THE FRÖLICHER-NIJENHUIS BRACKET

Basic information. Let M be a smooth manifold and let $\Omega^k(M;TM) = \Gamma(\bigwedge^k T^*M \otimes TM)$. We call $\Omega(M,TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M,TM)$ the space of all *vector valued differential forms*. The *Frölicher-Nijenhuis bracket* $[\quad,\quad] : \Omega^k(M;TM) \times \Omega^l(M;TM) \to \Omega^{k+l}(M;TM)$ is a $\mathbb{Z}$-graded Lie bracket:

$$[K,L] = -(-1)^{kl}[L,K],$$
$$[K_1,[K_2,K_3]] = [[K_1,K_2],K_3] + (-1)^{k_1k_2}[K_2,[K_1,K_3]].$$

It extends the *Lie bracket of smooth vector fields*, since $\Omega^0(M;TM) = \Gamma(TM) = \mathfrak{X}(M)$. The identity on TM generates the 1-dimensional center. It is called the Frölicher-Nijenhuis bracket since it appeared with its full properties for the first time in [1], after some indication in [8]. One formula for it is

$$\begin{aligned}[\varphi\otimes X,\psi\otimes Y] &= \varphi\wedge\psi\otimes[X,Y] + \varphi\wedge\mathcal{L}_X\psi\otimes Y - \mathcal{L}_Y\varphi\wedge\psi\otimes X\\ &\quad + (-1)^k\left(d\varphi\wedge i_X\psi\otimes Y + i_Y\varphi\wedge d\psi\otimes X\right),\end{aligned}$$

where X and Y are vector fields, φ is a k-form, and ψ is an l-form. It is a bilinear differential operator of bidegree $(1,1)$.

The Frölicher-Nijenhuis bracket is natural in the same way as the Lie bracket for vector fields: if $f : M \to N$ is smooth and $K_i \in \Omega^{k_i}(M;TM)$ are f-related to $L_i \in \Omega^l(N;TN)$ then also $[K_1,K_2]$ is f-related to L_1, L_2.

More details. A convenient source is [3], section 8. The basic formulas of calculus of differential forms extend naturally to include the Frölicher-Nijenhuis bracket: Let $\Omega(M) = \bigoplus_{k\geq 0}\Omega^k(M) = \bigoplus_{k=0}^{\dim M}\Gamma(\bigwedge^k T^*M)$ be the algebra of differential forms. We denote by $\operatorname{Der}_k\Omega(M)$ the space of all *(graded) derivations* of degree k, i.e. all bounded linear mappings $D : \Omega(M) \to \Omega(M)$ with $D(\Omega^l(M)) \subset \Omega^{k+l}(M)$ and $D(\varphi\wedge\psi) = D(\varphi)\wedge\psi + (-1)^{kl}\varphi\wedge D(\psi)$ for $\varphi\in\Omega^l(M)$. The space $\operatorname{Der}\Omega(M) = \bigoplus_k \operatorname{Der}_k\Omega(M)$ is a $\mathbb{Z}$-graded Lie algebra with the graded commutator $[D_1,D_2] := D_1 \leq D_2 - (-1)^{k_1k_2}D_2 \leq D_1$ as bracket.

A derivation $D\in\operatorname{Der}_k\Omega(M)$ with $D \mid \Omega^0(M) = 0$ satisfies $D(f.\omega) = f.D(\omega)$ for $f\in C^\infty(M,\mathbb{R})$, thus D is of tensorial character and induces a derivation $D_x \in \operatorname{Der}_k\bigwedge T_x^*M$ for each $x\in M$. It is uniquely determined by its restriction to 1-forms $D_x|T_x^*M : T_x^*M \to \bigwedge^{k+1}T^*M$ which we may view as an element $K_x \in \bigwedge^{k+1}T_x^*M\otimes T_xM$ depending smoothly on $x\in M$; we express this by writing

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$D = i_K$, where $K \in C^\infty(\bigwedge^{k+1} T^*M \otimes TM) =: \Omega^{k+1}(M;TM)$, and we have

$$(i_K\omega)(X_1 \dots, X_{k+\ell}) = \\ = \tfrac{1}{(k+1)!\,(\ell-1)!} \sum_{\sigma\in \S_{k+\ell}} \operatorname{sign}\sigma\, .\omega(K(X_{\sigma 1},\dots,X_{\sigma(k+1)}), X_{\sigma(k+2)},\dots)$$

for $\omega \in \Omega^\ell(M)$ and $X_i \in \mathfrak{X}(M)$ (or T_xM).

By putting $i([K,L]^\wedge) = [i_K, i_L]$ we get a bracket $[\quad,\quad]^\wedge$ on $\Omega^{*+1}(M,TM)$ which defines a graded Lie algebra structure with the grading as indicated, and for $K \in \Omega^{k+1}(M,TM)$, $L \in \Omega^{\ell+1}(M,TM)$ we have

$$[K,L]^\wedge = i_K L - (-1)^{k\ell} i_L K,$$

where $i_K(\omega \otimes X) := i_K(\omega)\otimes X$. The bracket $[\quad,\quad]^\wedge$ is called the the *Nijenhuis-Richardson bracket*, see [6] and [7]. If viewed on a vector space V, it recognizes Lie alebra structures on V: A mapping $P \in L^2_{\text{skew}}(V;V)$ is a Lie bracket if and only if $[P,P]^\wedge = 0$. This can be used to study *deformations of Lie algebra structures*: $P+A$ is again a Lie bracket on V if and only if $[P+A,P+A]^\wedge = 2[P,A]^\wedge + [A,A]^\wedge = 0$; this can be written in *Maurer-Cartan equation* form as $\delta_P(A) + \frac12[A,A]^\wedge = 0$, since $\delta_P = [P,\quad]^\wedge$ is the coboundary operator for the *Chevalley cohomology* of the Lie algebra (V,P) with values in the adjoint representation V. See [4] for a multigraded elaboration of this.

The exterior derivative d is an element of $\operatorname{Der}_1\Omega(M)$. In view of the formula $\mathcal{L}_X = [i_X, d] = i_X\, d + d\, i_X$ for vector fields X, we define for $K \in \Omega^k(M;TM)$ the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \operatorname{Der}_k\Omega(M)$ by $\mathcal{L}_K := [i_K, d]$. The mapping $\mathcal{L} : \Omega(M,TM) \to \operatorname{Der}\Omega(M)$ is injective. We have $\mathcal{L}(\mathrm{Id}_{TM}) = d$.

For any graded derivation $D \in \operatorname{Der}_k\Omega(M)$ there are unique $K \in \Omega^k(M;TM)$ and $L \in \Omega^{k+1}(M;TM)$ such that

$$D = \mathcal{L}_K + i_L.$$

We have $L = 0$ if and only if $[D,d] = 0$. Moreover, $D|\Omega^0(M) = 0$ if and only if $K = 0$.

Let $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$. Then obviously $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K,L])$$

for a uniquely defined $[K,L] \in \Omega^{k+\ell}(M;TM)$. This vector valued form $[K,L]$ is the *Frölicher-Nijenhuis bracket* of K and L.

For $K \in \Omega^k(M;TM)$ and $L \in \Omega^{\ell+1}(M;TM)$ we have

$$[\mathcal{L}_K, i_L] = i([K,L]) - (-1)^{k\ell}\mathcal{L}(i_L K).$$

The space $\operatorname{Der}\Omega(M)$ is a graded module over the graded algebra $\Omega(M)$ with the action $(\omega\wedge D)\varphi = \omega\wedge D(\varphi)$, because $\Omega(M)$ is graded commutative. Let the degree of ω be q, of φ be k, and of ψ be ℓ. Let the other degrees be as indicated. Then we have:

$$\begin{aligned}
&[\omega\wedge D_1,D_2]=\omega\wedge[D_1,D_2]-(-1)^{(q+k_1)k_2}D_2(\omega)\wedge D_1.\\
&i(\omega\wedge L)=\omega\wedge i(L)\\
&\omega\wedge\mathcal{L}_K=\mathcal{L}(\omega\wedge K)+(-1)^{q+k-1}i(d\omega\wedge K).\\
&[\omega\wedge L_1,L_2]^\wedge=\omega\wedge[L_1,L_2]^\wedge-\\
&\qquad-(-1)^{(q+\ell_1-1)(\ell_2-1)}i(L_2)\omega\wedge L_1.\\
&[\omega\wedge K_1,K_2]=\omega\wedge[K_1,K_2]-(-1)^{(q+k_1)k_2}\mathcal{L}(K_2)\omega\wedge K_1\\
&\qquad+(-1)^{q+k_1}d\omega\wedge i(K_1)K_2.
\end{aligned}$$

For $K\in\Omega^k(M;TM)$ and $\omega\in\Omega^\ell(M)$ the Lie derivative of ω along K is given by:

$$\begin{aligned}
&(\mathcal{L}_K\omega)(X_1,\dots,X_{k+\ell})=\\
&=\tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ \mathcal{L}(K(X_{\sigma1},\dots,X_{\sigma k}))(\omega(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)}))\\
&+\tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\ \omega([K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots)\\
&+\tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ \omega(K([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(k+2)},\dots).
\end{aligned}$$

For $K\in\Omega^k(M;TM)$ and $L\in\Omega^\ell(M;TM)$ the Frölicher-Nijenhuis bracket $[K,L]$ is given by:

$$\begin{aligned}
&[K,L](X_1,\dots,X_{k+\ell})=\\
&=\tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ [K(X_{\sigma1},\dots,X_{\sigma k}),L(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)})]\\
&+\tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\ L([K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots)\\
&+\tfrac{(-1)^{k\ell}}{(k-1)!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ K([L(X_{\sigma1},\dots,X_{\sigma\ell}),X_{\sigma(\ell+1)}],X_{\sigma(\ell+2)},\dots)\\
&+\tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ L(K([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(k+2)},\dots)\\
&+\tfrac{(-1)^{(k-1)\ell}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ K(L([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(\ell+2)},\dots).
\end{aligned}$$

The Frölicher-Nijenhuis bracket expresses obstructions to integrability in many different situations: If $J:TM\to TM$ is an almost complex structure, then J is complex structure if and only if the Nijenhuis tensor $[J,J]$ vanishes (theorem of Newlander and Nirenberg, [5]). If $P:TM\to TM$ is a fiberwise projection on the tangent spaces of a fiber bundle $M\to B$ then $[P,P]$ is a version of the curvature (see [3], sections 9 and 10). If $A:TM\to TM$ is fiberwise diagonalizable with all eigenvalues real and of constant multiplicity, then eigenspace of A is integrable if and only if $[A,A]=0$.

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