G is abelian.)

If we consider the chain complex $(\Omega^\bullet(P), d)$, the basic forms are a subcomplex $(\Omega^\bullet_{\text{bas}}(P), d)$. Since d commutes with pullbacks, the cohomology of the base is the same as the cohomology of the basic forms. This allows us in principle to express everything in terms of P without passing to the quotient.

- Curvature

We define curvature to be $\mu := d\theta + \theta \wedge \theta$.

(Aside:) Shouldn't $\theta \wedge \theta$ vanish? Write $\theta = \theta_i dx^i$, where $\theta_i \in \mathfrak{g}$. Then

$$\theta \wedge \theta = \theta_i \theta_j \, dx^i \wedge dx^j = \sum_{i<j} (\theta_i \theta_j - \theta_j \theta_i)\, dx^i \wedge dx^j = \sum_{i<j} [\theta_i, \theta_j]\, dx^i \wedge dx^j \in \mathfrak{g} \otimes \Omega^2(P).$$

Normally, when $\theta_i \in \mathbb{R}$, this will vanish. However, for a non-abelian Lie algebra, it generally will not vanish.

The curvature μ turns out to be horizontal. However, like θ, its components transform by the adjoint action; generally they're not invariant.

- Invariant polynomials

Although the curvature μ is not generally invariant, any adjoint-invariant combination of components is.

Suppose that $G = U(n)$. As for θ, we may think of μ as a skew-Hermitian matrix of 2-forms

$$\begin{pmatrix} \mu_{11} & \mu_{12} & \cdots \\ \mu_{21} & \ddots & \\ \vdots & & \end{pmatrix}.$$

Then

$$\mu_{11} + \cdots + \mu_{nn} = \mathrm{Tr}\ \mu \in \Omega^2(P)$$

is basic, as is $\det \mu \in \Omega^{2n}(P)$. It also happens that these forms are closed, and the cohomology classes they represent on the base are (up to a constant) the Chern classes $c_1 \in H^2(B)$ and $c_n \in H^{2n}(B)$.

More generally, characteristic classes correspond to adjoint-invariant polynomials in the components of curvature, so we wish to determine all such polynomials, at least in the case $G = U(n)$.

By diagonalizing this matrix, μ is conjugate to

$$\mu \sim \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}.$$

with eigenvalues $\lambda_1, \ldots, \lambda_n \in \Omega^2(P)$. μ is also conjugate to

$$\sim \begin{pmatrix} \lambda_2 & 0 & \cdots \\ 0 & \lambda_1 & \\ \vdots & & \ddots \end{pmatrix},$$

or generally any permutation of the eigenvalues. Thus any Ad-invariant function must be a symmetric function in the eigenvalues. Conversely, any symmetric function in the eigenvalues will be Ad-invariant. For example, $\mathrm{Tr}(\mu) = \lambda_1 + \cdots + \lambda_n$ is invariant, as is $\det(\mu) = \lambda_1 \cdots \lambda_n$.

The *total Chern class* is defined as

$$\det(I+\frac{i}{2\pi}\mu) = \prod_{k=1}^{n}(1+\frac{i}{2\pi}\lambda_k) = 1+\frac{i}{2\pi}(\sum \lambda_k)-\frac{1}{4\pi}\sum_{j\neq k=1}^{n}\lambda_j\lambda_k+\cdots+\left(\frac{i}{2\pi}\right)^n\prod_{k=1}^{n}\lambda_k.$$

The *k-th Chern class* is defined to be the degree $2k$ homogeneous component of the total Chern class. (Recall $\deg \lambda_k = 2$ since μ is a 2-form.) The homogeneous components are the elementary symmetric polynomials in the λ_i, and these generate the ring of symmetric polynomials in λ_i. Since Ad-invariant polynomials in the μ_{ij} correspond to symmetric polynomials in the λ_i, the Chern classes are a basis for the ring of basic forms coming from μ.

The symmetric polynomial ring is isomorphic to $\mathbb{C}[c_0 = 1, c_1, \ldots, c_n]$, $\deg c_i = 2i$. The corresponding topological result is $BU(n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$, and $H^\bullet_{\text{sing}}(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{C}) =$

$\mathbb{C}[x_1, \ldots, x_n]$, $\deg x_i = 2i$. The topological interpretation is that the Chern classes are the pullbacks of cohomology classes living in a universal object.

- Abstract Chern-Weil theory

The key ingredient of the prior construction was existence of the connection form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$. Similarly, we have $\mu :\in \Omega^2(P) \otimes \mathfrak{g}$. Then, given an Ad-invariant polynomial $P \in \mathrm{Sym}(\mathfrak{g}^*)$, we have the corresponding characteristic class $P(\mu) \in \Omega^{\mathrm{even}}(P)$ defined by contracting the $\mathfrak{g}$ in μ with the $\mathfrak{g}^*$ in P.

Consider the free supercommutative algebra W generated by the components of θ and μ. The components of θ are naturally parameterized by elements of $\mathfrak{g}^*$ via the map $\nu \mapsto \nu(\theta)$ for any $\nu \in \mathfrak{g}^*$. Since these components are of odd degree, the free algebra generated by θ is isomorphic to $\Lambda^*(\mathfrak{g}^*)$, in which elements of $\mathfrak{g}^*$ have degree one. Similarly, since the components of μ are two-forms, the corresponding algebra is $\mathrm{Sym}(\mathfrak{g}^*)$, where elements of $\mathfrak{g}^*$ have degree two. Thus

$$W \cong \Lambda^*(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*).$$

The condition "G acts freely on M" is too unwieldly in de Rham theory. Much better is the condition "G acts locally freely on M" because this has an infinitesimal expression: namely that the map $\mathfrak{g} \to T_pP$ given by $\xi \mapsto \xi_P|_p$ should be injective for all $p \in P$. This is equivalent to "P admits a connection." Since W is a universal construction based on the existence of a connection, this further translates to the condition "$\Omega^\bullet(P)$ admits a W-module structure."

In addition to being a supercommutative algebra, W has additional structure. For example, a differential is given by the Cartan equations

$$d\theta = \mu - \theta \wedge \theta,$$

$$d\mu = \mu \wedge \theta - \theta \wedge \mu.$$

In addition to the differential, one can define on W the notions of Lie derivative and contraction with respect to the vector fields ξ_P. An algebra with these structures is called a $G^\star$-algebra in Guillemin & Sternberg's book. Implicit in the statement "$\Omega^\bullet(P)$ admits a W-module structure" is the requirement that the module structure preserve the differential, Lie algebra, and contraction. For more details, refer to the book, or my separate notes on "definitions."

Since we have a differential on the graded algebra W, we may ask what is $H^\bullet(W)$? It's not hard to show that $H^\bullet(W) = \mathbb{C}$ in degree zero and is zero elsewhere. Thus, algebraically, W looks like a point. One can make sense of the statement that $W_{\mathrm{hor}} = \mathrm{Sym}(\mathfrak{g}^*)$ is generated by the components of curvature (degree 2), and $W_{\mathrm{bas}} = \mathrm{Sym}(\mathfrak{g}^*)^G$, the Ad-invariant polynomials on $\mathfrak{g}$. (All details are carefully done in Guillemin's book.) One then shows that when restricted to W_{bas}, $d = 0$ so that $H^\bullet(W_{\mathrm{bas}}) = \mathrm{Sym}(\mathfrak{g}^*)^G$.

All this shows that $H^\bullet(W) = H^\bullet_{\text{sing}}(\text{pt})$ and $H^\bullet(W_{\text{bas}}) = \text{Sym}(\mathfrak{g}^*)^G = H^\bullet_G(\text{pt})$, and in this sense, W serves as an algebraic model for EG. The precise topological analogy is as follows. We have four differential graded algebras:

$$\begin{array}{lccc} \text{Differential:} & W_{\text{bas}} & \subset & W, \\ \text{Topological:} & \mathcal{C}^\bullet_{\text{sing}}(BG;\mathbb{C}) & \subset & \mathcal{C}^\bullet_{\text{sing}}(EG;\mathbb{C}). \end{array}$$

Taking cohomology, we get

$$\begin{array}{lccc} \text{Differential:} & \text{Sym}(\mathfrak{g}^*)^G & \longrightarrow & \mathbb{C}, \\ \text{Topological:} & \text{Sym}(\mathfrak{g}^*)^G & \longrightarrow & \mathbb{C}, \end{array}$$

where the map to $\mathbb{C}$ is the restriction to the degree zero constant term. Thus, cohomologically, W is identical to EG.

- Weil model

We are now in a position to construct the Weil model for equivariant cohomology. We have

$$H^\bullet_G(M) = H^\bullet_{\text{sing}}\left(\frac{M \times EG}{G};\mathbb{C}\right).$$

As algebraic models, we can substitute the deRham model $\Omega^\bullet(M)$ for M, and W for EG. For the quotient, we take the basic subcomplex.

$$H^\bullet_G(M) = H^\bullet((\Omega^\bullet(M) \otimes W)_{\text{bas}}).$$

Note that tensoring with W tautologically turns $\Omega^\bullet(M)$ into a W-module, and corresponds topologically to multiplying by EG. Taking the basic cohomology corresponds to taking the quotient by G. Thus our topological functors have differential geometric analogues.

- Cartan model

Using an algebraic trick called the Mathai-Quillen isomorphism, we can simplify this further. It allows us to "rotate" so that the notion of horizontal affects only the W component:

$$(\Omega^\bullet(M) \otimes W)_{\text{hor}} \cong \Omega^\bullet(M) \otimes W_{\text{hor}} = \Omega^\bullet(M) \otimes S(\mathfrak{g}^*).$$

Such an element is a polynomial on $\mathfrak{g}$ with coefficients in $\Omega^\bullet(M)$.

Now $(\Omega^\bullet(M) \otimes W)_{\text{bas}} \cong (\Omega^\bullet(M) \otimes S(\mathfrak{g}^*))^G$ corresponds to the subset of equivariant polynomials. Furthermore, if ω is a polynomial in the variable ξ with $\Omega^\bullet(M)$ coefficients, then the differential turns out to be

$$(d_G\omega)(\xi) = d(\omega(\xi)) - i_\xi(\omega(\xi)).$$

To understand why this particular differential is the correct one, we would have to go through the Matthai-Quillen isomorphism, which we won't do. Anyway, we have arrived at the Cartan model.