CALT-68-1057
DOE RESEARCH AND
DEVELOPMENT REPORT

Symmetries of Coset Spaces and Kaluza-Klein Supergravity *

L. Castellani **, L. J. Romans and

N. P. Warner †

California Institute of Technology, Pasadena, CA 91125

ABSTRACT

We review known theorems about the isometry group of a general coset space G/H. The Killing vectors on G/H are explicitly constructed. Rescalings of the coset vielbeins are discussed, and we give a simple criterion to find which rescalings preserve the isometry group. We derive a general expression for the Riemann and Ricci tensors in terms of the rescaled vielbeins and the group structure constants.

These results have useful applications in Kaluza-Klein theories. As an example, we discuss the round and the squashed seven-spheres that have been used to compactify d = 11 supergravity: we show that they can be identified with two appropriately rescaled coset spaces SO(5)/SO(3).

November 9, 1983

* Work supported in part by the U. S. Department of Energy under contract No. DEAC-03-81-ER40050.

** On leave from the Instituto di Fisica Teorica, Torino University. Work supported in part by the Fleischmann Foundation.

† Weingart Foundation Fellow.

1. Introduction

Generalized Kaluza-Klein theories [1] offer a natural formalism for unified theories of gravity, gauge fields and scalar fields. One starts from 4 + N-dimensional gravity and, through the Kaluza-Klein mechanism, arrives at a 4-dimensional theory of gravitons, gauge bosons and scalars. The isometry group of the internal N-dimensional manifold becomes the gauge group of the reduced 4-dimensional Lagrangian. This encourages the study of homogeneous manifolds G/H: in the present paper G will always be a compact, connected, semisimple Lie Group, and H a (possibly nonmaximal) Lie subgroup. In general, if G/H is the space of right cosets gH, the left action of G and the right action of N(H)/H (N(H) being the normalizer of H in G) generate isometries of G/H. A detailed discussion is given in Sect. 2.

In pure multidimensional gravity one needs additional matter fields to trigger the compactification of the theory on the internal N-manifold. Also, the introduction of spin 1/2 fields in higher dimensions does not lead to massless fermions in 4-dimensions, the Dirac operator having no zero modes on positively curved manifolds (Lichnerowicz theorem). The situation is far better in Kaluza-Klein supergravity [1d, 1e, 2]: the Rarita-Schwinger operator can have zero modes, and the extra fields necessary for the spontaneous compactification are dictated by supersymmetry. In d = 11 supergravity, the nonvanishing fourth-rank "photon" curl drives the compactification to $M_4 \times M_7$, where M_4 is a 4-dimensional Einstein space-time with negative cosmological constant, and M_7 a 7-dimensional Einstein space, with positive curvature. The dimension of space-time is predicted to be four.

Eleven is a kind of magical number for supergravity: it is both the highest dimension for a consistent supergravity Lagrangian [3] and the minimum dimension for the compactification on a SU(3) × SU(2) × U(1) - isometric internal manifold [4]. The existence of SU(3) × SU(2) × U(1) compactifications has been proved in ref. [5].

A classification of compactifying solutions for d = 11 supergravity in which M_7 is a homogeneous space is given in [6].

Thus, a crucial question is whether a given coset space G/H can be given an Einstein metric, without losing its original symmetry. In Sec. 3 we describe the most general rescaling, or "squashing," that one can do to the metric of a homogeneous space and still preserve the symmetry. One finds that if the adjoint representation of N(H) is block-diagonal, then independent rescalings are allowed for vielbeins corresponding to different blocks. The general expressions for the rescaled Riemann and Ricci tensors are derived. The condition that the metric be Einstein becomes an algebraic constraint on the rescaling parameters. These results are applied in Sec. 4 to the round and squashed S^7 compactifications of d = 11 supergravity. We find that the squashed seven-sphere is a (rescaled) coset space SO(5)/SO(3), with SO(3) embedded as one of the factors in the SO(3) × SO(3) subgroup of SO(5). Another rescaling brings the squashed S^7 to the round S^7.

2. The isometries of G/H

We consider the right cosets G/H = {gH}, parameterized by the coordinates {y}. The left action of $g \in G$ on a coset representative L_y is given by (see e.g., [1f]):

$$g\, L_y \;=\; L_{y'}\, h\,, \tag{2.1}$$

where $L_{y'}$ and h are functions of L_y and g; the explicit form of these functions depends on the embedding of G/H in G, i.e. on the rule used in choosing the coset representatives.

The left-invariant Lie Algebra-valued 1-form:

$$e(y) = L_y^{-1}\, dL_y \,. \tag{2.2}$$

transforms under left multiplication by a constant $g \in G$ in the following way:

$$e(y') = hL_y^{-1}\, g^{-1}\, d(gL_y h^{-1}) = h\, e(y)h^{-1} - hdh^{-1}. \tag{2.3}$$

In particular, its projection on the coset generators, which defines the covariant frame (vielbein) on G/H, transforms as:

$$e^a(y') = (h\, e(y)\, h^{-1})^a = e^b(y)\, D_b{}^a(h^{-1}), \tag{2.4}$$

where $D_A{}^B(g)$ is the adjoint representation of G defined by

$$g^{-1}\, T_A\, g = D_A{}^B(g)\, T_B \quad (T_A \in \mathbb{G}). \tag{2.5}$$

The index conventions are:

$$\begin{array}{ll} a,\, b,\, \cdots & : \text{flat coset indices} \\ \alpha,\, \beta,\, \cdots & : \text{curved coset indices} \\ i,\, j,\, \cdots & : H \text{ indices} \\ A,\, B,\, \cdots & : G \text{ indices} \end{array} \tag{2.6}$$

The infinitesimal form of (2.4) is obtained by taking

$$g = 1 + \delta g^A\, T_A$$

$$h = 1 + \delta h^i\, T_i, \tag{2.7}$$

and reads:

$$e^a(y + \delta y) - e^a(y) = +\delta h^i\, C_{ib}{}^a\, e^b(y), \tag{2.8}$$

easily derived from the fact that $C_{iA}{}^B$ are the generators of the adjoint representation of H, and $C_{ij}{}^a = 0$. For compact semisimple Lie Algebras, the Killing metric can always be diagonalized, and for simplicity, we will assume it to be diagonal. Then one has $C_{ia}{}^{j} = 0$, and the algebra is reductive. Eq. (2.8) shows that the left action of G on $e^{a}(y)$ is equivalent to an SO(N) rotation of $e^{a}(y)$ (N = dim G/H). Thus the G-invariant metric on G/H is:

$$g_{\alpha\beta}(y) = \gamma_{ab}\, e_{\alpha}{}^{a}(y)\, e_{\beta}{}^{b}(y), \tag{2.9}$$

where γ_{ab} is the Killing group metric restricted to G/H.

The Killing vectors $K_A{}^{\beta}(y)$ associated with the left isometry group G are given by

$$K_A{}^{\beta}(y) = D_A{}^{a}(L_y)\, e_a{}^{\beta}(y), \tag{2.10}$$

$e_a{}^{\beta}$ is defined as the inverse of the coset vielbein $e_{\beta}{}^{a}$. Recall the definition of $K_A{}^{\beta}(y)$:

$$\begin{aligned} L_{y+\delta y} &= L_y + \delta y^{\beta}\, \partial_{\beta} L_y \\ &= L_y + \delta g^{A}\, K_A{}^{\beta}(y) \partial_{\beta} L_y . \end{aligned} \tag{2.11}$$

On the other hand, from (2.1) one finds

$$\begin{aligned} L_{y+\delta y} &\simeq (1 + \delta g^{A}\, T_A)\, L_y\, (1 - \delta h^{i}\, T_i) \\ &\simeq L_y (1 + \delta g^{A}\, D_A{}^{B}(L_y)\, T_B - \delta h^{i}\, T_i). \end{aligned} \tag{2.12}$$

Comparing (2.11) and (2.12), one arrives at the expression (2.10) for $K_A{}^{\beta}(y)$. Moreover, projecting on T_i yields:

$$\delta h^{i} = \delta g^{A}\, [D_A{}^{i}(L_y) - K_A{}^{\beta}(y)\, \omega_{\beta}{}^{i}(y)], \tag{2.13}$$

which relates the parameters of the infinitesimal transformations g and h in

(2.1). $\omega^i(y)$ is the projection of $e(y)$ on the H generators.

We now turn to the right isometries on G/H, and their corresponding Killing vectors.

The right action of $g \in G$ on L_y is given by:

$$L_y\, g = L_{y'} \cdot h. \tag{2.14}$$

For the expression $L_y\, g$ to make sense, it should not depend on the choice of coset representatives. This happens if and only if g belongs to the normalizer of H in G, denoted N(H). Indeed if

$$g\, H\, g^{-1} = H, \tag{2.15}$$

then $L_y\, g$ and $L_y{}'\, g$ are in the same coset ($L_y{}'$ is another representative of the same coset of L_y). Notice that for left multiplication [Eq. (2.1)], $g\, L_y$ is well defined for every $g \in G$.

It is clear that if $g \in H$, its right action on L_y is trivial: it does not move the point on the coset space. Thus we need only consider elements of N(H)/H, which has a natural group structure. We prove below that N(H)/H is the right isometry group of G/H.

Consider the transformation law of the 1-form (2.2) under right multiplication:

$$e(y') = L_{y'}{}^{-1}\, dL_{y'} = h\, g^{-1} (y^{-1} dy)\, g\, h^{-1} - h\, d h^{-1}. \tag{2.16}$$

Projecting on the coset generators T_a one finds

$$e^a(y') = e^b(y)\, D_b{}^a\, (g\, h^{-1})$$

$$= e^b(y)\, D_b{}^C\, (g)\, D_C{}^a (h^{-1})$$

$$= e^b(y)\, D_b{}^c\,(g)\, D_c{}^a(h^{-1}). \tag{2.17}$$

Infinitesimally, taking g and h as is (2.7), one has,

$$\begin{aligned} e^a(y + \delta y) - e^a(y) &= -\delta g^A\, e^B(y)\, C_{AB}{}^a - \delta h^i\, e^b(y)\, C_{ib}{}^a \\ &= (-\delta g^A\, C_{Ab}{}^a + \delta h^i C_{ib}{}^a) e^b(y) - \delta g^b\, C_{bi}{}^a \omega^i(y). \end{aligned} \tag{2.18}$$

Thus, the right action of g on the vielbein will induce an SO(N) rotation of $e^b(y)$ if and only if $\delta g^b\, C_{bi}{}^a = 0$, for every i, a. This happens if the generators $\mathbb{K}_b$ of the transformation $g \in$ N(H)/H commute with H: then

$$C_{bi}{}^a = 0 \quad \forall\ i, a. \tag{2.19}$$

If one decomposes $\mathbb{G}$

$$\mathbb{G} = \mathbb{H} + \mathbb{K} + \mathbb{L} \tag{2.20}$$

where $\mathbb{K}$ is the set of generators of N(H)/H, then eq. (2.19), and reductivity of $\mathbb{G}$ imply the following commutation relations:

$$\begin{aligned} &[\mathbb{K}, \mathbb{K}] \subset \mathbb{K} \\ &[\mathbb{K}, \mathbb{H}] = 0 \\ &[\mathbb{K}, \mathbb{L}] \subset \mathbb{L} \end{aligned} \tag{2.21}$$

In ref [7] it is shown that (2.21) are also the conditions for $\mathbb{K}$ to be in the Lie Algebra of N(H)/H. Therefore, N(H)/H is the right isometry group of G/H.

We now compute the corresponding Killing vectors on G/H, defined by

$$\begin{aligned} L_{y+\delta y} &= L_y + \delta g^A\, \tilde{K}_A{}^\beta(y) \partial_\beta L_y = \\ &= L_y(1 + \delta g^A\, \tilde{K}_A{}^\beta(y)\, e_\beta{}^B(y)\, T_B). \end{aligned} \tag{2.22}$$

From (2.14), we have

$$L_{y+\delta y} = L_y g h^{-1} = L_y(1 + \delta g^A T_A - \delta h^i T_i). \tag{2.23}$$

Projecting (2.22) and (2.23) on T_a yields:

$$\widetilde{K}_i{}^\beta(y) = 0 \tag{2.24}$$

$$\widetilde{K}_a{}^\beta(y) = e_a{}^\beta(y). \tag{2.25}$$

Eq. (2.24) is consistent with the fact that the right action of H is trivial on G/H. Eq. (2.25) tells us that the Killing vectors $\widetilde{K}_a{}^\beta$ corresponding to the right action of $\mathbf{K}_a$ on G/H are just the inverse vielbeins $e_a{}^\beta(y)$. As in the case of left isometries, one can compute the H parameter δh^i in (2.18) by projecting (2.22) and (2.23) on T_i. We find:

$$\delta h^i = -\delta g^b e_b{}^\beta(y)\omega_\beta{}^i(y). \tag{2.26}$$

It is easy to see from eqs. (2.1) and (2.14) that the isometries G and N(H)/H commute. This can be checked from the commutation relations between the left and right Killing vectors. As $\widetilde{K}$ is the inverse vielbein, the commutator $[K, \widetilde{K}]$ was in fact given in eq. (2.8). Indeed

$$e^a(y + \delta y) - e^a(y) = \delta g^A l_{\vec{K}_A} e^a(y) = \delta g^A (K_A{}^\alpha \partial_\alpha e_\beta{}^a + e_\alpha{}^a \partial_\beta K_A{}^\alpha)\, dy^\beta. \tag{2.27}$$

Multiplying by $-e_a{}^\gamma e_b{}^\beta$, and using (2.13), one derives the commutator

$$K_A{}^\alpha \partial_\alpha e_b{}^\gamma - e_b{}^\beta \partial_\beta K_A{}^\gamma = -[D_A{}^i(L_y) - K_A{}^\beta(y)\omega_\beta{}^i(y)]C_{ib}{}^c e_c{}^\gamma. \tag{2.28}$$

When the index b of $e_b{}^\gamma$ corresponds to the generators in K, $e_b{}^\gamma$ is the right Killing vector. But in that case $C_{ib}{}^c = 0\ \forall\ i, c$ (eq. 2.19), and therefore the right hand side of (2.28) vanishes: left and right Killing vectors commute. A criterion to find elements in the Lie Algebra of N(H)/H is given in eq. (2.19) or (2.21). [Cf.

ref [7], eqs (3.17) and (3.1.12).] The Killing vectors corresponding to left and right isometries are respectively

$$G: \; K_A{}^\beta(y) = D_A{}^\alpha(L_y) e_\alpha{}^\beta(y)$$

$$N(H)/H: \; \widetilde{K}_b{}^\beta(y) = e_b{}^\beta(y). \tag{2.29}$$

One should not conclude from the preceding discussion that the isometry group of G/H is always

$$G \times N(H)/H. \tag{2.30}$$

In most cases this is indeed correct. However, there are two ways the above construction can fail to give the actual isometry group:

1) Some of the right Killing vectors may coincide with left Killing vectors. As each right isometry commutes with each left isometry, these common Killing vectors can only correspond to explicit U(1) factors occurring in G and in N(H)/H. The actual isometry of G/H is therefore reduced to

$$G \times N'/H, \tag{2.31}$$

where $N = N' \times$ (common U(1)-factors). An example is provided by the coset spaces

$$\frac{\mathrm{SU(3)} \times \mathrm{U(1)}}{\mathrm{U(1)} \times \mathrm{U(1)}}, \tag{2.32}$$

discussed in ref [12]. For a generic embedding, N(H)/H = U(1) yields the same Killing vector as the explicit U(1) factor in G = SU(3) × U(1), which is indeed the true isometry group.

2) The symmetry may be larger than (2.30). This can happen when the coset manifold can be described by more than one quotient G/H. If $G/H = \widetilde{G}/\widetilde{H}$, with $\widetilde{G} \supset G$ the maximal group for which this is possible, the true isometry group of

the coset manifold will be

$$\tilde{G} \times N(\tilde{H})/\tilde{H}, \tag{2.33}$$

modulo the considerations in 1) above. A classic example is given by the seven-sphere: as a coset space, S^7 can be written in many ways:

$$\frac{SO(5)}{SO(3)} = \frac{SU(4)}{SU(3)} = \frac{SO(7)}{G_2} = \frac{SO(8)}{SO(7)}. \tag{2.34}$$

In the first two cases, the isometry group is in general $G \times N(H)/H$, but is increased to SO(8) for a particular rescaling of the vielbeins. On $SO(7)/G_2$ the unique SO(7)-invariant metric is also SO(8) invariant, and $SO(7)/G_2$ is indeed the round S^7.

3. Symmetric rescalings of G/H vielbeins.

By symmetric rescaling we mean a rescaling of the coset vielbein $e_\alpha{}^a$ which preserves the "natural" isometry $G \times N(H)/H$ of the coset metric $g_{\alpha\beta}(y) = \gamma_{ab}\, e_\alpha{}^a(y) e_\beta{}^b(y)$. For example, on a round sphere $S^n = SO(n+1)/SO(n)$, there is only one such rescaling, i.e. the trivial one which dilates uniformly all the directions. If some directions expand differently from others, the $SO(n+1)$ symmetry of the original S^n is lost, and the resulting squashed S^n has a lower symmetry.

Let us recall the transformation laws of the coset vielbein under the left and right action of G:

$$\begin{aligned}(\text{left } G):\ & e^a(y+\delta y) - e^a(y) \\ &= -\delta g^A [D_A{}^i(L_y) - K_A{}^\beta(y) e_\beta{}^i(y)]\, C_{ib}{}^a\, e^b(y)\end{aligned} \tag{3.1}$$

(right N(H)/H) : $e^a(y+\delta y) - e^a(y)$

$$= -\delta g^b [C_{bc}{}^a + e_b{}^\beta(y)\,\omega_\beta{}^i(y)\, C_{ic}{}^a] e^c(y), \qquad (3.2)$$

obtained from eqs. (2.8), (2.13) and (2.18), (2.26). Also, in the case of right isometries generated by T_b, $C_{bi}{}^a = 0$ has been used.

Both (3.1) and (3.2) are isometries of G/H: indeed they describe the effect of (left G) and (right N(H)/H) as SO(N)-rotations on the vielbein. It is clear that if $(C_i)_b{}^a$ is block diagonal in some subspaces $S_1, S_2, \ldots$ of K, then the vielbeins spanning these subspaces can be independently rescaled with no loss of left G symmetry. Indeed in this case eq. (3.1) will hold also for the rescaled vielbeins. A similar argument holds for the right N(H)/H-symmetry, when $(C_b)_c{}^a$ *and* $(C_i)_c{}^a$ are block diagonal in the same subspaces.

Thus we have the following

Theorem: a rescaling

$$\begin{aligned} &e^{a_1} \to r_1 e^{a_1}, \quad a_1 \text{ runs on } S_1 \\ &e^{a_2} \to r_2 e^{a_2}, \quad a_2 \text{ runs on } S_2 \\ &\quad\vdots \end{aligned} \qquad (3.3)$$

is a G × N(H)/H-symmetric rescaling if and only if

$$(C_D)_b{}^a \qquad D \text{ runs on N(H)}, \qquad (3.4)$$

is block diagonal in the spaces spanned by $e^{a_1}, e^{a_2} \ldots .$

4. Rescaled connection and curvature of G/H.

The Maurer-Cartan equations for e^a and ω^i are easily derived from $e = L_y^{-1} dL_y$:

$$de^a + \frac{1}{2}C^a{}_{bc}\, e^b \wedge e^c + C^a{}_{bi}\, e^b \wedge \omega^i = 0$$

$$d\omega^i + \frac{1}{2}C^i{}_{ab}\, e^a \wedge e^b + \frac{1}{2}C^i{}_{jk}\, \omega^j \wedge \omega^k = 0. \qquad (4.1)$$

Under a rescaling of e^a:

$$e^{a_1} = \mathcal{e}^{a_1} r(a_1)$$
$$e^{a_2} = \mathcal{e}^{a_2} r(a_2)$$
$$\vdots$$

eqs. (4.1) become:

$$d\mathcal{e}^a + \frac{1}{2}\frac{r(b)\,r(c)}{r(a)} C^a{}_{bc}\, \mathcal{e}^b \wedge \mathcal{e}^c + \frac{r(b)}{r(a)} C^a{}_{bi}\, \mathcal{e}^b \wedge \omega^i = 0$$

$$d\omega^i + \frac{1}{2} r(a)\, r(b)\, C^i{}_{ab}\, \mathcal{e}^a \wedge \mathcal{e}^b + \frac{1}{2} C^i{}_{jk}\, \omega^j \wedge \omega^k = 0. \qquad (4.2)$$

The connection one-form $B^a{}_b$ on G/H can be defined by

$$d\mathcal{e}^a + B^a{}_b \wedge \mathcal{e}^b = 0. \qquad (4.3)$$

Combining (4.3) and (4.1) together yields

$$B^a{}_b = -\frac{1}{2}\frac{r(b)\,r(c)}{r(a)} C^a{}_{bc}\, \mathcal{e}^c - \frac{r(b)}{r(a)} C^a{}_{bi}\, \omega^i + K^a{}_{bc}\, \mathcal{e}^c. \qquad (4.4)$$

The tensor $K^a{}_{bc}$, symmetric in b, c, is determined by the requirement of antisymmetry $B^a{}_b + B_b{}^a = 0$:

$$K^a{}_{bc} = -\frac{r(a)}{2} C^a{}_{bc}\left[\frac{r(c)}{r(b)} - \frac{r(b)}{r(c)}\right]. \qquad (4.5)$$

Thus the antisymmetric connection $B^a{}_b$ is given by

$$B^a{}_b = -\frac{1}{2}C^a{}_{bc}\, \mathcal{e}^c\left[\frac{r(b)\,r(c)}{r(a)} + \frac{r(a)\,r(c)}{r(b)} - \frac{r(b)\,r(a)}{r(c)}\right] - C^a{}_{bi}\, \frac{r(b)}{r(a)}\, \omega^i. \qquad (4.6)$$

The Riemann curvature is defined in terms of $B^a{}_b$ as:

$$R^a{}_b \equiv dB^a{}_b + B^a{}_c \wedge B^c{}_b \equiv R^a{}_{bde}\, e^d \wedge e^e. \tag{4.7}$$

Substituting (4.6) in (4.7), using the Maurer-Cartan eqs. (4.2) for the differentiated vielbeins, and Jacobi identities for products of structure constants, one arrives at

$$\begin{aligned} R^a{}_{bde} = {} & \frac{1}{4} C^a{}_{bc} C^c{}_{de} \begin{bmatrix} a\,b \\ c \end{bmatrix} \frac{r(d) r(e)}{r(c)} + \frac{1}{2} C^a{}_{bi} C^i{}_{de} r(d) r(e) \\ & + \frac{1}{8} C^a{}_{cd} C^c{}_{be} \begin{bmatrix} a\,c \\ d \end{bmatrix} \begin{bmatrix} b\,c \\ e \end{bmatrix} - \frac{1}{8} C^a{}_{ce} C^c{}_{bd} \begin{bmatrix} a\,c \\ e \end{bmatrix} \begin{bmatrix} b\,c \\ d \end{bmatrix}, \end{aligned} \tag{4.8}$$

with

$$\begin{bmatrix} a\,b \\ c \end{bmatrix} \equiv \frac{r(a)\, r(c)}{r(b)} + \frac{r(b)\, r(c)}{r(a)} - \frac{r(a)\, r(b)}{r(c)}. \tag{4.9}$$

Antisymmetry of $R^{ab}{}_{cd}$ in a, b, and c, d is manifest. The symmetry under (a, b) and (c, d) interchange is not so obvious, but can be checked with a little work.

The Ricci tensor R_{be} is easily obtained by contracting a and d in (4.8). Notice that for symmetric algebras its expression reduces to

$$R_{be} = \frac{1}{2} C^a{}_{bi}\, C^i{}_{ae}\, r(a) r(e) = \frac{1}{4} \gamma_{be} [r(b)]^2 = \frac{1}{4} \gamma_{be} [r(e)]^2, \tag{4.10}$$

because

$$\gamma_{be} \equiv -C^A{}_{bD}\, C^D{}_{eA} = -C^i{}_{bd}\, C^d{}_{ei} - C^a{}_{bi}\, C^i{}_{ea} = +2\, C^a{}_{bi}\, C^i{}_{ae}, \tag{4.11}$$

and the $C^a{}_{bi}$ is block diagonal.

5. Kaluza-Klein supergravity and the squashed seven-sphere.

An interesting compactification of d = 11 supergravity occurs on the squashed S^7, henceforth denoted J^7, obtained from the round S^7 through a continuous deformation. It was originally constructed by Jensen [8] and shown to be an Einstein space, i.e., a space for which the Ricci tensor is proportional to the metric. As previously mentioned, such spaces are important in d = 11 supergravity: indeed they give solutions of the field equations [9]. J^7 was investigated as a compactification of d = 11 supergravity by Awada, Duff and Pope in ref [10], and its supersymmetry content was shown to be either N = 0 or N = 1 depending upon orientation [11]. Subsequently Bais, Nicolai and van Nieuwenhuizen [12] showed that J^7 can be identified with the coset space

$$\frac{SO(5) \times SO(3)}{SO(3) \times SO(3)}, \tag{5.1}$$

where SO(3) × SO(3) is embedded as $SO(3)^I \times SO(3)^{J+L}$, $SO(3)^I \times SO(3)^J$ being the SO(3) × SO(3) subgroup of SO(5), $SO(3)^L$ the factorized SO(3) and $SO(3)^{J+L}$ the diagonal subgroup of $SO(3)^J \times SO(3)^L$.

Here we will show that J^7 can also be viewed as the coset space

$$\frac{SO(5)}{SO(3)}, \tag{5.2}$$

where SO(3) is embedded in one of the SO(3)'s of the SO(3) × SO(3) subgroup of SO(5).

The root diagram of SO(5) is given in Figure 1.

The $SO(3)^I \times SO(3)^J$ subalgebra is associated to the generators $E_1, E_2, F_1, F_2, H_1, H_2$. More precisely, it is generated by the following combinations:

$$SO(3)^I: \ \mathbf{E}_1 = \frac{i\sqrt{6}}{2}(E_1 + E_2), \ \mathbf{E}_2 = \frac{\sqrt{6}}{2}(E_1 - E_2), \ \mathbf{E}_3 = -\frac{i\sqrt{6}}{2}(H_1 + H_2) \quad (5.3)$$

$$SO(3)^J: \ \mathbf{F}_1 = \frac{i\sqrt{6}}{2}(F_1 + F_2), \ \mathbf{F}_2 = \frac{\sqrt{6}}{2}(F_1 - F_2), \ \mathbf{F}_3 = \frac{i\sqrt{6}}{2}(H_1 - H_2). \quad (5.4)$$

The remaining 4 generators Q in SO(5) split into a singlet S and a triplet T_a of the $SO(3)^{I+J}$ subgroup:

$$\begin{aligned} S &= i(Q_5 + Q_7) \\ T_1 &= Q_6 - Q_8 \\ T_2 &= -i(Q_6 + Q_8) \\ T_3 &= Q_5 - Q_7. \end{aligned} \quad (5.5)$$

A list of the structure constants of SO(5) in the basis (5.3), (5.4), (5.5) follows:

$$\begin{aligned} & C^{i}{}_{jk} = \varepsilon_{ijk}, \quad C^{\hat{i}}{}_{\hat{j}\hat{k}} = \varepsilon_{\hat{i}\hat{j}\hat{k}} \\ & C^{0}{}_{ia} = \frac{1}{2}\delta_{ai}, \ C^{a}{}_{i0} = -\frac{1}{2}\delta_{ai}, \ C^{i}{}_{a0} = \frac{1}{3}\delta_{ai} \\ & C^{0}{}_{\hat{i}a} = -\frac{1}{2}\delta_{a\hat{i}}, \ C^{a}{}_{\hat{i}0} = \frac{1}{2}\delta_{a\hat{i}}, \ C^{\hat{i}}{}_{a0} = -\frac{1}{3}\delta_{a\hat{i}} \\ & C^{b}{}_{ia} = \frac{1}{2}\varepsilon_{iab}, \ C^{i}{}_{ab} = \frac{1}{3}\varepsilon_{iab} \\ & C^{b}{}_{\hat{i}a} = \frac{1}{2}\varepsilon_{\hat{i}ab}, \ C^{\hat{i}}{}_{ab} = \frac{1}{3}\varepsilon_{\hat{i}ab}. \end{aligned} \quad (5.6)$$

The index conventions are

$$\begin{array}{ll} i, j, k & \text{run on } SO(3)^J \\ \hat{i}, \hat{j}, \hat{k} & \text{run on } SO(3)^I \\ a, b, c & \text{correspond to } T_a \\ 0 & \text{corresponds to } S\ . \end{array} \quad (5.7)$$

The Killing metric

$$\gamma_{AB} \equiv -C^{D}{}_{AC}\,C^{C}{}_{BD} \qquad A, B \cdots \text{ run on SO(5)}, \tag{5.8}$$

is given by:

$$\gamma_{ij} = 3\delta_{ij},\ \gamma_{\hat{i}\hat{j}} = 3\delta_{\hat{i}\hat{j}},\ \gamma_{ab} = 2\delta_{ab},\ \gamma_{oo} = 2, \tag{5.9}$$

with vanishing non-diagonal parts.

We choose here to embed SO(3) in the $SO(3)^I$ subgroup of SO(5). The coset indices are therefore $a, i, 0$.

According to our discussion in Section 2, the symmetry of this SO(5)/SO(3) coset space is

$$SO(5) \times SO(3)^J, \tag{5.10}$$

the $SO(3)^J$ being the normalizer of $SO(3)^I$ in SO(5). The rescalings that preserve (5.10) are

$$e^i \to \frac{1}{a} e^i,\quad e^a \to \frac{1}{b} e^a,\quad e^o \to \frac{1}{b} e^o. \tag{5.11}$$

Indeed

$$C_{D\beta}{}^{\alpha} \qquad \begin{matrix} \alpha,\beta = a, i, 0 \to G/H \\ D = \hat{i}, i \to N(H) \end{matrix} \tag{5.12}$$

is block diagonal in the spaces $\{i\}$ and $\{a, 0\}$.

Applying formula (4.8) of Section 4, the rescaled Riemann curvature is found to be

$$R^{a}{}_{bde} = \frac{1}{48}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})\left[8 - \frac{3b^2}{a^2}\right] b^2$$

$$R^{a}{}_{obo} = \frac{1}{48}\delta_{ab}\left[8 - \frac{3b^2}{a^2}\right] b^2$$

$$R^{i}{}_{ojo} = \frac{1}{48}\delta_{ij}\,\frac{b^4}{a^2}$$

$$
\begin{aligned}
R^{a}{}_{ibj} &= +\frac{1}{32}\delta_{ij}\delta_{ab}\frac{b^4}{a^2} - \frac{1}{32}(\delta_{aj}\delta_{ib} - \delta_{ai}\delta_{bj})\left(2-\frac{b^2}{a^2}\right)b^2 \\
R^{i}{}_{jkl} &= \frac{1}{4}\delta^{ij}_{kl}\, a^2 \\
R^{i}{}_{jab} &= \frac{1}{12}(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{aj})\left(\frac{b^4}{a^2} - 2b^2\right) \\
R^{i}{}_{joa} &= -\frac{1}{24}\varepsilon_{ija}\left(2 - \frac{b^2}{a^2}\right)b^2.
\end{aligned}
\tag{5.13}
$$

The corresponding Ricci tensor is

$$
\begin{aligned}
R_{ab} &= \delta_{ab}\left(\frac{1}{2} - \frac{1}{8}\frac{b^2}{a^2}\right)b^2 \\
R_{ij} &= \delta_{ij}\left(\frac{a^2}{4} + \frac{1}{8}\frac{b^4}{a^2}\right) \\
R_{oo} &= \left(\frac{1}{2} - \frac{1}{8}\frac{b^2}{a^2}\right)b^2.
\end{aligned}
\tag{5.14}
$$

We now look for rescalings a, b such that the resulting space becomes an Einstein space. This requires the coefficients in front of the Kronecker deltas in (5.14) to be equal. This happens only in two cases:

$$i)\quad \frac{b^2}{a^2} = 2 \tag{5.15}$$

$$ii)\quad \frac{b^2}{a^2} = \frac{2}{5}. \tag{5.16}$$

As it is easy to verify by reinserting (5.15) and (5.16) into the Riemann curvature (5.13), $b^2/a^2 = 2$ corresponds to the round S^7:

$$R^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{24}\,\delta^{\alpha\beta}_{\gamma\delta} b^2. \tag{5.17}$$

To put (5.13) in the form $R^{\alpha\beta}{}_{\gamma\delta}$ one uses the metric in (5.9). The second rescaling $b^2/a^2 = 2/5$ corresponds to the squashed seven-sphere. Indeed (5.13) reduces to

$$R^{a}{}_{bde} = \frac{1}{3}\,\frac{17}{20}\,\frac{1}{2}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})b^2$$

$$R^{a}{}_{obo} = \frac{1}{6}\,\frac{17}{20}\,\delta_{ab}\,b^2$$

$$R^{i}{}_{ojo} = \frac{1}{120}\,\delta_{ij}\,b^2$$

$$R^{a}{}_{ibj} = \frac{b^2}{16}\,\frac{1}{5}\,\delta_{ij}\delta_{ab} - \frac{1}{20}\,(\delta_{aj}\delta_{ib} - \delta_{ai}\delta_{bj})b^2$$

$$R^{i}{}_{jkl} = \frac{5}{8}\,\delta^{ij}_{kl}$$

$$R^{i}{}_{jab} = -\frac{8}{60}\,(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{aj})b^2$$

$$R^{i}{}_{joa} = -\frac{1}{15}\,\varepsilon_{ija}\,b^2, \tag{5.18}$$

which coincides with the curvature tensor of the squashed seven-sphere (cf. refs [10,11]).

This concludes our proof that SO(5)/SO(3), with the embedding discussed above and the rescaling $b^2/a^2 = 2/5$, is indeed J^7.

When one rescales with $b^2/a^2 = 2$, the round S^7 is recovered. This is an interesting illustration of how the symmetry G × N(H)/H of a coset space G/H can be increased by a rescaling that brings G/H to be equivalent to $\tilde{G}/\tilde{H}$, with $G \subset \tilde{G}$. Here the SO(5) × SO(3) symmetry becomes the full SO(8) of the round S^7

Acknowledgements

It is a pleasure to thank Dr. Krzyszthof Pilch for valuable discussions on the geometry of coset spaces.

References

[1] a) Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1 (1921) 966;

b) O. Klein, Z. Phys. 37 (1926), 895;

For a modern discussion, see:

c) B. de Witt, in Lectures at Les Houches, Relativity, groups and topology, 1963;

d) E. Cremmer, in "Supergravity '81," Cambridge University Press, eds. S. Ferrara and J. G. Taylor, 1982;

e) M. Duff and C. N. Pope, "Kaluza-Klein supergravity and the seven-sphere," Imperial College preprint ICTP/82/83-7, 1983;

f) A. Salam and J. Strathdee, Ann. of Phys. 141 (1982) 316.

[2] For a review on Supergravity, see P. van Nieuwenhuizen, Phys. Rep. 68 (1981), 192.

[3] W. Nahm, Nucl. Phys. B135 (1978) 149;
L. Castellani, P. Fré, F. Giani, K. Pilch and P. van Nieuwenhuizen, Phys. Rev. D, 26 (1982) 1481.

[4] E. Witten, Nucl. Phys. B186 (1981) 412.

[5] L. Castellani, R. D'Auria and P. Fré, "SU(3) × SU(2) × U(1) from d = 11 Supergravity," Torino preprint IFTT 427 (1983), to appear in Nucl. Phys. B.

[6] L. Castellani, L. J. Romans and N. P. Warner, "A Classification of Compactifying Solutions for d = 11 Supergravity," Caltech preprint CALT-68-1055, 1983.

[7] R. Coquereaux and A. Jadczyk, "Geometry of multidimensional universes," CERN preprint ref. TH. 3483, 1982.

[8] G. R. Jensen, J. Diff. Geom. 8 (1973), 599.

[9] P. G. O. Freund and M. A. Rubin, Phys. Lett. 97B (1980) 233;
M. J. Duff and D. J. Toms, in the Second Europhysical Study Conference in Unification, Erice 1981; CERN preprint TH-3259 (1982).

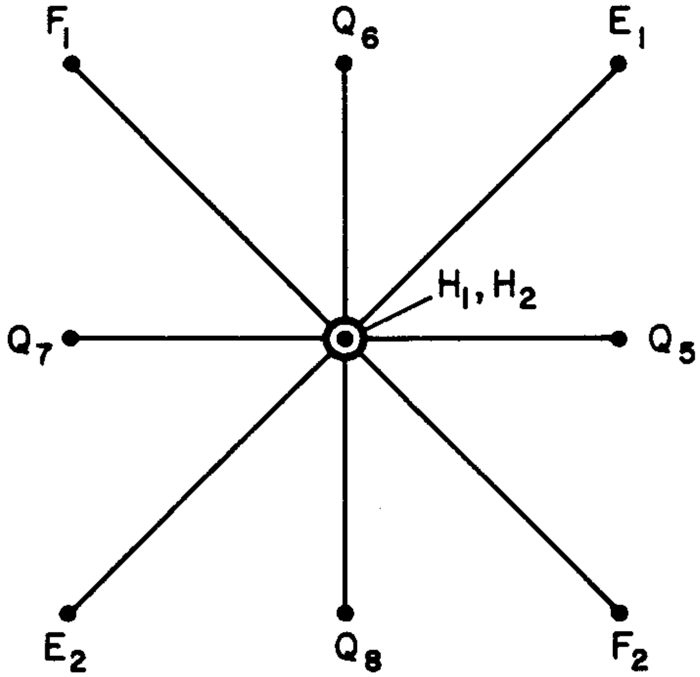
[10] M. A. Awada, M. J. Duff and C. N. Pope, Phys. Rev. Lett. 50 (1983) 294.

[11] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Phys. Rev. Lett. 50 (1983), 2043.

[12] F. A. Bais, H. Nicolai and P. van Nieuwenhuizen, "Geometry of coset spaces and massless modes of the squashed seven sphere in supergravity," CERN preprint ref. TH. 3577, 1983.

Figure Caption

[1] The root diagram for SO(5).



F_1
Q_6
E_1
H_1, H_2
Q_7
Q_5
E_2
Q_8
F_2