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The BRST complex and the cohomology of compact Lie algebras

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* *Research supported by the Stichting F.O.M.*

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1 Preliminaries

1.1 Lie algebras

In this paper we consider compact Lie algebras[1] $\mathcal{G}$ which are direct sums of a finite number of simple Lie algebras (i.e. a semi-simple algebra), and one or more abelian $u(1)$ algebras:

$$\mathcal{G} = \bigoplus_{a=1}^{r} \mathcal{G}_a \tag{1.1}$$

Thus each of these algebras $\mathcal{G}_a$ is an *ideal* of $\mathcal{G}$. We assume there is a representation of the algebra in which the generators G_α of $\mathcal{G}$ are hermitean, with the Lie bracket

$$[G_\alpha, G_\beta] = i\, f_{\alpha\beta}^{\;\;\gamma}\, G_\gamma. \tag{1.2}$$

Then the structure constants $f_{\alpha\beta}^{\;\;\gamma}$ are real and anti-symmetric in (α, β). When necessary we take an orthonormal basis for the semi-simple subalgebra:

$$g_{\alpha\beta} = -\frac{1}{2}\, f_{\alpha\sigma}^{\;\;\tau} f_{\beta\tau}^{\;\;\sigma} = \delta_{\alpha\beta}. \tag{1.3}$$

For a semi-simple algebra the inverse $g^{\alpha\beta}$ exists and the two forms can be used to raise and lower indices. This leads to a completely anti-symmetric form for the structure constants:

$$f_{\alpha\beta\gamma} = f_{\gamma\alpha\beta} = f_{\alpha\beta}^{\;\;\sigma} g_{\sigma\gamma}. \tag{1.4}$$

1.2 Clifford algebras

In the following we also encounter $2n$-dimensional Clifford algebras[2] with generators $\gamma_k, k = 1, ..., 2n$:

$$\{\gamma_k, \gamma_l\} = \gamma_k\gamma_l + \gamma_l\gamma_k = 2\delta_{kl}. \tag{1.5}$$

[1] See for example refs.[1, 2]

[2] Our treatment largely follows ref.[3].

The irreducible representation of this algebra is defined by $2n$ hermitean Dirac-matrices of dimension $2^n \times 2^n$. Out of these we can construct a 2^n-dimensional spinor representation of the so($2n$) Lie algebra with generators:

$$\sigma_{kl} = \frac{i}{4}\left[\gamma_k, \gamma_l\right]. \tag{1.6}$$

The $2n$-dimensional Clifford algebra naturally splits into two anti-commuting n-dimensional Clifford algebras, generated by elements Γ_α and $\tilde{\Gamma}_\alpha$ with $\alpha = 1, ..., n$:

$$\Gamma_\alpha = \gamma_\alpha, \qquad \tilde{\Gamma}_\alpha = \gamma_{\alpha+n}, \tag{1.7}$$

satisfying the anti-commutation relations

$$\begin{gathered} \left\{\Gamma_\alpha, \Gamma_\beta\right\} = 2\delta_{\alpha\beta} \qquad \left\{\tilde{\Gamma}_\alpha, \tilde{\Gamma}_\beta\right\} = 2\delta_{\alpha\beta}, \\ \left\{\Gamma_\alpha, \tilde{\Gamma}_\beta\right\} = 0. \end{gathered} \tag{1.8}$$

Under this decomposition the so($2n$)-spinor representation splits into two commuting so(n) algebras generated by

$$\Sigma_{\alpha\beta} = \frac{i}{4}\left[\Gamma_\alpha, \Gamma_\beta\right], \qquad \tilde{\Sigma}_{\alpha\beta} = \frac{i}{4}\left[\tilde{\Gamma}_\alpha, \tilde{\Gamma}_\beta\right]. \tag{1.9}$$

From the last of eqs.(1.8) it follows directly that

$$\left[\Sigma_{\alpha\beta}, \tilde{\Sigma}_{\gamma\delta}\right] = 0. \tag{1.10}$$

We can now construct an alternative form of the original $2n$-dimensional Clifford algebra in terms of operators (c_α, π_α) defined by

$$c_\alpha = \frac{1}{2}\left(\Gamma_\alpha - i\tilde{\Gamma}_\alpha\right), \qquad \pi_\alpha = \frac{1}{2}\left(\Gamma_\alpha + i\tilde{\Gamma}_\alpha\right). \tag{1.11}$$

These operators satisfy the anti-commutation relation

$$\left\{c_\alpha, \pi_\beta\right\} = \delta_{\alpha\beta}, \tag{1.12}$$

which defines an algebra of n fermionic co-ordinates and their conjugate momenta. In this form, the $2n$-dimensional Clifford algebra is decomposed into two n-dimensional Grassmann algebras, which are conjugate to each other in the sense of the anti-commutation relation (1.12). In the Berezin representation [4] they correspond to anti-commuting c-numbers and derivatives, respectively.

1.3 Spinor representation of semi-simple Lie algebras

In the previous section we saw that a Clifford algebra can be used to construct representations of so(n) on spinors. This construction can be generalized to any semi-simple Lie algebra, as we now show. We start from the adjoint representation defined in terms of the structure constants:

$$(G_\alpha)_\beta^{\ \gamma} = -i\, f_{\alpha\beta}^{\ \ \gamma}. \tag{1.13}$$

If the Lie algebra has n dimensions, the adjoint representation acts on an n-dimensional vector space. We obtain a representation on a $2^{[n/2]}$-dimensional spinor space by defining

$$\Sigma_\alpha = -\frac{1}{2} f_\alpha^{\ \beta\gamma}\, \Sigma_{\beta\gamma}. \tag{1.14}$$

Like for the adjoint representation, the proof that these operators define a representation of the Lie algebra (1.2) is a direct consequence of the Jacobi identity. This embedding of an n-dimensional Lie algebra in the spinor representation of so(n) can be used for example to define a Dirac operator on the group manifold.

In the BRST cohomology theory we encounter realizations of the Lie algebra in the spinor representation of so($2n$), using its decomposition into so(n) $\oplus$ so(n), eq.(1.9). This implies a trivial doubling of the algebra, with two commuting sets of generators $\{\Sigma_\alpha\}$ and $\{\tilde{\Sigma}_\alpha\}$. The result can be written conveniently in terms of the operators (c^α, π_α) defined above, as follows:

$$\Sigma_\alpha + \tilde{\Sigma}_\alpha = -ic^\beta\, f_{\alpha\beta}^{\ \ \gamma}\, \pi_\gamma. \tag{1.15}$$

This expression defines the image of the semi-simple Lie algebra in the spinor representation of so($2n$).

2 The BRST operator

As in sect.(1), $\mathcal{G}$ denotes a compact n-dimensional Lie algebra which is a direct sum of a semi-simple Lie algebra and a finite number of abelian u(1) algebras with generators G_α:

$$[G_\alpha, G_\beta] = i f_{\alpha\beta}^{\ \ \gamma}\, G_\gamma. \tag{2.1}$$

In addition consider a $2n$-dimensional Clifford algebra:

$$\{c^\alpha, \pi_\beta\} = \delta^\alpha_\beta. \tag{2.2}$$

The BRST operator Ω [5, 6] corresponding to $\mathcal{G}$ is defined by[3]

$$\Omega = c^\alpha\, G_\alpha + \frac{i}{2}\, c^\gamma c^\beta f_{\beta\gamma}^{\ \ \alpha} \pi_\alpha. \tag{2.3}$$

By construction, the BRST operator is nilpotent:

$$\Omega^2 = 0. \tag{2.4}$$

We observe that the definition of Ω is unambiguous, since a change in the ordering of the operators (c^α, π_α) leads to terms proportional to $f_{\alpha\beta}^{\ \ \beta}$, and this vanishes for compact semi-simple and abelian Lie algebras. The operators c^α define a Grassmann algebra and are referred to as *ghosts* [12], whilst the conjugate operators π_α, which define an isomorphic Grassmann algebra, are called the *ghost momenta.*

The generators of the Lie algebra G_α being hermitean by assumption, the BRST operator Ω is self-adjoint with respect to any inner product such that the ghosts and ghost momenta are self-adjoint. Consider the space of polynomials in the ghost variables, with complex co-efficients:

$$\psi[c] = \sum_{k=0}^{n} \frac{1}{k!}\, c^{\alpha_1} ... c^{\alpha_k} \psi^{(k)}_{\alpha_1 ... \alpha_k}. \tag{2.5}$$

Modulo non-singular redefinitions of the ghost variables there is only one such inner product on this space, which is defined by the Berezin integral [4] over the anti-commuting ghosts c^α:

[3]For some reviews, consult refs.[7, 8, 9, 10, 11]

$$\langle\phi,\psi\rangle = \int dc^n ... dc^1 \, \phi^\dagger \psi. \tag{2.6}$$

In terms of the components, eq.(2.5), this expression reads

$$\langle\phi,\psi\rangle = \frac{1}{n!}\,\epsilon^{\alpha_1...\alpha_n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \phi^{\dagger(n-k)}_{\alpha_{n-k}...\alpha_1} \psi^{(k)}_{\alpha_{n-k+1}...\alpha_n}. \tag{2.7}$$

The action of the ghosts and their momenta on the components is given by

$$(c^\alpha\psi)^{(k)}_{\alpha_1...\alpha_k} = \delta^\alpha_{\alpha_1}\,\psi^{(k-1)}_{\alpha_2\alpha_3...\alpha_k} - \delta^\alpha_{\alpha_2}\,\psi^{(k-1)}_{\alpha_1\alpha_3...\alpha_k} + ... + (-1)^{k-1}\delta^\alpha_{\alpha_k}\,\psi^{(k-1)}_{\alpha_1\alpha_2...\alpha_{k-1}}, \tag{2.8}$$

for $k = 1, ..., n$, and

$$(\pi_\alpha\psi)^{(k)}_{\alpha_1...\alpha_k} = \psi^{(k+1)}_{\alpha\alpha_1...\alpha_k}, \tag{2.9}$$

for $k = 0, ..., n-1$. It may now be checked directly from the component expression (2.7), that

$$\langle\phi, c^\alpha\psi\rangle = \langle c^\alpha\phi, \psi\rangle, \tag{2.10}$$

and similarly

$$\langle\phi, \pi_\alpha\psi\rangle = \langle\pi_\alpha\phi, \psi\rangle. \tag{2.11}$$

Therefore the BRST operator is self-adjoint as well:

$$\langle\phi, \Omega\psi\rangle = \langle\Omega\phi, \psi\rangle. \tag{2.12}$$

3 The anti-BRST operator

From the symmetry between the ghosts c^α and the ghost momenta π_α in the Clifford algebra (2.2) we infer the existence of a second BRST operator ${}^*\Omega$ defined by

$${}^*\Omega = G^\alpha\pi_\alpha + \frac{i}{2}\,\pi_\gamma\pi_\beta f^{\beta\gamma}{}_\alpha c^\alpha. \tag{3.1}$$

It is nilpotent as well:

$$ {}^{*}\Omega^2 = 0. \tag{3.2} $$

In the following it is referred to as the anti-BRST operator[4]. Where appropriate, we have raised and lowered indices on the Lie algebra using the Killing-Cartan form of the semi-simple Lie subalgebra, whilst there is no distinction between upper and lower indices for the generators of abelian ideals, since the the structure constants involving these vanish identically.

Obviously, the anti-BRST operator ${}^{*}\Omega$ is self-adjoint with respect to the inner product $\langle\ ,\ \rangle$ defined in eqs. (2.6), (2.7). We note in passing that the BRST operator Ω and the anti-BRST operator ${}^{*}\Omega$ are both self-adjoint also with respect to the analogous inner product defined in the space of polynomials of the ghost momenta, $\psi[\pi]$, by the Berezin integral over this conjugate Grassmann algebra. These two spaces of polynomials in c^α and π_α are isomorphic by a generalization of the Fourier transformation to anti-commuting variables and are referred to as the *co-ordinate* and *momentum pictures* of the state space associated with the Clifford algebra of the ghosts.

Taking the anti-commutator of the BRST and anti-BRST operator we obtain a new operator W which is even in the number of Clifford generators:

$$ W = \{{}^{*}\Omega, \Omega\} = G_\alpha G^\alpha + \dots , \tag{3.3} $$

where the dots denote terms involving the ghosts. Since Ω and ${}^{*}\Omega$ are nilpotent, the graded Jacobi identity for three such operators implies that W is both BRST *and* anti-BRST invariant:

$$ [\Omega, W] = 0, \qquad\qquad [{}^{*}\Omega, W] = 0. \tag{3.4} $$

Therefore W is the BRST and anti-BRST invariant generalization of the quadratic Casimir of the Lie algebra. We return to a more detailed discussion of this operator in a later section.

[4]Note that this definition of the anti-BRST operator differs in important respects from the one in ref.[13]

4 Duality and the scalar product

The co-efficients $\psi^{(k)}_{\alpha_1...\alpha_k}$ in the expansion of the ghost polynomial $\psi[c]$ are completely anti-symmetric in the indices $\alpha_1, ..., \alpha_k$. Therefore they may be regarded as k-forms on the Lie algebra [14]. We introduce the usual Hodge star operation:

$$^*\psi^{(k)\alpha_1...\alpha_k} = \frac{1}{(n-k)!}\,\epsilon^{\alpha_1...\alpha_k\alpha_{k+1}...\alpha_n}\,\psi^{(n-k)}_{\alpha_{k+1}...\alpha_n}, \tag{4.1}$$

with the property

$$^*(^*\psi)^{(k)}_{\alpha_1...\alpha_k} = (-1)^{k(n-k)}\,\psi^{(k)}_{\alpha_1...\alpha_k}. \tag{4.2}$$

Using these definitions we now introduce a second inner product on the space of ghost polynomials, which we will refer to as the *scalar product* in order to distinguish it from the product (2.6). The scalar product is defined by

$$(\phi, \psi) = \langle \mathcal{P}^*\phi,\, \psi\rangle = (-1)^{[n/2]}\,\langle \phi,\, \mathcal{P}^*\psi\rangle, \tag{4.3}$$

where the operator $\mathcal{P}$ denotes multiplication of each k-form by a k-dependent sign, as follows:

$$(\mathcal{P}\phi)^{(k)}_{\alpha_1...\alpha_k} = (-1)^{[k/2]}\,\phi^{(k)}_{\alpha_1...\alpha_k} = \phi^{(k)}_{\alpha_k...\alpha_1}. \tag{4.4}$$

In components the scalar product is

$$(\phi, \psi) = \sum_{k=0}^{n} \frac{1}{k!}\,\phi^{\dagger(k)\alpha_1...\alpha_k}\psi^{(k)}_{\alpha_1...\alpha_k}. \tag{4.5}$$

Contrary to the inner product (2.6), the scalar product (4.3), (4.5) is positive definite. However, with respect to the scalar product the ghost operators (c^α, π_α) are no longer self-adjoint. In stead, they are adjoint to each other:

$$c^\dagger_\alpha = \pi_\alpha, \tag{4.6}$$

or

$$(\phi, c^\alpha\psi) = (\pi^\alpha\phi, \psi). \tag{4.7}$$

As a result, the BRST operator Ω and the anti-BRST operator ${}^*\Omega$ are adjoint under the scalar product as well:

$$(\phi, \Omega\psi) = ({}^*\Omega\phi, \psi)\,. \tag{4.8}$$

This result has important implications for the zero modes of the operator W introduced in eq.(3.3), to wit we can prove the following theorem:

Every solution of the equation $W\psi = 0$ is BRST- and anti-BRST invariant: $\Omega\psi = 0$ and ${}^*\Omega\psi = 0$.

Proof: recall that $W = {}^*\Omega\,\Omega + \Omega\,{}^*\Omega$; therefore

$$(\psi, W\psi) = (\Omega\psi, \Omega\psi) + ({}^*\Omega\psi, {}^*\Omega\psi)\,. \tag{4.9}$$

Since all terms on the right-hand side are manifestly positive definite, the left-hand side can equal zero only if these terms vanish separately:

$$\Omega\psi = 0, \qquad {}^*\Omega\psi = 0. \tag{4.10}$$

This proves the theorem. Note, that the positivity of W follows from eq.(4.9) as a corrollary.

5 BRST cohomology

The single most important property of the BRST operators is their nilpotence:

$$\Omega^2 = 0, \qquad {}^*\Omega^2 = 0. \tag{5.1}$$

Therefore these operators formally behave like exterior derivatives[5] [14], with the generalized Casimir operator W playing the role of the Laplacian:

$$W = \{{}^*\Omega, \Omega\} = (\Omega + {}^*\Omega)^2\,. \tag{5.2}$$

[5]The structure of Lie algebra cohomology has been discussed in the mathematics literature using methods similar to the BRST approach in ref.[16]; I am indebted to J. Kowalski-Glikman for bringing this to my attention

As a result, one can define a BRST cohomology in the space of ghost states $\psi[c]$ (or $\psi[\pi]$ in the momentum picture).

First we introduce some terminology. A state which is BRST invariant:

$$\Omega\,\psi = 0, \tag{5.3}$$

is called *BRST closed.* Similarly, a state which is anti-BRST invariant:

$$^*\Omega\,\psi = 0, \tag{5.4}$$

is called *co-closed.* A state which is the BRST-transform of another state:

$$\psi = \Omega\,\chi, \tag{5.5}$$

is called *BRST exact*, and a state which is the anti-BRST transform of another state:

$$\psi = {}^*\Omega\,\chi, \tag{5.6}$$

is called *co-exact.* Any BRST-exact state is BRST closed, and any co-exact state is co-closed, but the inverse is not necessarily true. The BRST cohomology is the set of states which are BRST closed, but not exact:

$$H(\Omega) = Ker\Omega/Im\Omega. \tag{5.7}$$

Similarly we define the anti-BRST cohomology by

$$H(^*\Omega) = Ker^*\Omega/Im^*\Omega. \tag{5.8}$$

Note that as regards BRST-cohomology all states which differ by a BRST-exact state are considered equivalent:

$$\psi \sim \psi' \Leftrightarrow \psi' = \psi\, +\, \Omega\,\chi, \tag{5.9}$$

since ψ' is closed whenever ψ is. A similar statement holds for co-closed states. Therefore BRST cohomology deals only with equivalence classes of states. In the language of quantum field theory, the BRST operator generates a kind of *gauge transformations* on the space of states (as opposed to, say, the configuration space), and the cohomology classes (5.9) identify all states which differ by a gauge transformation.

In addition to BRST exact and co-exact states, we distinguish also *BRST harmonic* states. These are the zero modes of the operator W:

$$W\,\psi = 0. \tag{5.10}$$

In the previous section it has already been proven, that such states are both closed *and* co-closed.

With the above definitions it is now straightforward to prove the following decomposition theorem:

Any state $\psi[c]$ can be decomposed into a BRST-exact, a co-exact and a harmonic state:

$$\psi \;=\; \omega \,+\, \Omega\,\chi \,+\, {}^{*}\Omega\,\phi, \tag{5.11}$$

where

$$W\,\omega = 0. \tag{5.12}$$

To prove this theorem, one shows that the graded Lie algebra defined by eqs.(5.1),(5.2) only has the following irreducible representations[6]:

(*i*) *Singlets*; these are harmonic forms ω such that $\Omega\,\omega = 0$ and ${}^{*}\Omega\,\omega = 0$.

(*ii*) *Doublets*; pairs of states which have positive eigenvalues under W and which are the BRST/anti-BRST transform of each other.

(*iii*) *Quartets*; sets of four states with positive eigenvalues under W, which are linear combinations of BRST-exact and co-exact states, and which transform into each other under BRST/anti-BRST transformations.

The details of the proof are given in appendix A.

From the theorem (5.11) it follows, that any BRST-closed state differs from a harmonic state by at most an exact state. In field theory language:

[6]The representation theory of Ω itself has been discussed in ref.[8]

any BRST-invariant state is gauge equivalent to a harmonic state. The proof goes as follows: decompose ψ as in eq.(5.11); then

$$\Omega\,\psi = \Omega\,{}^*\Omega\,\phi, \tag{5.13}$$

because the harmonic state and the BRST-exact state are closed. Now if ψ is closed, this implies in particular that

$$(\phi, \Omega\psi) = 0. \tag{5.14}$$

Inserting eq. (5.13) gives

$$(\phi, \Omega\,{}^*\Omega\phi) = ({}^*\Omega\phi,\, {}^*\Omega\phi) = 0, \tag{5.15}$$

which is true if and only if ${}^*\Omega\phi = 0$. Hence we establish the result that

$$\Omega\,\psi = 0 \;\Leftrightarrow\; \psi \;=\; \omega \,+\, \Omega\,\chi. \tag{5.16}$$

This result may also be stated as follows: any non-trivial solution of the equation

$$\Omega\,\psi = 0, \tag{5.17}$$

can be transformed into a non-trivial solution of

$$W\,\psi = 0, \tag{5.18}$$

by addition of a BRST-exact state; in addition it then satisfies the condition

$${}^*\Omega\,\psi = 0. \tag{5.19}$$

6 Harmonic states

Having established the result (5.16), it follows that the BRST cohomology can be found by solving for all harmonic states. Therefore we now turn to study the solutions of equation (5.10):

$$W\,\psi = 0.$$

The complete solution of this equation is characterized by the following two conditions:

$$G_\alpha \psi = 0 \qquad \text{and} \qquad c^\beta f_{\alpha\beta}{}^\gamma \pi_\gamma \psi = 0. \tag{6.1}$$

Therefore harmonic states are singlets of the Lie algebra *and* of its image in the spinor representation of so($2n$) defined by the ghosts, cf. eq.(1.15).

The result (6.1) is most conveniently derived in the representation of the ghost algebra defined by $(\Gamma_\alpha, \tilde{\Gamma}_\alpha)$, eqs.(1.11). In this representation we have

$$\Omega + {}^*\Omega = \Gamma \cdot \left(G + \Sigma/2 + \tilde{\Sigma}/2\right), \tag{6.2}$$

where the dot denotes contraction over the Lie algebra index α. Squaring this operator and using the result

$$\Sigma_\alpha^2 = \tilde{\Sigma}_\alpha^2 = \frac{n}{4}\mathbf{1}, \tag{6.3}$$

which expresses the fact that both Σ_α^2 and $\tilde{\Sigma}_\alpha^2$ are the quadratic Casimir operators of the image of the Lie algebra $\mathcal{G}$ in the spinor representation of so(n), we obtain

$$\begin{aligned} W &= (\Omega + {}^*\Omega)^2 \\ &= \tfrac{1}{2}G^2 + \tfrac{1}{2}\left(G + \Sigma + \tilde{\Sigma}\right)^2. \end{aligned} \tag{6.4}$$

Again we find that W, being a sum of squares, is non-negative. Moreover, harmonic states correspond to zero-modes and hence must satisfy

$$G_\alpha \psi = 0, \qquad (\Sigma_\alpha + \tilde{\Sigma}_\alpha)\psi = 0. \tag{6.5}$$

Using eq.(1.15) the result (6.1) then follows.

Below we refer to states satisfying the first condition (6.1) as *G-singlets.* The second condition, written in components, becomes:

$$f_{\alpha[\alpha_1}{}^\gamma \, \psi^{(k)}_{\alpha_2\ldots\alpha_k]\gamma} = 0, \tag{6.6}$$

where the square brackets denote complete anti-symmetrization of all indices enclosed with unit total weight. We conclude,that the BRST harmonic states are *all completely anti-symmetric invariant tensors of rank k which are also G-singlets.*

Eq.(6.6) is always satisfied trivially for $k = 0$ and $k = n$, which are singlets under the adjoint representation of the Lie group. It is known, that no solutions exist for $k = 1$ and, by duality, for $k = n - 1$ [7]. A simple proof of this statement can be given in our formalism by observing, that the n conditions (6.6) in the form:

$$(\Sigma_\alpha + \tilde{\Sigma}_\alpha)\,\psi = -ic^\beta\, f_{\alpha\beta}^{\;\;\gamma}\,\pi_\gamma\,\psi[c] = 0, \tag{6.7}$$

can be summarized by the single equation

$$\left(\Sigma + \tilde{\Sigma}\right)^2\,\psi = 0, \tag{6.8}$$

or

$$\left(c^\alpha\,\pi_\alpha - \frac{1}{4} f_{\alpha\beta\gamma}\,f^{\alpha\sigma\tau}\,c^\beta c^\gamma\,\pi_\sigma\pi_\tau\right)\,\psi[c] = 0. \tag{6.9}$$

This last equation reads in components:

$$\psi^{(k)}_{\alpha_1\ldots\alpha_k} = \frac{(k-1)}{4}\,f^{\alpha\beta\gamma}\,f_{\alpha[\alpha_1\alpha_2}\,\psi^{(k)}_{\alpha_3\ldots\alpha_k]\beta\gamma}. \tag{6.10}$$

Obviously, the right-hand side of this equation vanishes for $k = 1$. However, it, also vanishes for $k = n - 1$ because the right-hand side is equal to

$$\frac{(k+2)(k+1)}{4(n-k)}\,f^{\alpha\beta\gamma}\,f_{\alpha[\beta\gamma}\,\psi^{(k)}_{\alpha_1\ldots\alpha_k]}, \tag{6.11}$$

and for $k = n - 1$ this involves an anti-symmetrization over $(n + 1)$ indices taking only n values.

One consequence of these results is, that for semi-simple algebras of dimension $n \leq 3$ there are no BRST-harmonic states other than the singlets with $k = (0, n)$, just because there are no antisymmetric tensors of rank $2 \leq k \leq n - 2$. This applies to the algebra su(2) $\cong$ so(3), and the result is confirmed by an explicit calculation in the next section.

Similarly, there are no solutions with $k = 2$ [15], but for $k = 3$ there is always a solution defined by the structure constants themselves:

[7] See for example ref.[15], ch.5

$$\psi^{(3)}_{\alpha\beta\gamma} = f_{\alpha\beta\gamma}\,\chi, \tag{6.12}$$

where χ is a G-singlet. This follows directly from the Jacobi identity.

A useful result in looking for non-trivial solutions of the BRST-cohomology is the following lemma:

Let the compact Lie algebra $\mathcal{G}$ be a direct sum of r simple and/or abelian ideals:

$$\mathcal{G} = \bigoplus_{a=1}^{r} \mathcal{G}_a. \tag{6.13}$$

Then the only BRST-harmonic states of $\mathcal{G}$ are the direct products of the BRST-harmonic states of each ideal $\mathcal{G}_a$.

To prove this assertion, note that for the abelian part of the algebra (i.e. the maximal abelian ideal) eq.(6.6) is satisfied trivially, because all structure constants pertaining to this part of the algebra vanish. Hence we only have to consider the semi-simple part of the algebra. We show, that the solutions of eq.(6.6) reduce to solutions of the same equation for each simple Lie algebra (i.e. each non-abelian ideal) separately. This results immediately from the property of a semi-simple Lie algebra that it can be decomposed into a direct sum of simple ideals, hence the structure constants vanish whenever two of their indices take values in different subalgebras. Then for α having a value in $\mathcal{G}_a$, eq.(6.6) gets contributions only from terms with the other index α_i of $f_{\alpha\alpha_i}^{\;\;\gamma}$ in the same $\mathcal{G}_a$. Define the subset $(\alpha_{i_1}, ..., \alpha_{i_m})$ of the indices on $\psi^{(k)}_{\alpha_1 ... \alpha_k}$ as those indices which take values in the same ideal as α; then eq.(6.6) reduces to

$$f_{\alpha[\alpha_{i_1}}^{\;\;\;\gamma}\;\psi^{(k)}_{\gamma\alpha_{i_2}...\alpha_{i_m}]\alpha_{i_{m+1}}...\alpha_{i_k}}. \tag{6.14}$$

This is just eq.(6.6) restricted to a single simple ideal $\mathcal{G}_a$. Taking all possible values of α in all ideals, the solutions of (6.6) reduce to a direct product of those for the simple algebras $\mathcal{G}_a$. Therefore our lemma is proven.

Summarizing, we observe that for any semi-simple Lie algebra a BRST-harmonic state represents a solution to the equation

$$G_{[\alpha_1}\,\psi^{(k)}_{\alpha_2\ldots\alpha_{k+1}]} + (-1)^k\,\frac{ik}{2}\,f_{[\alpha_1\alpha_2}{}^{\beta}\,\psi^{(k)}_{\alpha_3\ldots\alpha_{k+1}]\beta} = 0. \tag{6.15}$$

In field theory, the infinite-dimensional generalization of these equations are known for $k = 1$ as the Wess-Zumino consistency conditions. These equations admit trivial solutions given by the BRST-exact states of the form

$$\psi^{(k)}_{\alpha_1\ldots\alpha_k} = G_{[\alpha_1}\,\phi^{(k-1)}_{\alpha_2\ldots\alpha_k]} + (-1)^{k-1}\,\frac{i}{2}(k-1)\,f_{[\alpha_1\alpha_2}{}^{\beta}\,\phi^{(k-1)}_{\alpha_3\ldots\alpha_k]\beta}, \tag{6.16}$$

with $\phi^{(k-1)}$ arbitrary. Non-trivial solutions are provided by all direct products of all those invariant anti-symmetric tensors of rank $0 \leq k \leq n_a$ associated with the abelian and simple ideals $\mathcal{G}_a$, which are G-singlets as well. Examples are provided by the singlet states with $k = 0$ or $k = n_a$, and states transforming as the structure constants $f_{\alpha\beta\gamma}$ themselves.

7 Examples

We now discuss some examples illustrating the general analysis presented in the earlier sections.

• u(1).

First, let us consider the case of a compact abelian u(1) algebra. There is only one generator A, trivially commuting with itself, and the structure constants vanish. Correspondingly, there is only one ghost c, with conjugate momentum π. The BRST and anti-BRST operator read

$$\Omega = c\,A, \qquad\qquad {}^*\Omega = \pi\,A, \tag{7.1}$$

and we have

$$W = \{\,{}^*\Omega, \Omega\,\} = A^2. \tag{7.2}$$

In the co-ordinate representation, the state space consists of functions

$$\psi[c] = \psi^{(0)} + c\psi^{(1)}, \tag{7.3}$$

with A acting on the components $\psi^{(i)}, i = (1,2)$. The BRST-harmonic states are simply *all* A-invariant states:

$$A\,\psi[c] = 0 \;\Leftrightarrow\; A\,\psi^{(i)} = 0, \qquad i = (1,2). \tag{7.4}$$

Note, that $\psi^{(0)}$ corresponds to the state with all ghost levels *empty*, and $\psi^{(1)}$ to the state with all ghost levels *filled*; ψ has no further components.

• su(2) ≅ so(3).

The Lie algebra su(2) ≅ so(3) is given by

$$[G_i, G_j] = i\,\varepsilon_{ijk}\,G_k, \qquad i,j,k = (1,2,3). \tag{7.5}$$

The Killing-Cartan form is

$$g_{ij} = -\frac{1}{2}\,\varepsilon_{ikl}\,\varepsilon_{jlk} = \delta_{ij}. \tag{7.6}$$

The Lie algebra has dimension $n = 3$, hence we introduce a 6-dimensional Clifford algebra

$$\{c_i, \pi_j\} = \delta_{ij}. \tag{7.7}$$

An explicit representation is given in appendix B. Defining

$$\Sigma_i = -\frac{i}{4}\,\varepsilon_{ijk}\,\Gamma_j\Gamma_k, \qquad \tilde{\Sigma}_i = -\frac{i}{4}\,\varepsilon_{ijk}\,\tilde{\Gamma}_j\tilde{\Gamma}_k, \tag{7.8}$$

with $(\Gamma_i, \tilde{\Gamma}_i)$ given in terms of (c_i, π_i) as in eq.(1.11), we have

$$\Sigma_i^2 = \tilde{\Sigma}_i^2 = \frac{3}{4}\mathbf{1}, \tag{7.9}$$

and in the representation of appendix B:

$$\left(\Sigma+\tilde{\Sigma}\right)^2=\frac{1}{2}\begin{pmatrix} 3-\sigma_3 & 0 & -\sigma_1-i\sigma_2 & 0 \\ 0 & 3+\sigma_3 & 0 & -\sigma_1+i\sigma_2 \\ -\sigma_1+i\sigma_2 & 0 & 3+\sigma_3 & 0 \\ 0 & -\sigma_1-i\sigma_2 & 0 & 3-\sigma_3 \end{pmatrix}. \quad (7.10)$$

Its zero-modes form a 2-dimensional space spanned by

$$\psi_+=\frac{1}{2}\begin{pmatrix}1\\0\\0\\1\\0\\1\\1\\0\end{pmatrix}, \qquad \psi_-=\frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\\0\\1\\-1\\0\end{pmatrix}. \quad (7.11)$$

These are precisely the states annihilated by π_i or c_i as defined in appendix B:

$$\pi_i\,\psi_+=0, \qquad c_i\,\psi_-=0. \quad (7.12)$$

The BRST-harmonic states now take the form

$$\psi=f\,\psi_-\;+\;g\,\psi_+, \quad (7.13)$$

with

$$G_i\,f=G_i\,g=0. \quad (7.14)$$

Again, we find that the only non-trivial solutions of the BRST-cohomology are the G-invariant states with either all ghost levels empty (ψ_-), or all ghost levels filled (ψ_+).

• so(4) = so(3) ⊕ so(3).

The Lie algebra so(4) is not simple: so(4) = so(3) ⊕ $\tilde{\text{so}}(3)$. Here the tilde only serves to distinguish the two different so(3) algebras. The BRST cohomology of so(4) is composed directly out of the two so(3) cohomologies. That is, each non-trivial BRST-harmonic state is a G-singlet under both so(3) algebras:

$$G_i\,\psi = \tilde{G}_i\,\psi = 0, \tag{7.15}$$

and the two ghost states $\psi_\pm$ of the first so(3) algebra can be combined with each of the two ghost states $\tilde{\psi}_\pm$ of the other $\tilde{\text{so}}(3)$ algebra, giving rise to four different ghost states which are solutions to the ghost cohomology condition (6.7).

These examples sufficiently illustrate the power of our approach.

A Representations of the BRST algebra

In this appendix we study the representations of the BRST algebra

$$\Omega^2 = 0, \qquad {}^*\Omega^2 = 0,$$
$$\{\Omega, {}^*\Omega\} = W, \tag{A.1}$$
$$[W, \Omega] = 0, \qquad [W, {}^*\Omega] = 0.$$

We show, that these representations consist of singlets, doublets and quartets with the properties used in sect. 5 to prove eq.(5.11).

We begin by noting that W is a hermitean and semi-positive definite operator with respect to the scalar product (,) defined in eqs.(4.3), (4.5). Therefore it has real non-negative eigenvalues w and it can be diagonalized in the space of ghost states $\psi[c]$:

$$W\psi = w\psi. \tag{A.2}$$

We now distinguish two cases: $w = 0$ and $w > 0$. In the first case ψ is BRST harmonic and we know that it is both BRST and anti-BRST invariant. Hence we can write

$$\psi = \omega, \qquad \Omega\omega = {}^*\Omega\omega = 0. \tag{A.3}$$

This is what we call a *singlet* representation of the algebra (A.1). It trivially satisfies the decomposition theorem (5.11):

$$\psi = \omega + \Omega\phi + {}^*\Omega\chi, \tag{A.4}$$

with $\phi = \chi = 0$.

Next we consider the second case, $w > 0$. Since now $W\psi \neq 0$, at least one of the operators (Ω, ${}^*\Omega$) does not annihilate ψ:

$$\Omega\psi \neq 0, \qquad \text{or} \qquad {}^*\Omega\psi \neq 0. \tag{A.5}$$

Suppose $\Omega\psi \neq 0$. Define

$$\phi = \frac{1}{\sqrt{w}}\Omega\psi. \tag{A.6}$$

Then ϕ is BRST exact, hence closed:

$$\Omega\phi = 0. \tag{A.7}$$

Next define

$$\psi' = \frac{1}{\sqrt{w}}{}^*\Omega\phi. \tag{A.8}$$

Then ψ' is co-exact. Now there are again two cases to consider. First, let

$${}^*\Omega\,\psi = 0. \tag{A.9}$$

In this case

$$\psi' = \frac{1}{w}{}^*\Omega\,\Omega\,\psi = \frac{1}{w}W\,\psi = \psi. \tag{A.10}$$

As a result there are only two states (ψ, ϕ) and we obtain a *doublet* representation of the BRST algebra, with

$$\Omega\,\psi = \sqrt{w}\,\phi, \qquad {}^*\Omega\,\phi = \sqrt{w}\,\psi. \tag{A.11}$$

Note that both states are eigenstates of W with the same eigenvalue w, as expected. Of course we could equally well have started the construction of the doublet with a state ϕ which is BRST-closed but not co-closed. This leads to the same representation, hence there is only one type of doublet. The decomposition theorem (5.11) is satisfied for doublets; specifically, ψ is co-exact:

$$\psi = \frac{1}{\sqrt{w}}{}^*\Omega\,\phi, \tag{A.12}$$

whilst ϕ is BRST exact:

$$\phi = \frac{1}{\sqrt{w}}\Omega\,\psi. \tag{A.13}$$

The second case we must consider is ${}^*\Omega\psi \neq 0$. Define

$$\chi = \frac{1}{\sqrt{w}}{}^*\Omega\,\psi, \tag{A.14}$$

and

$$\rho = \frac{1}{w}\,[{}^*\Omega, \Omega]\,\psi. \tag{A.15}$$

The states (ψ, ϕ, χ, ρ) now form a closed representation space for the BRST algebra (A.1), which is called the *quartet* representation. To see this, note that

$$\begin{aligned} {}^*\Omega\,\phi &= \frac{1}{\sqrt{w}}{}^*\Omega\,\Omega\psi \\ &= \frac{1}{2\sqrt{w}}\left(\{{}^*\Omega, \Omega\} + [{}^*\Omega, \Omega]\right)\psi \\ &= \frac{1}{2}\sqrt{w}\,(\psi + \rho). \end{aligned} \tag{A.16}$$

Similarly one derives

$$\Omega\,\chi = \frac{1}{2}\sqrt{w}\,(\psi - \rho). \tag{A.17}$$

These equations can be inverted to give:

$$\psi = \frac{1}{\sqrt{w}}({}^*\Omega\,\phi + \Omega\,\chi), \tag{A.18}$$

and

$$\rho = \frac{1}{\sqrt{w}}({}^*\Omega\,\phi - \Omega\,\chi). \tag{A.19}$$

Hence ϕ is BRST exact, χ is co-exact, and ψ and ρ are linear combinations of exact and co-exact states. In addition, all these states are eigenstates of W with the same eigenvalue w. We conclude, that in the quartet representation the decomposition theorem (5.11) is also satisfied.

B d=6 Clifford algebra

In this appendix we give an explicit representation of the six-dimensional Clifford algebra used in the construction of the ghost states for so(3). Let

$$\{\gamma_\alpha, \gamma_\beta\} = 2\,\delta_{\alpha\beta}, \tag{B.1}$$

where $(\alpha, \beta) = 1, ..., 6$. We use an irreducible (8×8) representation in which

$$\gamma_\alpha = \begin{pmatrix} 0 & \gamma_\alpha^{(4)} \\ \gamma_\alpha^{(4)} & 0 \end{pmatrix}, \qquad \alpha = 1, ..., 4;$$

$$\gamma_5 = \begin{pmatrix} 0 & i\mathbf{1}_{(4)} \\ -i\mathbf{1}_{(4)} & 0 \end{pmatrix}; \tag{B.2}$$

$$\gamma_6 = \begin{pmatrix} \mathbf{1}_{(4)} & 0 \\ 0 & -\mathbf{1}_{(4)} \end{pmatrix},$$

with the super/subscript (4) denoting the Dirac and unit matrices in four dimensions. The four-dimensional Dirac matrices we use are

$$\gamma_i^{(4)} = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3;$$

$$\gamma_4^{(4)} = \begin{pmatrix} \mathbf{1}_{(2)} & 0 \\ 0 & -\mathbf{1}_{(2)} \end{pmatrix}, \tag{B.3}$$

where the σ_i are the standard Pauli matrices.

Now define

$$\Gamma_i = \gamma_i, \qquad \tilde{\Gamma}_i = \gamma_{i+3}, \qquad i = 1, 2, 3, \tag{B.4}$$

and

$$c_i = \frac{1}{2}(\Gamma_i - i\tilde{\Gamma}_i), \qquad \pi_i = \frac{1}{2}(\Gamma_i + i\tilde{\Gamma}_i), \tag{B.5}$$

as in eq.(1.11). Then we have

$$-\,i\epsilon_{ijk}\,c_j\,\pi_k = \Sigma_i + \tilde{\Sigma}_i, \tag{B.6}$$

with

$$\Sigma_1 + \tilde{\Sigma}_1 = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 & -\mathbf{1} & 0 \\ 0 & \sigma_1 & 0 & -\mathbf{1} \\ -\mathbf{1} & 0 & \sigma_1 & 0 \\ 0 & -\mathbf{1} & 0 & \sigma_1 \end{pmatrix},$$

$$\Sigma_2 + \tilde{\Sigma}_2 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 & -i\mathbf{1} & 0 \\ 0 & \sigma_2 & 0 & i\mathbf{1} \\ i\mathbf{1} & 0 & \sigma_2 & 0 \\ 0 & -i\mathbf{1} & 0 & \sigma_2 \end{pmatrix}, \tag{B.7}$$

$$\Sigma_3 + \tilde{\Sigma}_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 - \mathbf{1} & 0 & 0 & 0 \\ 0 & \sigma_3 + \mathbf{1} & 0 & 0 \\ 0 & 0 & \sigma_3 + \mathbf{1} & 0 \\ 0 & 0 & 0 & \sigma_3 - \mathbf{1} \end{pmatrix}.$$

Squaring these expressions and summing the results gives eq.(7.10).

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