LPTHE 86/16

The geometry of the space of fields in Yang-Mills theory.

Claude-M. Viallet

Lectures given at the XXII Karpacz Winter School of Theoretical Physics (1986)

Physique Théorique et des Hautes Energies (Unité Associée CNRS UA 280)
Université Pierre et Marie Curie
Tour 16. 1° Etage.
4 Place Jussieu
75230 PARIS CEDEX 05.

Abstract.

In the framework of Yang-Mills gauge theories, we describe the main features of constrained systems, and the geometry introduced by their symmetries.

The structures we describe exist in all kinds of such systems, having an infinite number of degrees of freedom or not.

Among the features which these lectures explain:

-The group of gauge transformations arises from the analysis of constraints in the definition of the hamiltoninan à la Dirac.

-The space of gauge potentials is a (infinite dimensional) principal fibre bundle with group the (infinite dimensional) group of gauge transformations and base space the "true" configuration space of the theory (orbit space).

-There is a deep link between the Faddeev-Popov determinant and the natural riemannian metric on the orbit space.

-The origin of the Gribov ambiguity is a property of the geodesics on the orbit space (existence of focal points).

-The Becchi-Rouet-Stora operator is a part of the exterior differential on the space of connections, and the ghost field lives naturally on this space.

-The consistency equation for anomalies is a cohomological problem on the space of fields.

It has been realized for a long time that any proper description of gauge fields [1] requires some basic notions of differential geometry (like principal fibre bundles, connections,etc...),and that these concepts flourished both in mathematics and physics [2].

It will not be my purpose to explain why a gauge potential $A_\mu(x)$ is a component of a connection in some finite dimensional principal bundle, why the notion of a bundle is just the right one to accommodate fancy boundary conditions on the fields, or things of the sort. There is a number of references on that [3] to [18]. See also [19].

My purpose is to describe the geometry of the entire space of all gauge fields (an infinite dimensional space, since we are dealing with field theory).

What is remarkable is that basic notions of differential geometry applied to this space are the suitable concepts to understand the characteristic features of gauge theory, and in fact all kinds of theories with symmetry.

The lectures are organized as follows:

1. Notations and basic objects.
2. The space of connections $\mathcal{C}$ and the action of the group of gauge transformations $\mathcal{G}$.
3. The metric on $\mathcal{C}$, the gauge fixing problem, and $\mathcal{C}$ as a $\mathcal{G}$-bundle. The connection on the space of connections.
4. The metric on the orbit space $\eta=\mathcal{C}/\mathcal{G}$.
5. Dirac's analysis of the lagrangian.
6. The riemannian geometry of η.
7. Functional measures on the orbit space: the origin of the Dirac-Faddeev determinant, and the geometrical meaning of the Faddeev-Popov determinant.
8. More on the geometry of the orbit space.
9. The Gribov ambiguity in gauge fixing.
10. The Becchi-Rouet-Stora operator and the ghost field.
11. The anomaly problem as a cohomological problem on $\mathcal{C}$.
12. Conclusion.

1. Notations and basic objects [11].

We deal with gauge theory over space-time M. Space-time will be of any dimension, especially 4-dim euclidean space-time in the covariant case, and M=RxV with V=3-dim euclidean space in the hamiltonian formalism (resp. M or V will be supposed compact and without boundary, which is a way of introducing a volume cut-off into the theory).

It is important to notice that the geometry of the space of fields is essentially <u>not</u> sensitive to the dimension of M, except in 1+1 dimensions where it somewhat degenerates.

The structure group G will be a compact Lie group. Its Lie algebra is denoted by $\mathbf{g}$.

Gauge potentials are connections on a principal fibre bundle P(M,G).

We use two interesting associated bundles, constructed from P:

$$E=P\times_{Ad}\mathbf{g}.$$

(vector bundle with fibre $\mathbf{g}$, with the adjoint action of G on $\mathbf{g}$),and

$$F=P\times_{ad}G.$$

(bundle with fibre the group G, with the adjoint action).

We also introduce spaces of forms on M with values in E:

$$\mathbf{A}^p=\Gamma(E\otimes\Lambda^p(M)),\quad \mathbf{A}=\oplus\mathbf{A}^p.$$

If ω is a connection on P, we have a corresponding covariant derivative ∇ acting on $\mathbf{A}$.

$$\nabla:\ \mathbf{A}^p\to\mathbf{A}^{p+1}.$$

With the metric on M and a biinvariant metric on G (denoted tr), we may define a scalar product in $\mathbf{A}^p$, using the Hodge $*$ operator:

$$\forall\alpha\in\mathbf{A}^p,\ \forall\beta\in\mathbf{A}^p\quad (\alpha,\beta)=\int_M \mathrm{tr}(\alpha\wedge*\beta).$$

The covariant derivative ∇ has an adjoint with respect to the scalar product (,), the covariant divergence ∇^*: $\mathbf{A}^{p+1}\to\mathbf{A}^p$, such that

$$\forall\tau\in\mathbf{A}^p,\ \forall\xi\in\mathbf{A}^{p+1}\qquad (\tau,\nabla\xi)=(\nabla^*\tau,\xi).$$

We will use the covariant laplacian on $\mathfrak{A}^0$

$$\Box_\omega = \nabla_\omega{}^* \nabla_\omega : \mathfrak{A}^0 \to \mathfrak{A}^0.$$

When the laplacian is invertible, we denote its inverse by G_ω:

$$G_\omega . \Box_\omega = \Box_\omega . G_\omega = \mathbb{1}.$$

2. The space $\mathcal{C}$ of connections and the group $\mathcal{G}$ of gauge transformations.

The local expression of gauge transformations is very well known: the transformation is given by a G-valued function g on M, and the action on the components $A_\mu(x)$ is $A_\mu(x) \to g^{-1}(x)A_\mu(x)g(x)+g^{-1}(x)\partial_\mu g(x)$.

The correct way of describing such a transformation on a connection ω on P is the following:

Strictly speaking, a gauge transformation is an automorphism of P, which induces the identity mapping on the base space.

Phrased differently, it is a mapping f of P into itself, which moves the points of P along fibres, and commutes with the group action on P: $\forall p \in P$, f(p) belongs to the same fibre as p, and $\forall a \in G$, $\forall p \in P$, $f(p.a)=f(p).a$ (where p.a denotes the right action of $a \in G$ on $p \in P$).

A gauge transformation may equivalently be described by a G-valued function φ on P, since we can always write $f(p)=p.\varphi(p)$.

The equivariance property of f reads

$$\varphi(p.a)=a^{-1}\varphi(p)a=ad_{a^{-1}}(\varphi(p)).$$

This last relation shows that we may consider gauge transformations as defined on M, provided their values are taken not in G, but in the bundle F introduced above.

The product of gauge transformations is just the composition of mapping on P, and gives the pointwise product in G.

We denote by $\mathcal{G}$ the group of gauge transformations.

$\mathcal{G}$ acts naturally on any connection on P by pull-back.

Clearly, we recover that an element of $\mathcal{G}$ is, locally, a G-valued function on M, and that the usual gauge transformation formula is a change of coordinates under a change of sections of P.

It is possible to show that $\mathcal{G}$ (= space of sections of F = $\Gamma(F)$) is a true Lie group (although infinite dimensional).

Its Lie algebra is the space of sections of E ($=\Gamma(E)=\mathcal{A}^0$).

The action of $\mathcal{G}$ on $\mathcal{C}$ is: $\omega \to \omega^g = \omega + g^{-1}\nabla g.$ (eq. 1)

What is noticeable in this transformation law is that $\mathcal{C}$ is not a vector space. It is an affine space, since the difference τ of any two connections transforms covariantly. Actually $\tau \in \mathcal{A}^1$, and thus the tangent space to $\mathcal{C}$ is canonically $\mathcal{A}^1$.

We shall denote by $T_\omega(\mathcal{C})$ the tangent space to $\mathcal{C}$ at ω.

The gauge transformation formula (eq. 1) reduces, for an infinitesimal gauge transformation $\xi \in \mathcal{A}^0$, to

$$\omega \to \omega + \nabla\xi.$$

This gives the form of the elements of $T_\omega(\mathcal{C})$ which are tangent to the fibre through ω. These are the vertical vectors at ω. We denote by $V_\omega(\mathcal{C})$ the vector space of vertical vectors at ω.

From the expression of vertical vectors, it easy to see that the action of $\mathcal{G}$ on $\mathcal{C}$ has no fixed point, if for example we impose some normalization to the gauge transformations. It is sufficient to suppose that gauge transformations are normalized to unity at some point,or equivalently that infinitesimal transformations vanish at this point. For ω to be a fixed point of the infinitesimal transformation ξ, we have to have $\omega = \omega + \nabla\xi$, or $\Box_\omega\xi=0$, which implies, with our hypothesis, $\xi=0$.

3. The metric on $\mathcal{C}$, the gauge fixing problem, and $\mathcal{C}$ as a (big) fibre bundle. The connection on the space of connections.

The scalar product (,) on $\mathbf{A}^1 \approx \mathbf{T}_\omega(\mathbf{C})$ is a metric on $\mathbf{C}$.

With that metric, $\mathbf{C}$ is flat, since (,) does not depend on ω.

Moreover the metric on $\mathbf{C}$ is <u>gauge invariant</u>.

This is the basic point for what we will say in these lectures.

It is a fundamental principle of the theory that two gauge potentials related by a gauge transformation are equivalent and describe the same physical reality [1]. This will also appear in the analysis of the lagrangian (see § 5.).

The gauge fixing is just the choice of one representative in each equivalence class (orbit). We want to draw a surface in $\mathbf{C}$ which cuts all orbits once (define a section of the $\mathbf{G}$-bundle $\mathbf{C}$).

We may do this locally around an origin ω_0 (reference connection) as follows:

Define the affine subspace $\mathbf{S}_0$ of $\mathbf{C}$

$\mathbf{S}_0 = \{\omega \in \mathbf{C}$ s.t. $\tau = \omega - \omega_0$ is orthogonal to the orbit through $\omega_0\}$.

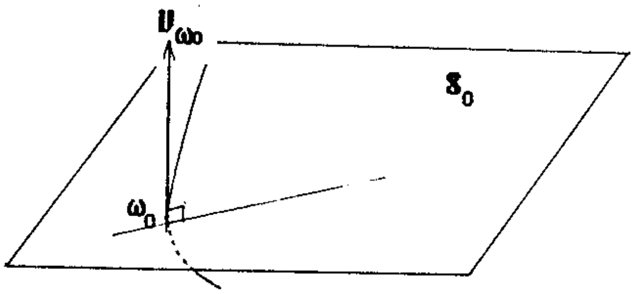
$\mathbf{S}_0$ is made out of points which depart from ω_0 perpendicularly to the orbit. These points verify

$$(\tau, \nabla_0 \xi) = 0 \quad \forall \xi \in \mathbf{A}^0, \text{ or}$$

$$(\xi, \nabla_0{}^* \tau) = 0 \quad \forall \xi \in \mathbf{A}^0, \text{ or equivalently}$$

$$\nabla_0{}^* \tau = 0. \qquad \text{(eq. 2)}$$

This is a linear condition on τ, and defines what we call <u>horizontal</u> vectors at ω_0. We denote by $\mathbf{H}_{\omega_0}$ the space of solutions of equation 2: $\mathbf{S}_0$ is the affine space generated by $\mathbf{H}_{\omega_0}$ when ω_0 is taken as origin.



<u>Claim:</u> $\mathcal{S}_0$ is a good gauge section around ω_0(covariant background gauge condition around ω_0)[20][21][22][23].

It was shown by topological methods [23][24] that there is no global section: if one goes sufficiently far away from ω_0 (within $\mathcal{S}_0$) one has to meet a point gauge related to ω_0. Our claim is that there is a region <u>of finite radius</u> around ω_0 in $\mathcal{S}_0$, where no two gauge related point exist, and that all orbits in the vicinity of the orbit through ω_0 cut $\mathcal{S}_0$inside that region.

We will return to the problem of gauge fixing later.

The previous result is the property of local triviality, basis of the stucture of fibre bundle of $\mathcal{C}$.

Actually, with some care taken of the spaces of functions we work with (Sobolev spaces), one can show that the action of $\mathcal{G}$ on $\mathcal{C}$ does define a nice fibre bundle, and that the orbit space is modelled on $\mathcal{S}_0$. [see [23] [25] [26] [27], especially for the more delicate points of the definition of normalized group and of the restriction to irreducible connections].

Notice that a similar structure exists on the space of metrics on a riemannian manifold [28][29]: gauge transformations are replaced by diffeomorphisms, irreducible connections are replaced by metrics without isometries, and the same kind of objects on the spaces of metrics have exactly the same kind of structure. This is used in gravity theory and in string theory.

We denote by p the projection: $\mathcal{C} \to \mathcal{C}/\mathcal{G}$.

We have already defined a <u>connection on $\mathcal{C}$</u>, with our horizontality condition:

Indeed if we define the 1-form χ on $\mathcal{C}$ with values in the Lie algebra of $\mathcal{G}$

(i.e. $\mathfrak{A}^0$) by:

$$\chi = G_\omega \nabla_\omega^*.$$

then:

-the kernel $\mathbf{H}_\omega$ of χ at each point ω in $\mathcal{C}$ defines a distribution of horizontal spaces invariant by $\mathcal{G}$.

-the value of χ on a fundamental vector field ξ^* (vertical vector field on $\mathcal{C}$ generated by the infinitesimal action of $\xi \in \mathfrak{A}^0$) is ξ itself.

- χ transforms with the adjoint representation of $\mathcal{G}$.

-some regularity properties are required,which are satisfied.

We define the horizontal projection operator $\Pi_\omega : \mathbf{T}_\omega(\mathcal{C}) \to \mathbf{H}_\omega$ by:

$$\Pi_\omega = \mathbf{1} - \nabla_\omega G_\omega \nabla_\omega^* = \mathbf{1} - \nabla_\omega \chi_\omega .$$

(or the vertical projection operator $\nabla_\omega \chi_\omega : \mathbf{T}_\omega(\mathcal{C}) \to \mathbf{V}_\omega$).

The operator Π_ω verifies:

$$\Pi_\omega^2 = \Pi_\omega^* = \Pi_\omega .$$

$$\chi_\omega \Pi_\omega = 0.$$

4. The metric on the orbit space $\eta = \mathcal{C}/\mathcal{G}$.

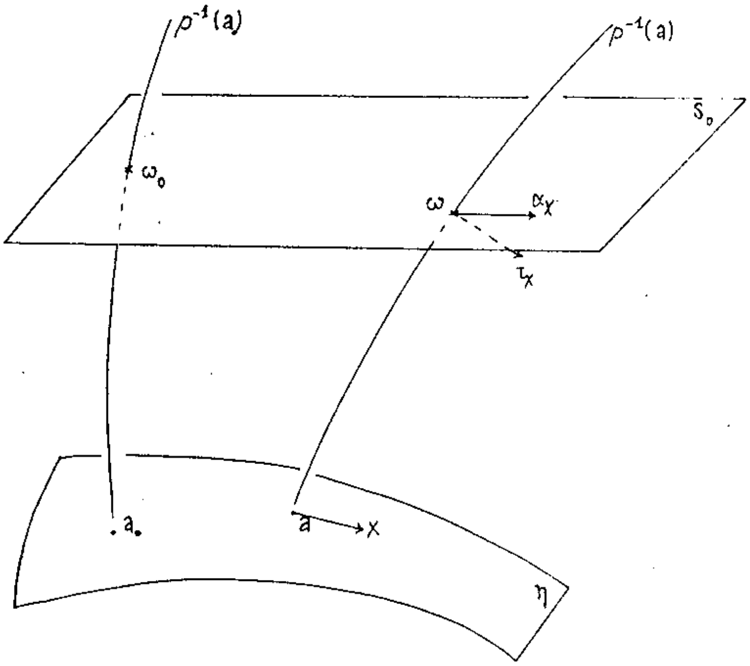
We define a scalar product in the tangent space $\mathbf{T}_a(\eta)$ at any point $a \in \eta$ as the one induced by (,):

If X, Y $\in \mathbf{T}_a(\eta)$, choose any point ω in the fibre $p^{-1}(a)$ above a. The vectors X and Y have horizontal lifts τ_X and τ_Y at ω. By definition the scalar product (metric on η) is:

$$g(X,Y) = (\tau_X, \tau_Y).$$

The gauge invariance of (,) ensures the independence of g on the choice of ω in $p^{-1}(a)$.

We can now compute the metric g in the local coordinate system centered at ω_0 and defined by $\mathbf{S}_0$.



The vectors X, Y $\in \mathbf{T}_a(\eta)$ have coordinates α_X and α_Y such that:

$$\nabla_0{}^*\alpha_X = \nabla_0{}^*\alpha_Y = 0 \text{ (or } \Pi_0\alpha_X=\alpha_X,\ \Pi_0\alpha_Y=\alpha_Y).$$

Clearly α_X is not the horizontal lift τ_X of X at $\omega\in\mathbf{S}_0$ (resp. α_Y...). These horizontal lifts are:

$$\tau_X = \Pi_\omega(\alpha_X).$$

$$\tau_Y = \Pi_\omega(\alpha_Y).$$

Thus $g(X,Y) = (\Pi_\omega\alpha_X, \Pi_\omega\alpha_Y)$, or

$$g(X, Y) = (\alpha_X\ ,\ \Pi_0\Pi_\omega\Pi_0\ \alpha_Y).$$

5. Dirac analysis of the lagrangian.

This analysis of the lagrangian leads to the construction of the hamiltonian of the theory. We thus use the canonical formalism (non covariant) where time is separated from space [30] [31] [32] [33] [34] [35]. Gauge potentials are time dependent connections on a bundle over 3-dimensional space V.

The action is:

$$S = {}^1/_4 \int dt \int_V dv \; tr(F_{\mu\nu}F^{\mu\nu}), \text{ with}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (\mu, \nu = 0,1,2,3).$$

With our notations, the lagrangian is:

$$\mathbf{L} = {}^1/_2 \; (\dot{A} - \nabla A_0, \dot{A} - \nabla A_0) - \mathbf{U}$$

where

$$\dot{A} = \partial A/\partial t, \text{ and}$$

$$\mathbf{U} = {}^1/_4 \; (\Omega, \Omega),$$

with Ω the curvature 2-form of A. ($\Omega \in \mathbf{A}^2$).

The conjugate momenta are:

$$\mathbf{p} = \dot{A} - \nabla A_0.$$

$$\mathbf{p}_0 = 0.$$

The last equation is the primary constraint and leads to the hamiltonian

$$H_0 = {}^1/_2 \; (\mathbf{p},\mathbf{p}) + \mathbf{U} + (\mathbf{p}, \nabla A_0) + (\lambda, \mathbf{p}_0),$$

where λ is a lagrange multiplier. ($\lambda \in \mathbf{A}^0$).

We get as a secondary constraint:

$$\{H_0, \mathbf{p}_0\} = \nabla^* \mathbf{p} = 0. \qquad \text{(Gauss condition)}$$

The hamiltonian becomes by incorporating Gauss condition:

$$H_T = H_0 + (\mu, \nabla^* \mathbf{p}),$$

yielding as equations of motion:

$$\dot{\mathbf{p}}_0 = 0.$$

$$\dot{A}_0 = \lambda.$$

A_0 appears as an unphysical degree of freedom, which we have to discard. The true hamiltonian is thus:

$$H = {}^1/_2\,(\mathbf{p},\mathbf{p}) + \mathcal{U} + (\xi,\nabla^*\mathbf{p}),$$

with ξ a lagrange multiplier. ($\xi \in \mathcal{A}^0$).

The equation of motion is

$$\dot{A} = \{H,A\} = \mathbf{p} + \nabla\xi.$$

The time evolution of A contains an horizontal part $\mathbf{p}$ ($\mathbf{p}$ is horizontal from Gauss condition) and a vertical part $\nabla\xi$ (pure gauge variation induced by the Lagrange multiplier). From Gauss condition we may express $\mathbf{p}$ in terms of A:

$$\mathbf{p} = (1-\nabla G\nabla^*)\dot{A} = \Pi_A\dot{A}.$$

and the true lagrangian is:

$$\mathcal{L} = {}^1/_2\,(\Pi_A\dot{A},\Pi_A\dot{A}) - \mathcal{U}.$$

The lagrangian $\mathcal{L}$ is naturally defined on the orbit space. Both parts of $\mathcal{L}$ are gauge invariant, and the true configuration space appears to be the orbit space. The first term is a kinetic energy term constructed with the metric g on η. The second term is a potential part (magnetic part).

On the true configuration space, the lagrangian is of course non singular and of the typical form:

$$\mathcal{L}(q,\dot{q}) = {}^1/_2\, g(\dot{q},\dot{q}) - \mathcal{U}(q),$$

where q denotes a generic point of η, and $\dot{q}$ denotes its velocity.

6. The riemannian geometry of η.

The previous paragraph shows that the classical evolution of the Yang-Mills fields is a motion on a non flat configuration space with a potential term $\mathcal{U}$.

This motivates a detailed study [35] of the riemannian geometry of η.

We will perform our computations in the local coordinate system given by the covariant background gauge condition around a reference connection ω_0.

We define the following operators, associated to a generic point ω ($\omega \in \mathbf{S}_0$).

$\gamma: \mathbf{A}^0 \to \mathbf{A}^0 \qquad \gamma=\nabla_0{}^*\nabla_\omega=\nabla_\omega{}^*\nabla_0$.

γ is the Faddeev-Popov operator in the coordinate system we consider.

γ is invertible if ω is sufficiently close to ω_0.

$\mathbf{P}: \mathbf{A}^1 \to \mathbf{A}^1 \qquad \mathbf{P}=\mathbf{1}-\nabla_0\gamma^{-1}\nabla_\omega{}^*$.

$\mathbf{P}$ is the projection on $\mathbf{H}_\omega$ along $\mathbf{U}_0$, and reduces to Π_0 if $\omega=\omega_0$.

We have a number of relations between Π and $\mathbf{P}$, especially:

$$\Pi_0\Pi_\omega\Pi_0\, \mathbf{P^*P} = \mathbf{P^*P}\, \Pi_0\Pi_\omega\Pi_0 = \Pi_0,$$

meaning that $\mathbf{P^*P}$ is the inverse of the metric in our coordinate system.

Finally define, for any $\tau \in \mathbf{A}^1$:

$\mathbf{K}_\tau: \mathbf{A}^p \to \mathbf{A}^{p+1} \qquad \mathbf{K}_\tau(\xi)=[\tau,\xi]$.

$\mathbf{K}_\tau{}^*: \mathbf{A}^{p+1} \to \mathbf{A}^p$ its adjoint.

The riemannian connection D on η may easily be written for vector fields having constant coordinates X, Z (and thus commuting).

$$D_X Z={}^1/_2\mathbf{P^*P}(-\chi_\omega{}^*\mathbf{K}_X{}^*\Pi_\omega Z-\Pi_\omega\mathbf{K}_X\chi_\omega Z-\chi_\omega{}^*\mathbf{K}_Z{}^*\Pi_\omega X-\Pi_\omega\mathbf{K}_Z\chi_\omega X+[\chi_\omega X,\Pi_\omega Z]+[\chi_\omega Z,\Pi_\omega X]).$$

The riemannian curvature tensor is

$$R(X,Y)Z = \Pi_0[-2\mathbf{K}_Z G\mathbf{K}_X{}^*(Z)-\mathbf{K}_Y G\mathbf{K}_X{}^*(Z)+\mathbf{K}_X G\mathbf{K}_Y{}^*(Z)].$$

(nb: this expression is valid at the center of coordinates).

The sectional curvature in the 2-plane generated by the two orthogonal vectors X and Y is:

$$\mathcal{K}(X,Y) = 3\ (\mathbf{K}_X{}^*(Y), G\mathbf{K}_X{}^*(Y)).$$

We see that η is of positive sectional curvature. However there is no strictly positive lower bound for $\mathcal{K}$.

7. Functional measures on the orbit space: the origin of the Dirac-Faddeev determinant and the geometrical meaning of the Faddeev-Popov determinant.

When one uses a functional integral formalism to write down rules of quantization for the Yang-Mills theory, one is lead to a functional measure which depends on the gauge condition [31] [36] [37] [38] [39].

It is very important to distinguish the hamiltonian formalism and the covariant formalism.

As a first step we will compare (formally) the spectra of the operators γ (on $\mathcal{A}^0$) and the metric operator g (in the tangent space to η).

The difficulty comes from the fact that γ essentially acts on vertical vectors, while g acts on horizontal vectors, and thus on spaces of different dimensions.

Let us introduce the operator $\mathbf{Q}$: $\mathcal{A}^1 \to \mathcal{A}^1$ defined by:

$$\mathbf{Q} = \Pi_0 \nabla_\omega G_\omega \nabla_\omega{}^* = \Pi_0(1-\Pi_\omega).$$

$\mathbf{Q}$ sends $\mathcal{V}_\omega$ in $\mathbf{H}_0$.

Its adjoint is $\mathbf{Q}^* = \nabla_\omega G_\omega \nabla_\omega{}^* \Pi_0 = (1-\Pi_\omega)\Pi_0$.

$\mathbf{Q}^*$ sends $\mathbf{H}_0$ in $\mathcal{V}_\omega$.

On $\mathbf{S}_0$, the metric can be written $g = 1 - \mathbf{Q}\mathbf{Q}^*$.

Let $\mathbf{h}$: $\mathcal{A}^0 \to \mathcal{A}^0$ be the operator:

$$\mathbf{h}=G_\omega\nabla_\omega{}^*\nabla_0 G_0\nabla_0{}^*\nabla_\omega=\chi_\omega(1-\Pi_0)\nabla_\omega.$$

There exists an isomorphism between $\mathbf{A}^0$ and $\mathbf{U}_\omega$ given by:

$$\nabla_\omega: \mathbf{A}^0 \rightarrow \mathbf{U}_\omega.$$

Its 'inverse' is given by $\chi_\omega: \mathbf{U}_\omega \rightarrow \mathbf{A}^0$.

Thus $\mathbf{h}$ is similar to $\mathbf{h}': \mathbf{U}_\omega \rightarrow \mathbf{U}_\omega$

$$\mathbf{h}'=\nabla_\omega \mathbf{h}\chi_\omega.$$

It is easy to check that $\mathbf{h}'=1-\mathbf{Q}^*\mathbf{Q}$.

From the fact that $\mathbf{QQ}^*$ and $\mathbf{Q}^*\mathbf{Q}$ have the same non zero spectrum, and that $\det_{A^1}\mathcal{g}=\det_{\mathbf{g}_0}\mathcal{g}$, we get:

$$\det_{\mathbf{g}_0}\mathcal{g} = \det_{\mathbf{U}_\omega}\mathbf{h}' = \det_{A^0}\mathbf{h}.$$

or, formally, by assuming that the determinant of a product is the product of determinants, we get the basic identity [39]:

$$\det \mathcal{g}\,.\,\det\Box_0\,.\,\det\Box_\omega = (\,\det \gamma\,)^2. \qquad \text{(eq. 3)}$$

We denote by $\mathcal{g}_3$ the metric on the true configuration space, and by $\mathcal{g}_4$ the metric on the orbit space of 4-dimensional potentials.

In the <u>canonical formalism</u>, we see that the measure (up to constant factors) is

$$\prod_{\text{time}}\sqrt{\det \mathcal{g}_3}\ ,$$

a naive natural volume element for paths over the orbit space.

In the <u>covariant formalism</u> however we have:

$$\text{Faddeev-Popov determinant } = \sqrt{\det \mathcal{g}_4}\,.\sqrt{\det\Box_\omega}$$

The factor $\sqrt{\det\Box_\omega}$ being the scale of the fibre through ω, the covariant functional integral is an integral <u>over the whole space of connections rather than over</u>

the orbit space.

Notice that the same phenomenon happens when one wants to integrate over the space of metrics an action which is invariant by diffeomorphisms, in gravity theory [40] [41] [42] [43] as well as in string theory [44, and L. Alvarez Gaumé's lectures in this School].

8. More on the riemannian geometry of the orbit space: geodesics.

The geodesics of $\mathbf{C}$ are all straight lines in $\mathbf{C}$.

It is a general property that, for any group action on a riemannian manifold, and provided the metric is invariant by the group action, if one geodesic cuts one orbit perpendicularly at some point, then it cuts all orbits it meets perpendicularly [45]. Some straight lines in $\mathbf{C}$ have this property, as we may see directly:

Suppose we consider the line through ω of unit vector τ:

$$\omega = \omega_0 + \lambda\tau \qquad (\lambda \in R).$$

Such a line is horizontal at ω_0 if $\nabla_0{}^*\tau=0$.

It is then horizontal at all its points since

$$\nabla_\omega{}^*\tau = \nabla_0{}^*\tau + \lambda K_\tau{}^*(\tau) = \nabla_0{}^*\tau = 0.$$

Therefore we have a notion of horizontal line in $\mathbf{C}$.

Claim: Geodesics on η are just the projection of horizontal lines.

The proof is immediate from the geodesics equation [35].

Remark 1: If a_1 and a_2 are two points in η, we may evaluate the distance between a_1 and a_2. Take a generic point ω_1 (resp $\omega_2=\omega_1+\tau$) in $\not{p}^{-1}(a_1)$ (resp $\not{p}^{-1}(a_2)$). The distance in $\mathbf{C}$ between ω_1 and ω_2 is $\alpha=\sqrt{(\tau,\tau)}$. It is invariant by a simultaneous gauge transformation of ω_1 and ω_2. To define a distance d_η on η, we may take the minimum of α when ω_2 runs along its fibre a_2. When α is minimized, we have $\nabla_2{}^*\tau=0$ and thus, at least locally, d_η = geodesic distance.

Remark 2: Suppose we start from a point ω in $\mathbf{C}$, along some horizontal straight line, then the orbits we meet are all perpendicular to the line we follow, but they do not remain perpendicular to $\mathbf{S}_0$.

Remark 3: Since the metric g is defined via the connection χ (itself issued from the metric on $\mathbf{C}$), the projection p : $\mathbf{C} \to \eta$ of horizontal lines preserves length. Thus η is geodesically complete, for all straight lines are of infinite length.

Remark 4: The property of the geodesics shows that the covariant background gauge around ω_0 yields a normal coordinate system at ω_0.

Remark 5: Similar properties hold true for the space of moduli of metrics.

9. The Gribov ambiguity in gauge fixing.

Suppose we use the covariant backgroung gauge around ω_0.

The Faddeev-Popov operator γ is invertible as long as ω is in a neighbourhood of ω_0 (there exists such a neighbourhood); However if we go far enough from ω_0, then at the point $\omega=\omega_0+\lambda\tau$, the operator $\gamma(\lambda)=\Box_0+\lambda\nabla_0{}^*\mathbf{K}_\tau$ may become non invertible.

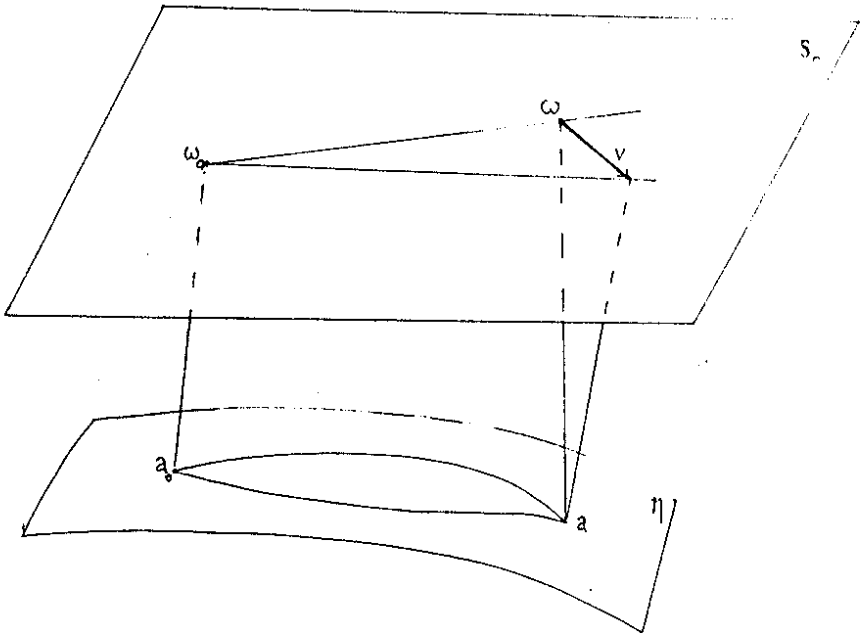
This is where the Gribov ambiguity appears [46] [24] [35] [47].

It is the point where the coordinate system becomes singular. At this point there exist vectors v, which are vertical, but verify the gauge condition $\nabla_0{}^*v=0$

(equivalent to saying that γ has a kernel: if $\gamma(\xi)=0$, then $v=\nabla_\omega\xi$ is such a vector).

The point ω is the first focal point of ω_0 in the direction τ.

The picture is the following:



The vector v projects to zero on η.

At the point ω, the projection ρ from $\mathbf{S}_0$ to η is singular.

The region of $\mathbf{S}_0$ where $\det g \geq 0$ is convex and is precisely the region where the coordinate system is non singular (the riemannian exponential is non singular) [48].

To know if and how that region covers the whole orbit space is an open question.

10. The Becchi-Rouet-Stora operator and the ghost field.

The behaviour under gauge transformations of any function of the connections is easy to test: we just have to compute the derivatives of the function along the fibres.

Notice that this will test infinitesimal gauge transformations. If $\mathcal{G}$ has more than one connected component, we stay within the component of the identity [49] [50].

For infinitesimal gauge transformations, we may do the following [51] [52] [53]:

Let $d_{\mathcal{C}}$ be the exterior derivative on $\mathcal{C}$. Define the vertical part δ of $d_{\mathcal{C}}$ by: if φ is a q-form on $\mathcal{C}$, then:

$$\delta\varphi(X_1,X_2,\dots,X_{q+1})=d_{\mathcal{C}}\varphi(V_1,V_2,\dots,V_{q+1}),$$

where V_i is the vertical part of X_i $(V_i=\nabla\chi X_i)$.

For a function on $\mathcal{C}$ (zero-form), we measure the variation along fibres.

Notice that this definition is similar to the definition of the covariant derivative (one would take horizontal parts and not vertical parts). However, contrarily to what happens for the covariant derivative, we have:

$$\delta^2=0,$$

by integrability of the distribution of vertical spaces.

δ is the Becchi-Rouet-Stora operator.

Let $\Omega^p(P)$ be the space of p-forms on P with values in the Lie algebra $\mathfrak{g}$, which transform by ad under $\mathcal{G}$.

Let $S^{p,q}$ be the space q-forms on $\mathcal{C}$ with values in $\Omega^p(P)$, and which are invariant by $\mathcal{G}$. (and $S=\oplus S^{p,q}$).

The exterior differential d_P of P acts on S, by acting on the values.

But the exterior derivative $d_{\mathcal{C}}$ of $\mathcal{C}$ also acts on S. So does δ. [We shall take into account the degree of the value by using $(-)^p d_{\mathcal{C}}$ on $S^{p,q}$ rather than $d_{\mathcal{C}}$ (resp. $(-)^p\delta$ rather than δ)].

The function ω, defined on $\mathcal{C}$, and which to any connection on P associates its connection 1-form belongs to $S^{1,0}$.

$\delta\omega$ is a 1-form on $\mathcal{C}$ with values $\Omega^1(P)$.

$$\delta\omega(\tau) = - \text{ vertical part of } \tau = -\nabla\chi(\tau).$$

Thus

$$\delta\omega = -\nabla\chi,$$

which is the B.R.S. transformation of the gauge potential.

The connection 1-form χ on $\mathbf{C}$ belongs to $S^{0,1}$.

Since the curvature 2-form $\mathbf{R}=d_{\mathbf{C}}\chi+{}^{1}/_{2}[\chi,\chi]$ of the connection χ is horizontal in $\mathbf{C}$, we have

$$\delta\chi = -{}^{1}/_{2}\ [\chi,\chi].$$

χ is the ghost field.

11. The anomaly problem as a cohomological problem on $\mathbf{C}$.

Quantum anomalies are the breaking, at the quantum level of the classical gauge symmetry: some quantum diagrams, involving fermion loops, generate after renormalisation, non invariant interactions [54] to [61].

For example, if we denote by $\Gamma(A)$ the quantum effective action of a background gauge potential in the presence of quantized Weyl fermions, $\Gamma(A)$ may not be gauge invariant . Equivalently

$$\Delta = \delta\Gamma \neq 0.$$

Δ is the anomaly.

From $\delta^2=0$, we see immediately that:

$$\delta\Delta = 0. \qquad\qquad \text{(eq.4)}$$

This is the Wess-Zumino consistency condition [62].

From the way the non invariance of Γ appears at the level of Feynman graphs, it is known that Δ is an integral over space-time of some polynomial in the fields and their derivatives. It is always possible to redefine Γ by such a polynomial: the anomaly Δ is spurious if it is of the form $\Delta = \delta$(polynomial).

The problem of finding the true anomalies is thus a cohomological problem: we have to find $\Delta(A)$ (a vertical 1-form on $\mathbb{C}$) verifying $\delta\Delta=0$ modulo the trivial solutions of the form $\delta M(A)$ with 'local' functions. [see [53] and R. Stora's lectures in this School].

There is a simple way of producing solutions of eq. 4, from the cohomology of the orbit space:

Suppose $[\varphi]$ is in $H^2(\eta)$, i.e. φ is a 2-form on η in the cohomology of η (e.g. de Rham cohomology although the precise definition of this cohomology needs some detail) [63][64].

Then the pull-back $\psi=p^*\varphi$ is a 2-form on $\mathbb{C}$ such that:

a) $$d_{\mathbb{C}}\,\psi = 0.$$

b) ψ vanishes on vertical vectors.

Since $d_{\mathbb{C}}$ has no cohomology on $\mathbb{C}$, there exists a 1-form θ on $\mathbb{C}$ such that:

$$\psi = d_{\mathbb{C}}\,\theta.$$

Restricting θ to vertical vectors produces a solution of eq. 4. of ghost degree one.

What is remarkable is that on S^4, we get the usual chiral anomaly [65] [66] [67], although <u>the condition of locality is absent in this approach</u>.

It is an open problem to define the part of $H^*(\eta)$ which will give the correct (local) cohomology of δ.

Two different paths have been followed:

-take in $H^*(\eta)$ only the Chern character of the appropriate bundle. This is the index theorem approach, and in fact it links directly to the original problem of definition of functional determinant, at least when space-time is compactified to a sphere [65][66]. This approach also applies to gravity [68].

-use a purely algebraic approach and limit oneself to some polynomials in the fields and their derivatives. This line was taken in [69] [70] [71] [72] , see M. Dubois-Violette's lectures in this School.

The importance of the consistency equation is revealed not only in the problem of the usual chiral anomaly (first cohomology group of δ), but also, and with possible drastic consequences on our understanding of quantum gauge theories, in the study of Schwinger terms in the commutation of quantum currents (second cohomology group of δ)

[73] [74], but this is beyond the scope of these lectures.

It is worth noticing that the covariant anomaly also has an interpretation on $\mathcal{C}$ [75].

One should also mention the very nice description of the relevance of the cohomology of η for the hamiltonian formulation of quantum Yang-Mills theory in 3+1 dimension with θ vacuum term, and 2+1 dimension with Chern-Simons mass term [76]. See M. Asorey's exposé in this School and [77].

Conclusion.

We have succintly described various features of the geometry of the space of fields in Yang-Mills theory, which is one of the best examples of constrained field theory.

In particular we have shown how the geometry of the classical theory governs many of the aspects of quantum theory, although a construction of non perturbative quantum theory (via for example a proper definition of the functional integral [78]), is still to come.

Rather than concluding, which would be foolhardy on such a vast subject, we can point directions for future developments. They should have non perturbative quantum theory as a goal, and should include:

-the continuation of the work of ref. [78], and definition of a Schrödinger equation on functional space.

-a better understanding of the coordinatization of the orbit space, especially for the study of the effect of its shape (the two questions are related).

-a direct computation of the local cohomology suitable for the problem of anomalies.

-a good control of the functional measures over fermions, especially Weyl fermions [79] [80].

-an understanding of the recent proposals to quantize anomalous gauge theories [73] (the two questions are related).

The list of references given below cannot be complete, since the literature on gauge theories is enormous and apologies are made to the Contributors to the field whose work remains unquoted here.

References.

[1] C. N Yang, R. L. Mills. *"Conservation of isotopic spin and isotopic gauge invariance".* Phys. Rev. 96 (1954), 191.

[2] C. N. Yang. in "Chern Symposium". June 1979. Berkeley.

[3] E. Lubkin. *"Geometric definition of gauge invariance".* Ann. Phys. (NY) 23 (1963), 233.

[4] R. Kerner. *"Generalization of the Kaluza-Klein theory for an arbitrary non abelian gauge group".* Ann. Inst. H. Poincaré A9 (1968), 143.

[5] A Trautman. *"Fibre bundles associated with space-time".* Rep. Math. Phys. 1 (1970), 29.

[6] T. T. Wu, C. N. Yang. *"Concept of non integrable phase factors and global formulation of gauge fields".* Phys. Rev. D12 (1975), 3845.

[7] Y. M. Cho. *"Higher dimensional unifications of gravitation and gauge theories".* J.M.P. 16 (1975), 2029.

[8] W. Dreschsler, M.E. Mayer. *"Fiber Bundle Techniques in Gauge Theories".* Lect. Notes in Physics 67. Springer (1977).

[9] S.J. Avis, C. Isham. Cargèse Lectures 1977.

[10] M.F. Atiyah. *"Geometry of Yang-Mills fields".* Acc. Naz. dei Lincei. Pisa (1979).

[11] M. Daniel, C.M. Viallet. *"The geometrical setting of gauge theories of the Yang-Mills type".* Rev. Mod. Phys. 52 (1980), 175.

[12] T. Eguchi, P.B. Gilkey, A.J. Hanson. *"Gravitation, gauge theory, and differential geometry".* Phys. Reports 66 (1980), 213.

[13] Y. Choquet-Bruhat, C. de Witt-Morette, M. Dillard-Bleick. *"Analysis, Manifolds and Physics",* 2nd. ed. (1982).

[14] J. Madore. "Geometric methods in classical field theory". Phys. Rep. 75 (1981), 125.

[15] A. Trautman. *"Differential geometry for physicists".* (1984) Bibliopolis.

[16] N. Steenrod. *"The topology of fibre bundles".* Princeton Math. Series 14 (P.U.P. 1951).

[17] A. Lichnerowicz. " *Théorie globale des connexions et des groupes d'holonomie".* Ed. Cremonese. Roma (1955).

[18] S. Kobayashi, N. Nomizu. *"Foundations of differential geometry".* Vol1(1963), Vol2(1969). Wiley.

[19] R. Utiyama. " *Invariant theoretical interpretation of interaction".* Phys. Rev. 101 (1956), 1597.

[20] M. F. Atiyah, N. Hitchin, I. M. Singer. *"Self duality in four dimensional riemannian geometry"*Proc. R. Soc. A362 (1978), 425.

[21] M. Daniel, C. M. Viallet. *"The gauge fixing problem around classical solutions of the Yang-Mills theory".* Phys. Lett. 76B (1978), 458.

[22] V. Moncrief. *"Gribov degeneracies: Coulomb gauge conditions and initial value constraint".* Journ. Math. Phys. 20 (1979), 579.

[23] M. S. Narasimhan, T. R. Ramadas. *"The geometry of SU(2) gauge fields".* Comm. Math. Phys. 67 (1979), 21.

[24] I. M. Singer. *"Some remarks on the Gribov ambiguity".* Comm. Math. Phys. 60 (1978), 7.

[25] P. K. Mitter, C.M. Viallet. *"On the bundle of connections and the gauge orbit manifold in Yang-Mills theory".* Comm. Math. Phys. 79 (1981), 43.

[26] W. Kondracki. J. S. Rogulski. " *On the stratification of the orbit space for the action of automorphisms of connections".*Preprint P. A. N. 1983.

[27] W. Kondracki. P. Sadowski. "Geometric structure of the orbit space of gauge connections". Inst. Math. Polish Ac. of Sciences (1984).

[28] D. Ebin. " *The manifold of riemannian metrics".* Proc. AMS Symp. Pure Math. XV (1968), 11.

[29] J. P. Bourguignon. " *Une stratification de l'espace des structures riemanniennes".* Comp. Math. 30 (1975), 1.

[30] P. A. M. Dirac. *"Lectures on quantum mechanics"* (1964) Belfer Series.

[31] L. D. Faddeev. " *The Feynman integral for singular lagrangians*". Theor. Math. Phys. 1 (1969), 3.

[32] A. Hanson, T. Regge, C. Teitelboim. " *Constrained hamiltonian systems*". Acc. Naz. dei Lincei.

[33] E. S. Fradkin, G. S. Vilkovisky. " *Quantization of relativistic systems with constraints. Equivalence of canonical and covariant formalims in quantum theory of gravitational field*". CERN Preprint TH 2332 (1977) (unpublished).

[34] K. Sundermeyer. " *Constrained dynamics*". Lect. Notes in Phys. Springer (1982).

[35] O. Babelon, C.M. Viallet. *"The riemannian geometry of the configuration space of gauge theories"*. Comm. Math. Phys. 81 (1981), 515.

[36] L. D. Faddeev, V.N. Popov. " *Feynman diagrams for the Yang-Mills field*". Phys. Lett. 25B (1967), 29.

[37] R. P. Feynman. " *Quantum theory of gravitation*". Acta Phys. Pol. 24 (1963), 697.

[38] B. S. de Witt in Relativity, groups and topology (1964) Blackie and son.

[39] O. Babelon, C.M. Viallet. *"The geometrical interpretation of the Faddev-Popov determinant"*. Phys. Lett. 85B (1979), 246.

[40] B. S. de Witt. " *Quantum theory of gravity I. The canonical theory*". Phys. Rev. 160 (1967), 1113.

[41] B. S. de Witt. " *Quantum theory of gravity II. The manifestly covariant theory*". Phys. Rev. 162 (1967), 1195.

[42] B. S. de Witt. " *Quantum theory of gravity III. Applications of the covariant theory*". Phys. Rev. 162 (1967), 1239.

[43] R. Catenacci, M. Martellini. " *On a geometrical interpretation of the Faddeev-Popov determinant for pure quantum gravity*". Phys. Lett. 138B (1984), 263.

[44] A. Polyakov. " *Quantum geometry of bosonic strings*". Phys. Lett. 103B (1981), 207.

[45] R. Bott in "Representation theory of Lie groups". London Math. Soc. Lect. Notes 34 Cambridge Univ. Press (1979).

[46] V. N. Gribov. "*Quantization of non abelian gauge theories*". Nucl. Phys. B139 (1978), 1.

[47] T. P. Killingback. "*The Gribov ambiguity in gauge theories on the four-torus*". Phys. Lett. 138B (1984), 87.

[48] D. Zwanziger. "*Non perturbative modification of the Faddeev-Popov formula and banishment of the naive vacuum*". Nucl. Phys. B209 (1982), 336.

[49] E. Witten. "*An SU(2) anomaly*". Phys. Lett. 117B (1982), 324.

[50] P. Nelson. L. Alvarez-Gaumé. "*Hamiltonian interpretation of anomalies*". Comm. Math. Phys. 99 (1985), 103.

[51] C. Becchi, A. Rouet, R. Stora."*Renormalization of gauge theories*". Ann. Phys.98 (1976),287.

[52] C. Becchi, A. Rouet, R. Stora. "*Renormalization of the abelian Higgs-Kibble model*". Comm. Math. Phys.42 (1975),127.

[53] L. Bonora, P. Cotta-Ramusino. "*Some remarks on BRS transformations, anomalies, and the cohomology of the Lie algebra of the group of gauge transformations*". Comm. Math. Phys. 87 (1983), 589.

[54] S. Adler, W. Bardeen. "*Absence of higher order corrections in the anomalous axial-vector divergence equation*". Phys. Rev. 182 (1969), 1517.

[55] W. Bardeen. "*Anomalous Ward identities in spinor field theory*". Phys. Rev. 184 (1969), 1848.

[56] R. Jackiw, K. Johnson. "*Anomalies of the axial-vector current*". Phys. Rev. 182 (1969), 1459.

[57] S. Adler. "*Perturbation theory anomalies*". Brandeis Summer School 1970. Deser, Grisaru, Pendleton eds. M.I.T. Press.

[58] R. Stora. Erice Lectures 1975 and Cargèse Lectures 1976.

[59] B. Zumino. Les Houches Lectures 1983.

[60] R. Stora. Cargèse Lectures 1983 and Gift Lectures 1985.

[61] C. Becchi. Les Houches Lectures 1983.

[62] J. Wess, B. Zumino. *"Consequences of anomalous Ward identities"*. Phys. Lett. 37B (1971), 95.

[63] M. F. Atiyah, R. Bott. *"The Yang-Mills equations over Riemann surfaces"*. Phil. Trans. R. Soc. A308 (1982), 523.

[64] M. Asorey, P. K. Mitter. *"On geometry, topology and θ sectors in a regularized quantum Yang-Mills theory"*. Preprint CERN TH3424 (1982).

[65] L. Alvarez-Gaumé, P. Ginsparg. *"The topological meaning of non-abelian anomalies"*. Nucl. phys. B243 (1984), 449.

[66] M. F. Atiyah, I. M. Singer. *"Dirac operators coulped to vector potentials"*. Proc. Nat. Ac. Sc. U.S.A. 81 (1984), 2597.

[67] I. M. Singer. *"Families of Dirac operators with applications to physics"*. Conference en l'Honneur d'E. Cartan. Astérisque(1984).

[68] O. Alvarez, I. M. Singer, B. Zumino. *"Gravitational anomalies and the family's index theorem"*. Comm. Math. Phys. 96 (1984), 409.

[69] J. A. Dixon. *"Cohomology and renormalisation of gauge theories"* (1976), unpublished, quoted in [58].

[70] J. Thierry-Mieg. *"Classification of the Yang-Mills anomalies in even and odd dimension"*. Phys. Lett. 147B (1984), 430.

[71] M. Dubois-Violette, M. Talon, C.M. Viallet. *"Results on BRS cohomologies in gauge theory"*. Phys. Lett. 158B (1985), 231.

[72] M. Dubois-Violette, M. Talon, C.M. Viallet. *"BRS algebras. Analysis of the consistency equations in gauge theory"*. Comm. Math. Phys. 102 (1985), 105.

[73] L. D. Faddeev. *"Operator anomaly for the Gauss law"*. Phys. Lett. 145B (1984), 81.

[74] R. Jackiw. *"Chern-Simons terms and cocycles in physics and mathematics"*. to appear in E. S. Fradkin Festschrift. Adam Hilger. Bristol (1986).

[75] D. Bao, V. P. Nair. "*A note on the covariant anomaly as an equivariant momentum mapping*." Comm. Math. Phys. 101 (1985), 437.

[76] S. Deser, R. Jackiw, S. Templeton. "*Topologically massive gauge theories*." Ann. Phys. 140 (1982), 372.

[77] M. Asorey, P. K. Mitter. "*Cohomology of the Yang-Mills gauge orbit space and dimensional reduction*". Phys. Lett. 153B (1985), 147.

[78] M. Asorey. P. K. Mitter. "*Regularized continuum Yang-Mills process and Feynman-Kac functional integral*." Comm. Math. Phys. 80 (1981), 43.

[79] K. Fujikawa. "*Path integral for gauge theories with fermions*." Phys. Rev. D21 (1980), 2848.

[80] R. Roskies, F. A. Schaposnik. "*Comments on Fujikawa's analysis applied to the Schwinger model*." Phys. Rev. D23 (1981), 558.