The Standard Model Lagrangian

Abstract

The Lagrangian for the Standard Model is written out in full, here.

The primary novelty of the approach adopted here is the deeper analysis of the fermionic space. Analogous to the situation in the 19th century in which Maxwell inserted the "displacement current" term in the field law for electromagnetism in order to retain a charge conservation law and bring out the symmetric structure of the equations, the right neutrinos play the corresponding role in the present situation. Here, the symmetric structure that emerges is that, with the inclusion of the extra terms, the fermion space factors significantly. By employing this symmetric structure, the Lagrangian may be written in a substantially more transparent fashion. Two bases for fermion space will be developed here: the "hypercolor basis" and the "Casimir basis". The Standard Model, itself, is included as a special case within an enveloping generalization of Yang-Mills-Higgs theories that provides room for future extensions. In particular, the Yukawa sector is developed from first principles.

1. Yang-Mills-Higgs Lagrangians

The Standard Model is an instance of a Yang-Mills-Higgs system which may also be extended below to include both curvilinear systems and, going further, the gravitational interaction. Fundamentally, it is a theory of spin ½ fermionic matter under the influence of a Yang-Mills field which is mediated by spin 1 gauge bosons. The full symmetry of the interaction is broken at the state space level, with the vacuum retaining only a residual symmetry. The broken symmetries lead to extra scalar modes out of which arise the Higgs field, which is minimally coupled to the gauge field, as well. The interaction of the Higgs and fermion fields can be determined primarily by the requirement that it be trilinear in the fields. As shown below, this is nearly sufficient to prove that the coupling must be of the Yukawa type. Both this derivation and the reduction of the fields to mass eigenmodes will be carried out in detail below.

With respect to the notation to be developed below, the Lagrangian for a Yang-Mills-Higgs theory may be written as

$$\mathfrak{L} = \varepsilon\left(\psi, \left(i\gamma^{\mu}\left(\partial_{\mu} + \mathbf{A}_{\mu}\right) - G(\varphi)\right)\psi\right) - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} k(\mathbf{F}_{\mu\nu}, \mathbf{F}_{\rho\sigma}) + \chi\left(\left(\partial_{\mu} + \mathbf{A}_{\mu}\right)\varphi, \left(\partial_{\nu} + \mathbf{A}_{\nu}\right)\varphi\right) - \lambda\left(\chi(\varphi,\varphi) - \frac{v^2}{2}\right)^2 .$$

An interesting possibility, not further developed here, arises of pulling the Lagrangian back to a *square root*, by making use of a fermion "potential" to generate the field ψ. This development has been discussed in another writeup, but is not fully developed here. It requires an interaction that is parity-symmetric, which ties in closely with the issue raised below in the section on the Casimir basis. Though the Standard Model, itself, is not parity symmetric, it admits a possible extension to an interaction that is, where parity is a broken symmetry. This is an issue that falls squarely in line with the *See-Saw* model of neutrino physics.

1.1. Yang-Mills Sector

The gauge field $\mathbf{A}_{\mu}$ associated with a symmetry group G may be written in terms of a basis

$$(\mathbf{Y}_a : a = 0, \ldots, \dim G - 1)$$

of the corresponding Lie algebra $L = \mathrm{Lie}(G)$ as

$$\mathbf{A}_{\mu} = \sum_{a=0}^{\dim G - 1} A_{\mu}^{a} \mathbf{Y}_a .$$

In a $U(1)$ field, such as the Maxwell field, in a Minkowski frame, the kinetic momentum P_{μ} of a test charge, its canonical momentum p_{μ} and the potential A_{μ} assume the respective forms

$$P_{\mu} = m\left(-\frac{d\mathbf{r}}{ds}, \frac{dt}{ds}\right), \quad p_{\mu} = \left(-\mathbf{p}, H\right), \quad A_{\mu} = (\mathbf{A}, -\varphi) ,$$

and are related by

$$\mathbf{p} = m\frac{d\mathbf{r}}{ds} + e\mathbf{A}, \quad H = m\frac{dt}{ds} + e\varphi ,$$

where s is the proper time of the test charge. These relations generalize in arbitrary coordinate frames to

$$p_\mu = P_\mu - eA_\mu .$$

Through the Equivalence Principle, they are generalized further to local coordinate frames for curved spacetimes. For a Yang-Mills field with a Lie group G and corresponding Lie algebra L, a similar relation holds, with the scalar charge e replaced by a charge co-vector θ_a and the simple product replaced by an inner product in the vector space of the Lie algebra L,

$$p_\mu = P_\mu - \sum_{a=0}^{\dim G-1} \theta_a A_\mu^a .$$

Under quantization, the canonical and kinetic momentum are replaced respectively by the ordinary derivstive ∂_μ and covariant derivative $D_\mu \equiv \partial_\mu + \mathbf{A}_\mu$ through the correspondences,

$$p_\mu \leftrightarrow i\hbar\partial_\mu , \quad P_\mu \leftrightarrow i\hbar D_\mu$$

This leads to the following representation for the charge

$$\theta_a = i\hbar \mathbf{Y}_a .$$

The charge operators are Hermitean and gauge generators anti-Hermitean,

$$\mathbf{Y}_a{}^+ = -\mathbf{Y}_a , \quad \theta_a{}^+ = \theta_a .$$

It is common practice to normalize the charge generator by explicitly bringing out whatever coupling constants are involved, so that one may then write

$$\mathbf{Y}_a = -ig\theta_a ,$$

instead. For a simple gauge group, there will only be one coupling, whereas for a semi-simple gauge group there will be a different coupling for each factor. By convention, units are generally chosen such that $\hbar = 1$, though we may equally well regard the extra $\hbar$ as having been absorbed in the definition of the coupling, g.

The gauge field for the Standard Model is that for the Lie group $S(U(2)\times U(3))$. By convention, it is written as

$$\mathbf{A}_\mu \equiv -ig'B_\mu \mathbf{Y} - ig\sum_{i=1}^{3} W_\mu^i \mathbf{I}_i - ig_s \sum_{a=1}^{8} G_\mu^a \mathbf{\Lambda}_a .$$

The charge generators are those of the covering group $U(1)_Y \times SU(2)_I \times SU(3)_\Lambda$ with the respective charge operators of the corresponding Lie algebras

$u(1)_Y$	$su(2)_I$	$su(3)_\Lambda$
$\mathbf{Y}$	$\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$	$\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3, \mathbf{\Lambda}_4, \mathbf{\Lambda}_5, \mathbf{\Lambda}_6, \mathbf{\Lambda}_7, \mathbf{\Lambda}_8$

The commutators for the $su(2)_I$ and $su(3)_\Lambda$ subalgebras are, respectively,

$$[\mathbf{I}_i, \mathbf{I}_j] = \sum_{k,l=1}^{3} i\varepsilon_{ijk}\delta^{kl}\mathbf{I}_l , \quad [\mathbf{\Lambda}_a, \mathbf{\Lambda}_b] = \sum_{c,d=1}^{8} if_{abc}\delta^{cd}\mathbf{\Lambda}_d .$$

The corresponding trilinear forms $[\mathbf{I}_i, \mathbf{I}_j, \mathbf{I}_k] \equiv \varepsilon_{ijk}$ and $[\mathbf{\Lambda}_a, \mathbf{\Lambda}_b, \mathbf{\Lambda}_c] \equiv f_{abc}$ are completely anti-symmetric, with

$$[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] = [\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3] = 1, \quad [\mathbf{\Lambda}_4, \mathbf{\Lambda}_5, \mathbf{\Lambda}_8] = [\mathbf{\Lambda}_6, \mathbf{\Lambda}_7, \mathbf{\Lambda}_8] = \frac{\sqrt{3}}{2},$$

$$[\mathbf{\Lambda}_1, \mathbf{\Lambda}_4, \mathbf{\Lambda}_7] = [\mathbf{\Lambda}_1, \mathbf{\Lambda}_6, \mathbf{\Lambda}_5] = [\mathbf{\Lambda}_2, \mathbf{\Lambda}_4, \mathbf{\Lambda}_6] = [\mathbf{\Lambda}_2, \mathbf{\Lambda}_5, \mathbf{\Lambda}_7] = [\mathbf{\Lambda}_3, \mathbf{\Lambda}_4, \mathbf{\Lambda}_5] = [\mathbf{\Lambda}_3, \mathbf{\Lambda}_7, \mathbf{\Lambda}_6] = \frac{1}{2}.$$

The field strengths are defined by

$$\mathbf{F}_{\mu\nu} \equiv \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] = -ig'B_{\mu\nu}\mathbf{Y} - ig\sum_{i=1}^{3} W_{\mu\nu}^i \mathbf{I}_i - ig_s \sum_{a=1}^{8} G_{\mu\nu}^a \mathbf{\Lambda}_a ,$$

with the components given explicitly by

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu ,$$

$$W_{\mu\nu}^{k} \equiv \partial_{\mu} W_{\nu}^{k} - \partial_{\nu} W_{\mu}^{k} + ig \sum_{k,l=1}^{3} \delta^{kl} \varepsilon_{lij} W_{\mu}^{i} W_{\nu}^{j} \ ,$$

$$G_{\mu\nu}^{c} \equiv \partial_{\mu} G_{\nu}^{c} - \partial_{\nu} G_{\mu}^{c} + ig_{s} \sum_{c,d=1}^{8} \delta^{cd} f_{dab} G_{\mu}^{a} G_{\nu}^{b} \ .$$

The field Lagrangian is given by

$$\mathfrak{L}_2 \equiv -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} k(\mathbf{F}_{\mu\nu}, \mathbf{F}_{\rho\sigma}) = \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} k(\mathbf{F}_{\mu\nu}{}^{+}, \mathbf{F}_{\rho\sigma}),$$

with the gauge group metric defined through the charge generators by

$$k(\mathbf{I}_i, \mathbf{Y}) = k(\mathbf{Y}, \mathbf{\Lambda}_a) = k(\mathbf{\Lambda}_a, \mathbf{I}_i) = 0,$$

$$k(\mathbf{Y}, \mathbf{Y}) = -\left(\frac{1}{g'}\right)^2, \quad k(\mathbf{I}_i, \mathbf{I}_j) = -\delta_{ij}\left(\frac{1}{g}\right)^2, \quad k(\mathbf{\Lambda}_a, \mathbf{\Lambda}_b) = -\delta_{ab}\left(\frac{1}{g_s}\right)^2.$$

An adjoint invariant metric is one satisfying the property

$$k(U\mathbf{u}U^{+}, U\mathbf{v}U^{+}) = k(\mathbf{u}, \mathbf{v}),$$

which implies,

$$k([\mathbf{u}, \mathbf{v}], \mathbf{w}) + k(\mathbf{v}, [\mathbf{u}, \mathbf{w}]) = 0 \ .$$

The most general non-degenerate adjoint invariant metric for $S(U(2)\times U(3))$ must take on the form just given, provided that the $U(1)$ mode is orthogonalized with respect to the other fields. This is accomplished by a transformation of the form

$$\mathbf{I}_i \rightarrow \mathbf{I}_i + w_i \mathbf{Y}, \quad \mathbf{\Lambda}_a \rightarrow \mathbf{\Lambda}_a + g_a \mathbf{Y} \ ,$$

which will not affect the underlying Lie algebra. The coupling coefficients are directly related to the gauge group metric, yielding its independent components.

Explicitly, the Lagrangian takes the form

$$\mathfrak{L}_B = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} \left(B_{\mu\nu} B_{\rho\sigma} + \delta_{ij} W_{\mu\nu}^{i} W_{\rho\sigma}^{j} + \delta_{cd} G_{\mu\nu}^{c} G_{\rho\sigma}^{d}\right).$$

In the classical field theory, the gauge group metric is assumed to be constant, though the assumption is not a necessary ingredient of classical gauge theory. In the quantized theory, the requirements of renormalization force one to endow it with a “scale dependency”. In general, “scale dependency” refers to the resolution at which the point-like sources represented by interacting quantum fields are probed in scattering experiments. In effect, the metric becomes dependent on the distance from a point-like source, making it (in fact) a function of position that tends toward a constant asymptotically.

In virtue of the close relation of the couplings to the gauge metric, this translates into “vertex” renormalization or (equivalently) associated with the scaling of the gauge fields.

1.2. Fermion Sector

The fermions are found in the following $SU(2)_I \times SU(3)_\Lambda \times U(1)_Y$ sectors

$$\begin{matrix} (\mathbf{1,1,6}) & (\mathbf{1,3,4}) & (\mathbf{1,\bar{3},2}) & (\mathbf{1,1,0}) \\ (\mathbf{2,1,3}) & (\mathbf{2,3,1}) & (\mathbf{2,\bar{3},-1}) & (\mathbf{2,1,-3}) \\ (\mathbf{1,1,0}) & (\mathbf{1,3,-2}) & (\mathbf{1,\bar{3},-4}) & (\mathbf{1,1,-6}) \end{matrix},$$

corresponding to

Left Positrons	Right Up Quarks	Left Anti-Down Quarks	(Right Neutrinos)
Right Anti-Leptons	Left Quarks	Right Anti Quarks	Left Leptons
(Left Anti-Neutrinos)	Right Down Quarks	Left Anti-Up Quarks	Right Electrons

Lepton refers collectively to electrons and neutrinos; anti-lepton to positrons and anti-neutrinos.

The two $(\mathbf{1,1,0})$ sectors are neutral and therefore do not participate in interactions, unless they have non-zero mass. There are not included in the Standard Model, but they will be retained here for the sake of simplicity. These correspond to the right-handed neutrino and left-handed anti-neutrinos. The question whether and how these sectors exist is wide open, particularly with the discovery of neutrino oscillation indicating the existence of non-zero neutrino masses. Also, because the additional sectors have zero charge, it turns out that there are more ways to endow them with mass than equating neutrinos with Dirac fields. This includes the possibility of Majorana fields or a combination of Majorana and Dirac fields.

There is also a 3-fold degeneracy of the charge spectrum, corresponding to what is called "generation". So the spectrum extends to equivalents involving two other varieties of neutrinos and with the following replacements { Electron, Up, Down } ↔ { Mu, Charm, Strange } ↔ { Tau, Top, Bottom }. The generations may be identified by their "charge eigenstates", which are defined as the normal modes of interaction with the gauge field. They may also be defined by their "mass eigenstates", defined as the normal modes of interaction with the Higgs field. The gauge field, itself, also has a similar dichotomy of representation. The names just mentioned refer to the mass eigenstates. The description immediately to follow refers to the charge eigenstates.

Since the gauge field associated with $SU(2)_I$ is non-abelian, and includes part of what we call electromagnetism, then the corresponding field equations are non-linear and inhomogeneous, containing on the right-hand sides of the equations governing ***both*** electric and magnetic sources constructed entirely from the fields. Magnetic monopole solutions can thus be derived.

In order for the classical theory to be consistently quantized, it must be free from anomalies. The one anomaly that occurs is directly associated with the left-right asymmetry of the $SU(2)_I$ sector. If the extra neutrino sectors are neutral, the requirement that the anomaly be absent uniquely specifies the $U(1)_Y$ charge up to a unit, which may be identified as the quantum. The spectrum given above for the hypercharge is written in terms of the smallest $U(1)_Y$ charge. The units adopted in the standard literature are either 3 or 6 times this value. If the extra neutrino sectors are included, then the anomaly removal condition allows for up to 2 separate $U(1)$ sectors (or combinations thereof), the second being associated with baryon number.

The spectrum is split between the Dirac spinor ψ and its conjugate $\overline{\psi}$ respectively into the "matter" and "anti-matter" sectors

$$\psi \leftrightarrow \begin{matrix} - & (\mathbf{1,3,4}) & - & (\mathbf{1,1,0}) \\ - & (\mathbf{2,3,1}) & - & (\mathbf{2,1,-3}) \\ - & (\mathbf{1,3,-2}) & - & (\mathbf{1,1,-6}) \end{matrix} \quad \overline{\psi} \leftrightarrow \begin{matrix} (\mathbf{1,1,6}) & - & (\mathbf{1,\overline{3},2}) & - \\ (\mathbf{2,1,3}) & - & (\mathbf{2,\overline{3},-1}) & - \\ (\mathbf{1,1,0}) & - & (\mathbf{1,\overline{3},-4}) & - \end{matrix}.$$

The symmetries of the Standard Model also include $SO(3,1) \to SU(2)_R \times SU(2)_L$, which pertains to the changes in the local spacetime frame, and is expressed in the decomposition respectively, for right and left handed states. The fermions occupy the sector $(\mathbf{2,1}) \oplus (\mathbf{1,2})$ and the full $SU(2)_I \times SU(3)_\Lambda \times U(1)_Y \times SU(2)_R \times SU(2)_L$ assignments are

$$\begin{matrix} (\mathbf{1,1,6,1,2}) & (\mathbf{1,3,4,2,1}) & (\mathbf{1,\overline{3},2,1,2}) & (\mathbf{1,1,0,2,1}) \\ (\mathbf{2,1,3,2,1}) & (\mathbf{2,3,1,1,2}) & (\mathbf{2,\overline{3},-1,2,1}) & (\mathbf{2,1,-3,1,2}) \\ (\mathbf{1,1,0,1,2}) & (\mathbf{1,3,-2,2,1}) & (\mathbf{1,\overline{3},-4,1,2}) & (\mathbf{1,1,-6,2,1}) \end{matrix}.$$

The matter-antimatter splitting is not unique, other splittings are possible. But the important element is that the Dirac spinor is being used to embody a metric $\varepsilon(\psi_1, \psi_2) \equiv \dfrac{\overline{\psi}_1 \psi_2 + (\psi_1)^T (\overline{\psi}_2)^T}{2}$ that associates the sectors in the following pairings

$$(\mathbf{2,1,-3,1,2}) \leftrightarrow (\mathbf{1,1,6,1,2}) \oplus (\mathbf{1,1,0,1,2}), \quad (\mathbf{2,\overline{3},-1,2,1}) \leftrightarrow (\mathbf{1,3,4,2,1}) \oplus (\mathbf{1,3,-2,2,1}),$$
$$(\mathbf{2,3,1,1,2}) \leftrightarrow (\mathbf{1,\overline{3},2,1,2}) \oplus (\mathbf{1,\overline{3},-4,1,2}), \quad (\mathbf{2,1,3,2,1}) \leftrightarrow (\mathbf{1,1,0,2,1}) \oplus (\mathbf{1,1,-6,2,1}).$$

The same comments made about the gauge metric apply here. In classical field theory, the fermion metric is constant, but renormalization in quantum field theory endows it with a scale dependency that makes the metric dependent on the distance from point-like sources, making it (in fact) a function of position that tends toward a constant away from the sources.

This translates into the renormalization factors associated with fermion scaling.

The fermion metric is ***not*** gauge invariant

$$\varepsilon(U\psi_1, U\psi_2) \neq \varepsilon(\psi_1, \psi_2)\,,$$

unless the gauge group acts in a parity symmetric way.

In retrospect, this may serve as an argument for parity being a broken symmetry, with the interactions actually being symmetric under parity.

1.3. The Hypercolor Basis

We will, here, adopt the "matter + anti-matter" decomposition expressing the associated Hilbert space in the following product basis

$$\{|+\rangle, |-\rangle\} \otimes \{|\mathbf{r}\rangle, |\mathbf{l}\rangle\} \otimes \{|\mathbf{u}\rangle, |\mathbf{d}\rangle\} \otimes \{|\mathbf{w}\rangle, |\mathbf{x}\rangle, |\mathbf{y}\rangle, |\mathbf{z}\rangle\} \otimes \{|\mathbf{1}\rangle, |\mathbf{2}\rangle, |\mathbf{3}\rangle\}.$$

Later, we will switch over to the 6-bit representation, which is better suited to factoring out the natural $2\times2\times2\times2\times2\times2$ structure contained within each generation of the fermion spectrum that, in turn, is strongly suggestive of an underlying basis in $SO(10,1)$.

The corresponding identity operators will be denoted by

$$I_S \equiv |+\rangle\langle+| + |-\rangle\langle-|, \quad I_P \equiv |\mathbf{r}\rangle\langle\mathbf{r}| + |\mathbf{l}\rangle\langle\mathbf{l}|, \quad I_I \equiv |\mathbf{u}\rangle\langle\mathbf{u}| + |\mathbf{d}\rangle\langle\mathbf{d}|,$$

$$I_C \equiv |\mathbf{w}\rangle\langle\mathbf{w}| + |\mathbf{x}\rangle\langle\mathbf{x}| + |\mathbf{y}\rangle\langle\mathbf{y}| + |\mathbf{z}\rangle\langle\mathbf{z}|, \quad I_G \equiv |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}| + |\mathbf{3}\rangle\langle\mathbf{3}|.$$

In the following, tensor products will be written as ordinary products, with the identity operators omitted. Thus, for instance,

$$|\mathbf{r}\rangle\langle\mathbf{r}| - |\mathbf{l}\rangle\langle\mathbf{l}| = I_S \otimes \left(|\mathbf{r}\rangle\langle\mathbf{r}| - |\mathbf{l}\rangle\langle\mathbf{l}|\right) \otimes I_I \otimes I_C \otimes I_G\,.$$

The first two factors $\{|+\rangle, |-\rangle\} \otimes \{|\mathbf{r}\rangle, |\mathbf{l}\rangle\}$ account for the $SO(3,1)$ decomposition with the respective assignments

$$\{|+\rangle, |-\rangle\} \otimes |\mathbf{r}\rangle \leftrightarrow (\mathbf{2},\mathbf{1}), \quad \{|+\rangle, |-\rangle\} \otimes |\mathbf{l}\rangle \leftrightarrow (\mathbf{1},\mathbf{2}).$$

With respect to this basis, the Dirac matrices assume the form

$$\gamma^0 = |\mathbf{l}\rangle\langle\mathbf{r}| + |\mathbf{r}\rangle\langle\mathbf{l}|, \quad \gamma^i = \sigma_i\left(|\mathbf{l}\rangle\langle\mathbf{r}| - |\mathbf{r}\rangle\langle\mathbf{l}|\right) \quad (i = 1,2,3)\,,$$

with the Pauli matrices assuming the form

$$\sigma_1 = |+\rangle\langle-| + |-\rangle\langle+|, \quad \sigma_2 = i\left(|-\rangle\langle+| - |+\rangle\langle-|\right), \quad \sigma_3 = |+\rangle\langle+| - |-\rangle\langle-|.$$

From this, we get

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = |\mathbf{r}\rangle\langle\mathbf{r}| - |\mathbf{l}\rangle\langle\mathbf{l}|.$$

This is the Weyl representation and it corresponds to the decomposition of the Dirac spinor into Weyl spinors as follows, using van der Waerden notation

$$\psi = \begin{pmatrix} \psi_R \\ \varepsilon\psi_L \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi^{\dot{0}} \\ \psi^{\dot{1}} \end{pmatrix} \in (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,.$$

with indices raised by

$$\varepsilon\psi_L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_{\dot{0}} \\ \psi_{\dot{1}} \end{pmatrix} = \begin{pmatrix} \psi_{\dot{1}} \\ -\psi_{\dot{0}} \end{pmatrix} = \begin{pmatrix} \psi^{\dot{0}} \\ \psi^{\dot{1}} \end{pmatrix}.$$

The third factor $\{|\mathbf{u}\rangle, |\mathbf{d}\rangle\}$ accounts for the electroweak sector, whose symmetry group is given by $U(2)_{I,Y}$, with the 2-fold covering group $SU(2)_I \times U(1)_Y$. For matter states, the basis effects the following $U(2)_{I,Y} \times SO(3,1)$ decomposition

$$\{|\mathbf{u}\rangle, |\mathbf{d}\rangle\} \otimes (\mathbf{1},\mathbf{2}) \leftrightarrow (\mathbf{2}, Y, \mathbf{1}, \mathbf{2}), \quad |\mathbf{u}\rangle \otimes (\mathbf{2},\mathbf{1}) \leftrightarrow (\mathbf{1}, Y+3, \mathbf{2}, \mathbf{1}), \quad |\mathbf{d}\rangle \otimes (\mathbf{2},\mathbf{1}) \leftrightarrow (\mathbf{1}, Y-3, \mathbf{2}, \mathbf{1}),$$

resulting in a grouping into quadruplets

$$(\mathbf{4}, Y) = (\mathbf{1}, Y+3, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, Y, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, Y-3, \mathbf{1}, \mathbf{2})$$

and leading to the following grouping of the matter fermion states into $(U(2)_{I,Y} \times SO(3,1)) \times SU(3)_\Lambda$ sectors

$$\left.\begin{matrix} (\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2},\mathbf{1}) \\ (\mathbf{2},\mathbf{1},-\mathbf{3},\mathbf{1},\mathbf{2}) \\ (\mathbf{1},\mathbf{1},-\mathbf{6},\mathbf{2},\mathbf{1}) \end{matrix}\right\} \to (\mathbf{4},-\mathbf{3},\mathbf{1}), \quad \left.\begin{matrix} (\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{2},\mathbf{1}) \\ (\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{2}) \\ (\mathbf{1},\mathbf{3},-\mathbf{2},\mathbf{2},\mathbf{1}) \end{matrix}\right\} \to (\mathbf{4},\mathbf{1},\mathbf{3}).$$

With respect to this basis, the $SU(2)_I$ generators are

$$\mathbf{I}_i = \frac{\tau_i}{2} |\mathbf{l}\rangle\langle\mathbf{l}|, \quad (i = 1,2,3),$$

where

$$\tau_1 = |\mathbf{u}\rangle\langle\mathbf{d}| + |\mathbf{d}\rangle\langle\mathbf{u}|, \quad \tau_2 = i(|\mathbf{d}\rangle\langle\mathbf{u}| - |\mathbf{u}\rangle\langle\mathbf{d}|), \quad \tau_3 = |\mathbf{u}\rangle\langle\mathbf{u}| - |\mathbf{d}\rangle\langle\mathbf{d}|.$$

The fourth factor $\{|\mathbf{w}\rangle, |\mathbf{x}\rangle, |\mathbf{y}\rangle, |\mathbf{z}\rangle\}$ accounts for the $SU(3)_\Lambda$ decomposition with the assignments

$$|\mathbf{w}\rangle \leftrightarrow \mathbf{1}, \quad (|\mathbf{x}\rangle, |\mathbf{y}\rangle, |\mathbf{z}\rangle) \leftrightarrow \mathbf{3}.$$

In this basis, the $SU(3)_\Lambda$ generators become

$$\mathbf{\Lambda}_a = \frac{\lambda_a}{2}, \quad (a = 1,2,3,4,5,6,7,8),$$

where

$$\begin{aligned} &\lambda_1 = |\mathbf{x}\rangle\langle\mathbf{z}| + |\mathbf{z}\rangle\langle\mathbf{x}|, \quad \lambda_2 = i(|\mathbf{z}\rangle\langle\mathbf{x}| - |\mathbf{x}\rangle\langle\mathbf{z}|), \quad \lambda_3 = |\mathbf{x}\rangle\langle\mathbf{x}| - |\mathbf{z}\rangle\langle\mathbf{z}|, \\ &\lambda_4 = |\mathbf{x}\rangle\langle\mathbf{y}| + |\mathbf{y}\rangle\langle\mathbf{x}|, \quad \lambda_5 = i(|\mathbf{y}\rangle\langle\mathbf{x}| - |\mathbf{x}\rangle\langle\mathbf{y}|), \quad \lambda_8 = \frac{|\mathbf{x}\rangle\langle\mathbf{x}| - 2|\mathbf{y}\rangle\langle\mathbf{y}| + |\mathbf{z}\rangle\langle\mathbf{z}|}{\sqrt{3}}. \\ &\lambda_6 = |\mathbf{z}\rangle\langle\mathbf{y}| + |\mathbf{y}\rangle\langle\mathbf{z}|, \quad \lambda_7 = i(|\mathbf{y}\rangle\langle\mathbf{z}| - |\mathbf{z}\rangle\langle\mathbf{y}|), \end{aligned}$$

The $U(1)_Y$ generator becomes

$$\mathbf{Y} = \frac{\tau_3}{2} |\mathbf{r}\rangle\langle\mathbf{r}| + G,$$

where

$$G \equiv \frac{|\mathbf{x}\rangle\langle\mathbf{x}| + |\mathbf{y}\rangle\langle\mathbf{y}| + |\mathbf{z}\rangle\langle\mathbf{z}| - 3|\mathbf{w}\rangle\langle\mathbf{w}|}{6}$$

is the baryon number operator (an additional factor of $1/2$ is added for future covenience).

Finally, the last factor $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle, |\mathbf{3}\rangle\}$ accounts for the generational degeneracy.

No theoretical weight is necessarily being given to this particular representation, but it is the most convenient way to write out the "matter + anti-matter" assignment of the fermion spectrum to the Dirac spinors. The explicit assignments are thus

$$\psi \leftrightarrow \begin{matrix} (\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{2},\mathbf{1}) & (\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2},\mathbf{1}) \\ (\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{2}) & (\mathbf{2},\mathbf{1},-\mathbf{3},\mathbf{1},\mathbf{2}) \\ (\mathbf{1},\mathbf{3},-\mathbf{2},\mathbf{2},\mathbf{1}) & (\mathbf{1},\mathbf{1},-\mathbf{6},\mathbf{2},\mathbf{1}) \end{matrix} \leftrightarrow \begin{matrix} |\mathbf{ur}\{\mathbf{x},\mathbf{y},\mathbf{z}\}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle & |\mathbf{urw}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle \\ |\{\mathbf{u},\mathbf{d}\}\mathbf{l}\{\mathbf{x},\mathbf{y},\mathbf{z}\}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle & |\{\mathbf{u},\mathbf{d}\}\mathbf{lw}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle \\ |\mathbf{dr}\{\mathbf{x},\mathbf{y},\mathbf{z}\}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle & |\mathbf{drw}\{\mathbf{1},\mathbf{2},\mathbf{3}\}\rangle \end{matrix}.$$

The dual basis is assigned to the conjugate spinor as follows

$$\overline{\psi} \leftrightarrow \begin{matrix} (\mathbf{1,1,0,1,2}) & (\mathbf{1,\overline{3},-4,1,2}) \\ (\mathbf{2,1,3,2,1}) & (\mathbf{2,\overline{3},-1,2,1}) \\ (\mathbf{1,1,6,1,2}) & (\mathbf{1,\overline{3},2,1,2}) \end{matrix} \leftrightarrow \begin{matrix} \langle \mathbf{ulw}\{\mathbf{1,2,3}\}| & \langle \mathbf{ul}\{\mathbf{x,y,z}\}\{\mathbf{1,2,3}\}| \\ \langle \{\mathbf{u,d}\}\mathbf{rw}\{\mathbf{1,2,3}\}| & \langle \{\mathbf{u,d}\}\mathbf{r}\{\mathbf{x,y,z}\}\{\mathbf{1,2,3}\}| \\ \langle \mathbf{dlw}\{\mathbf{1,2,3}\}| & \langle \mathbf{dl}\{\mathbf{x,y,z}\}\{\mathbf{1,2,3}\}| \end{matrix} .$$

The fermion part of the Lagrangian is

$$\mathcal{L}_\psi = \overline{\psi} i\gamma^\mu (\partial_\mu + \mathbf{A}_\mu)\psi .$$

Explicitly in terms of the fermion metric, this becomes

$$\mathcal{L}_\psi = \varepsilon(\psi, i\gamma^\mu (\partial_\mu + \mathbf{A}_\mu)\psi).$$

1.4. The Casimir Basis

The appearance of a right analogue to $\mathbf{I}_3$ in the representation of $\mathbf{Y}$ and decomposition of $(\mathbf{4}, Y)$, and the appearance of the baryon number in $\mathbf{Y}$ strongly suggests a more fundamental role should be played by these two operators.

The argument used in the standard model to arrive at the hypercharge spectrum involves a condition to remove a chiral anomaly present in the field theory. As shown by R.A. Bertlmann (1996) {*Anomalies in quantum field theory*. Clarendon Press. Oxford.} the gaussian constraint

$$\mathcal{G}_a = \nabla \cdot \mathbf{D}_\mathbf{a} - \rho_a \approx 0$$

classically satisfies the Poisson bracket relation

$$\{\mathcal{G}_a(x), \mathcal{G}_b(y)\} = f_{ab}^{\ c} \mathcal{G}_c(x)\delta(x,y) ,$$

but, upon quantization, acquires an extra term corresponding to what is known as the "Triangle Anomaly" in perturbation theory

$$\frac{[\mathcal{G}_a(x), \mathcal{G}_b(y)]}{i\hbar} = f_{ab}^{\ c} \mathcal{G}_c(x)\delta(x,y) + \frac{\gamma_5}{24\pi^2} Tr(\{\mathbf{Y}_a, \mathbf{Y}_b\})\varepsilon^{\mu\nu\rho}\partial_\mu \mathbf{A}_\nu \partial_\rho \delta(x,y) .$$

Ultimately, the requirement for removal comes down to the condition that

$$Tr(\gamma_5 \mathbf{Y}_k \{\mathbf{Y}_i, \mathbf{Y}_j\}) = 0 .$$

That is, these cubic combinations of the weights summed over each of the left-hand modes should add up to the corresponding cubic combinations summed over the right-hand modes.

In the absence of the right neutrino and left anti-neutrino sectors (or equivalently, if one assumes that their charges are all 0), this constraint uniquely assigns a generation-invariant charge up to an overall scale. However, everything changes when the extra neutrino sectors are brought in. Then one also finds that the baryon number is allowed. The most general resolution is a linear combination of the baryon number and hypercharge or (equivalently) the "right isospin",

$$X = I_{3R} \equiv \frac{\tau_3}{2} |\mathbf{r}\rangle\langle\mathbf{r}| .$$

Indeed, separating out the right isospin, we may write down the fermion spectrum in the $SU(2)_I \times U(1)_X \times SU(3)_\Lambda \times U(1)_G \times SU(2)_R \times SU(2)_L$ decomposition as

$$\begin{matrix} (\mathbf{1,3,1,3,1,2}) & (\mathbf{1,3,3,1,2,1}) & (\mathbf{1,3,\overline{3},-1,1,2}) & (\mathbf{1,3,1,-3,2,1}) \\ (\mathbf{2,0,1,3,2,1}) & (\mathbf{2,0,3,1,1,2}) & (\mathbf{2,0,\overline{3},-1,2,1}) & (\mathbf{2,0,1,-3,1,2}) \\ (\mathbf{1,-3,1,3,1,2}) & (\mathbf{1,-3,3,1,2,1}) & (\mathbf{1,-3,\overline{3},-1,1,2}) & (\mathbf{1,-3,1,-3,2,1}) \end{matrix} .$$

We then see clearly that there is a separate decomposition for $U(2)_{I,X}$ into

$$\mathbf{4} = (\mathbf{1,3}) \oplus (\mathbf{2,0}) \oplus (\mathbf{1,-3}),$$

and $U(3)_{\Lambda,G}$ into the "fermion cube"

$$\mathbf{8} = (\mathbf{1,3}) \oplus (\mathbf{3,1}) \oplus (\mathbf{\overline{3},-1}) \oplus (\mathbf{1,-3}).$$

Tables for the $U(2)_{I,X}$ weights may then be compiled

$$\begin{array}{cccc} & (\mathbf{1,3}) & (\mathbf{2,0}) & (\mathbf{1,-3}) \\ I^2 & 0 & 3/4 & 0 \\ I_3 & 0 & (1/2 \quad -1/2) & 0 \\ X & 1/2 & 0 & -1/2 \end{array}$$

thus establishing the Casimir invariant and two "spin" operators

$$I^2 + 3X^2 = \frac{3}{4}, \quad X \pm I_3 = \pm\frac{1}{2}.$$

It will turn out that the 3-fold generational degeneracy will be tied to the generators $\mathbf{I}_1, \mathbf{I}_2$, so that the full spectrum will consist of 12 members, rather than just four. This will lead to the Isocolor Lattice, depicted below.

The weights and invariants are consistent with the assignment of $X = I_{R3}$ in a right-handed analogue $SU(2)_{IR}$ of isospin $SU(2)_I$ with the inclusion $SU(2)_I \times SU(2)_{IR} \supset U(2)_{I,X}$. The apparent absence of I_{1R}, I_{2R} could then be explained, at least in part, by assuming the parity violation of isospin is a broken symmetry. The key points of unexplained regularity that lead toward this direction are

(a) the zero mass mode of the electroweak symmetry breaking is also the parity-symmetric mode
(b) the gauge-dependency of the fermion metric, in the absence of overall parity-symmetry, distinguishing it in contrast to the Higgs and gauge metrics, which are both gauge-invariant.

The most significant regularity that would emerge if parity-symmetry is restored at the level of interactions is that the fermion sector of the Lagrangian would factor into a form given by

$$\mathcal{L}_\psi = \overline{((\partial_\mu + \mathbf{A}_\mu)\chi)}((\partial_\mu + \mathbf{A}_\mu)\chi)$$

where the field ψ, itself, is treated as the "curvature" associated with a "fermion gauge potential" χ.

The corresponding tables for the $U(3)_{\Lambda,G}$ weights are

$$\begin{array}{ccccc} & (\mathbf{1,3}) & (\mathbf{3,1}) & (\overline{\mathbf{3}},\mathbf{-1}) & (\mathbf{1,-3}) \\ \Lambda^2 & 0 & 4/3 & 4/3 & 0 \\ \begin{pmatrix}\Lambda_3 \\ \Lambda_8\end{pmatrix} & \begin{pmatrix}0 \\ 0\end{pmatrix} & \begin{pmatrix} 1/2 & 0 & -1/2 \\ \sqrt{1/12} & -\sqrt{1/3} & \sqrt{1/12}\end{pmatrix} & \begin{pmatrix} 1/2 & 0 & -1/2 \\ -\sqrt{1/12} & \sqrt{1/3} & -\sqrt{1/12}\end{pmatrix} & \begin{pmatrix}0 \\ 0\end{pmatrix} \\ G & 1/2 & 1/6 & -1/6 & -1/2 \end{array},$$

which establishes a second Casimir invariant and three more "spin" operators

$$\Lambda^2 + 6G^2 = \frac{3}{2}, \quad G - \frac{\Lambda_8}{\sqrt{3}} \pm \Lambda_3 = \pm\frac{1}{2}, \quad G + \frac{2\Lambda_8}{\sqrt{3}} = \pm\frac{1}{2}.$$

This is consistent with the assignment of $G = 2\Lambda_{15}/\sqrt{6}$ in a "hypercolor" group $SU(4) \supseteq U(3)_{\Lambda,G}$.

The overall decomposition mixes with the local symmetry group $SO(3,1)$, the mixing of the two captured the $U(2)_{I,X} \times SO(3,1)$ decompositions

$$\mathbf{4} = (\mathbf{1,3,1,2}) \oplus (\mathbf{2,0,2,1}) \oplus (\mathbf{1,-3,1,2}), \quad \overline{\mathbf{4}} = (\mathbf{1,3,2,1}) \oplus (\mathbf{2,0,1,2}) \oplus (\mathbf{1,-3,2,1}).$$

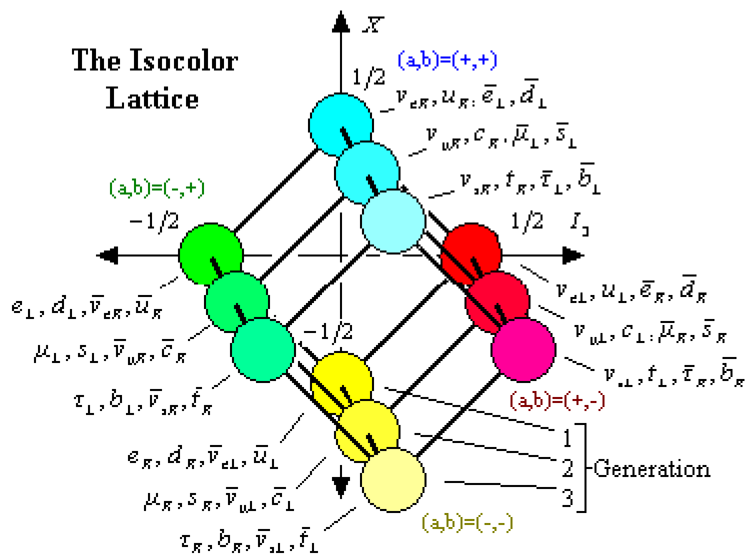
The $U(3)_{\Lambda,G}$ quadruplets $\mathbf{4} = (\mathbf{3,1}) \oplus (\mathbf{1,-3})$, $\overline{\mathbf{4}} = (\mathbf{1,3}) \oplus (\overline{\mathbf{3}},\mathbf{-1})$ would correspond to the fundamental quadruplets within $SU(4) \supseteq U(3)_{\Lambda,G}$; similarly, the $U(2)_{I,X} \times SO(3,1)$ quadriplets would correspond within $(SU(2)_I \times SU(2)_{IR}) \times (SU(2)_R \times SU(2)_L)$ respectively to $\mathbf{4} = (\mathbf{1,2,1,2}) \oplus (\mathbf{2,1,2,1})$ and $\overline{\mathbf{4}} = (\mathbf{1,2,2,1}) \oplus (\mathbf{2,1,1,2})$. Parity is intertwined with the internal symmetry space leading to an effective overall $SU(4)_{\Lambda,G} \times (SU(2)_I \times SU(2)_{IR} \times SU(2)_R \times SU(2)_L)$ decomposition of the fermion sector into effective right and left subspaces $(\mathbf{4,4}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{4}})$.

As a consequence of this, parity is already encapsulated by the 5 “spin” operators. Therefore, to specify the $SO(3,1)$ subspace, instead of using the product basis $\{|\mathbf{l}\rangle,|\mathbf{r}\rangle\}\otimes\{|+\rangle,|-\rangle\}$, one needs only the latter subbasis $\{|+\rangle,|-\rangle\}$. Alternatively, one of the “spin” operators may be eliminated in favor of the parity operator γ_5.

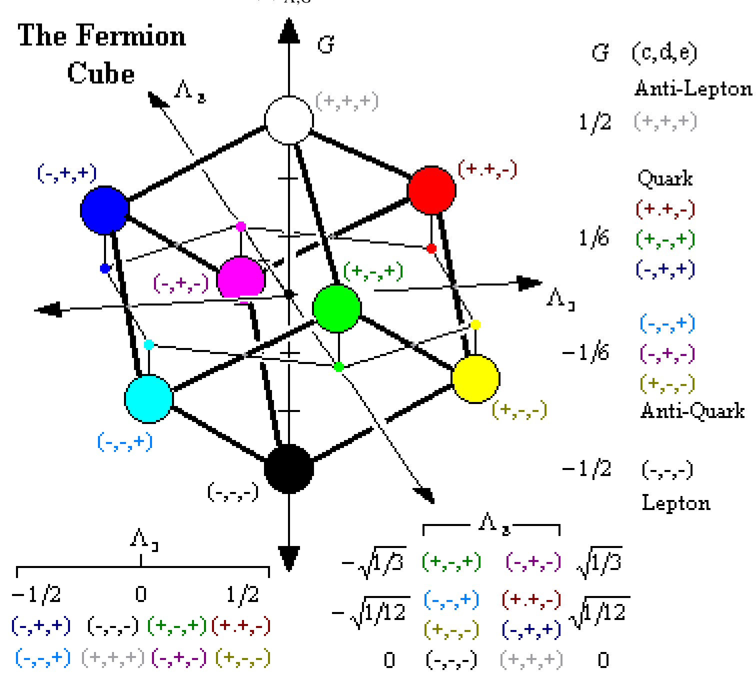
The remaining operators are

$$a \equiv X + I_3, \quad b \equiv X - I_3, \quad c \equiv G - \frac{\Lambda_8}{\sqrt{3}} + \Lambda_3, \quad d \equiv G + \frac{2\Lambda_8}{\sqrt{3}}, \quad e \equiv G - \frac{\Lambda_8}{\sqrt{3}} - \Lambda_3 \ .$$

The 32 combinations of the $\pm 1/2$ values of these operators will produce the charge eigenstates of each generation. As discussed later, the mass-energy eigenstates of a given isocolor, will be mixtures formed of the members of each isocolor triplet. This leads to a factoring of the fermion space into the *Isocolor Lattice*, associated with $U(2)_{I,X}$,



and the *Fermion Cube*, associated with $U(3)_{\Lambda,G}$,



The assignment of the units is given in the following table

Unit	X	I_3	Λ_3	Λ_8	G
a	$1/2$	$1/2$	0	0	0
b	$1/2$	$-1/2$	0	0	0
c	0	0	$1/2$	$-1/\sqrt{12}$	$1/3$
d	0	0	0	$1/\sqrt{3}$	$1/3$
e	0	0	$-1/2$	$-1/\sqrt{12}$	$1/3$

The triplet c,d,e is an instance of the $\mathbf{\overline{3}}$ representation of $SU(3)_\Lambda$ with $G=1/3$ and may be identified respectively with the colors amber, magenta and cyan. The doublet a,b is an instance of the $\mathbf{2}$ representation of $SU(2)_I$ with $X=1/2$. This will turn out to be the characteristic of the Higgs doublet which, therefore, may be identified as the fundamental charges corresponding to these units.

Parity is related to the other "spin" operators by the relation

$$\gamma_5 = \operatorname{sgn}(abcde)\ ,$$

and the one "spin" operator that changes with parity is b, which effectively represents the parity operator of the combination of the local spacetime frame and internal gauge bundle.

Therefore, we will replace the b operator by parity $p=\operatorname{sgn}(\gamma_5)$ and write the basis as $|acdeps\rangle$ with $a,c,d,e,p,s\in\{+,-\}$. This suffices to either define the 64 *real* components of the 32-component fermion spinor, or otherwise 64 complex components with a conjugacy relation. We adopt the latter approach, expressing a fermion spinor in the form

$$\psi \equiv \sum_{a,b,c,d,e,s} \psi_{acdeps} |acdeps\rangle\ ,$$

with the conjugacy operator $\mathcal{K}\psi$ and charge conjugacy operator $\mathcal{C}\psi$ relating the components. We will adopt the following conventions

$$\mathcal{C}\psi \equiv \sum_{a,b,c,d,e,s} \psi_{bcdeps} |a'c'd'e'p's'\rangle\ ,$$

and

$$\mathcal{K}\psi \equiv \sum_{a,b,c,d,e,s} \psi_{bcdeps}{}^{*} |acdeps\rangle = \sum_{a,b,c,d,e,s} ps\psi_{bcdeps} |a'c'd'e'p's'\rangle\ ,$$

where we use the signs of the respective bits as factors, e.g. $ps=\operatorname{sgn}(ps)$ and the prime to denote sign-reversal. That is, $(\psi_{acdeps})^* = ps\psi_{(-a)(-c)(-d)(-e)(-p)(-s)}$. The characteristics of the various states are given in the following table

Matter	$cde<0$,
Anti-Matter	$cde>0$,
Leptonic	$c=d=e$,
Baryonic	$c=-d$, $d=-e$ or $e=-c$,
Right	$p=\operatorname{sgn}(abcde)>0$,
Left	$p=\operatorname{sgn}(abcde)<0$.

The spinor components for each flavor are arranged as

$$\psi|acde\rangle = \begin{pmatrix} \psi|a+cde+\rangle \\ \psi|a+cde-\rangle \\ \psi|a-cde+\rangle \\ \psi|a-cde-\rangle \end{pmatrix},\quad \overline{\psi}^T|acde\rangle = \begin{pmatrix} \psi|(-a)+(-c)(-d)(-e)-\rangle \\ -\psi|(-a)+(-c)(-d)(-e)+\rangle \\ -\psi|(-a)-(-c)(-d)(-e)-\rangle \\ \psi|(-a)-(-c)(-d)(-e)+\rangle \end{pmatrix}.$$

The effect of the Dirac matrices on the basis is given by the following

$$\gamma_5|acdeps\rangle = p|acdeps\rangle,\quad \gamma^0|acdeps\rangle = |acdep's\rangle,$$

$$\gamma^1|acdeps\rangle = p|acdep's'\rangle,\quad \gamma^2|acdeps\rangle = ips|acdep's'\rangle,\quad \gamma^3|acdeps\rangle = ps|acdep's\rangle.$$

The effect of parity and time-reversal are given by

$$(\mathcal{P}\psi)(\mathbf{r},t)\left|acdeps\right\rangle=\psi(-\mathbf{r},t)\left|acdep's\right\rangle,\quad(\mathfrak{I}\psi)(\mathbf{r},t)\left|acdeps\right\rangle=ip\psi(\mathbf{r},-t)\left|a'c'd'e'p's\right\rangle.$$

In terms of the Casimir basis, the gauge generators take on a more interesting and revealing form. The $U(1)_Y$ generator mixes the actions of the right isospin and baryon-lepton number and has the following action

$$\mathbf{Y}\left|abcde\right\rangle=\left(\frac{a+b}{2}+\frac{c+d+e}{3}\right)\left|abcde\right\rangle.$$

The $SU(2)_I$ sector only acts on the a,b indices, with the following results

$$\mathbf{I}_1\left|abcde\right\rangle=\left|\frac{a-b}{2}\right|\left|bacde\right\rangle,\quad\mathbf{I}_2\left|abcde\right\rangle=i\frac{a-b}{2}\left|bacde\right\rangle,\quad\mathbf{I}_3\left|abcde\right\rangle=\frac{a-b}{2}\left|abcde\right\rangle.$$

Finally, the $SU(3)_\Lambda$ sector only acts on the c,d,e indices with the following results

$$\mathbf{\Lambda}_1\left|abcde\right\rangle=\left|\frac{c-e}{2}\right|\left|abedc\right\rangle,\quad\mathbf{\Lambda}_2\left|abcde\right\rangle=i\frac{c-e}{2}\left|abedc\right\rangle,\quad\mathbf{\Lambda}_3\left|abcde\right\rangle=\frac{c-e}{2}\left|abcde\right\rangle,$$

$$\begin{array}{ll}\mathbf{\Lambda}_4\left|abcde\right\rangle=\left|\frac{c-d}{2}\right|\left|abdce\right\rangle,\quad\mathbf{\Lambda}_5\left|abcde\right\rangle=i\frac{c-d}{2}\left|abdce\right\rangle, & \\ & \mathbf{\Lambda}_8\left|abcde\right\rangle=\frac{2d-c-e}{\sqrt{12}}\left|abcde\right\rangle. \\ \mathbf{\Lambda}_6\left|abcde\right\rangle=\left|\frac{e-d}{2}\right|\left|abced\right\rangle,\quad\mathbf{\Lambda}_7\left|abcde\right\rangle=i\frac{e-d}{2}\left|abced\right\rangle, & \end{array}$$

The generators $\mathbf{I}_1,\mathbf{I}_2$ perform exchanges $a\leftrightarrow b$, when the two qubits differ; while the generators $\mathbf{\Lambda}_1,\mathbf{\Lambda}_2$; $\mathbf{\Lambda}_4,\mathbf{\Lambda}_5$ and $\mathbf{\Lambda}_6,\mathbf{\Lambda}_7$ respective perform exchanges on $c\leftrightarrow e$, $c\leftrightarrow d$ and $e\leftrightarrow d$, when the qubits in the respective pairs differ. The remaining generators $\mathbf{Y},\mathbf{I}_3,\mathbf{\Lambda}_3,\mathbf{\Lambda}_8$ along with

$$G\left|abcde\right\rangle=\frac{c+d+e}{3}\left|abcde\right\rangle$$

produce the eigenvalue spectrum of the 5 qubits for the basis.

1.5. Higgs Sector

For the description in this and the remaining sections, we will use the hypercolor basis.

The Higgs is found in the following $SU(2)_I\times SU(3)_\Lambda\times U(1)_Y$ sector: $(\mathbf{2},\mathbf{1},\mathbf{3})$. The basis $\{\left|\mathbf{u}\right\rangle,\left|\mathbf{d}\right\rangle\}$ will therefore be used for the space, with the corresponding identity operator I_P defined as before, and with the following decomposition

$$\varphi=\varphi^+\left|\mathbf{u}\right\rangle+\varphi^0\left|\mathbf{d}\right\rangle,$$

as well as

$$\tilde{\varphi}=i\tau_2\varphi^*=\varphi^{0^*}\left|\mathbf{u}\right\rangle-\varphi^-\left|\mathbf{d}\right\rangle.$$

The action of the gauge group generators on this sector is thus

$$\mathbf{I}_i=\frac{\tau_i}{2},\quad\mathbf{Y}=\frac{I_P}{2},\quad\mathbf{\Lambda}_a=0.$$

The scalar part of the Lagrangian is

$$\mathcal{L}_\varphi=g^{\mu\nu}\left(\left(\partial_\mu+\mathbf{A}_\mu\right)\varphi\right)^+\left(\partial_\nu+\mathbf{A}_\nu\right)\varphi-\lambda\left(\varphi^+\varphi-\frac{v^2}{2}\right)^2,\quad(\lambda>0).$$

The field may be decomposed into polar form by writing

$$\varphi^+\equiv\frac{-\varphi_2-i\varphi_1}{\sqrt{2}},\quad\varphi^0\equiv\frac{\varphi_0+i\varphi_3}{\sqrt{2}}.$$

Then

$$\varphi=\Phi\left|\mathbf{d}\right\rangle,\quad\tilde{\varphi}=\Phi\left|\mathbf{u}\right\rangle,\quad\Phi=\tilde{\varphi}\left\langle\mathbf{u}\right|+\varphi\left\langle\mathbf{d}\right|,$$

where

$$\Phi = \frac{\varphi_0 - i(\varphi_1\tau_1 + \varphi_2\tau_2 + \varphi_3\tau_3)}{\sqrt{2}} = \frac{H+v}{\sqrt{2}} U_\Phi, \quad U_\Phi = \exp\left(\sum_a \lambda^a \mathbf{Y}_a\right) = \exp\left(-\frac{i}{2}\left(\lambda^1\tau_1 + \lambda^2\tau_2 + \lambda^3\tau_3\right)\right).$$

The effect of the specific representation is to embody the scalar metric

$$\chi(\varphi_1, \varphi_2) \equiv \varphi_1^{\,*}\varphi_2 = \chi^+ \Phi_1^{\,*} \Phi_2 \chi \quad \chi \equiv |\mathbf{d}\rangle .$$

The vector χ is therefore the cyclic vector generating the representation $(\mathbf{2,1,3})$. The Lagrangian may therefore be written as

$$\mathcal{L}_\varphi = g^{\mu\nu}\chi((\partial_\mu + \mathbf{A}_\mu)\varphi, (\partial_\nu + \mathbf{A}_\nu)\varphi) - \lambda\left(\chi(\varphi,\varphi) - \frac{v^2}{2}\right)^2 .$$

An important property implicit in this notation is that the metric is gauge invariant,

$$\chi(U\varphi_1, U\varphi_2) = \chi(\varphi_1, \varphi_2).$$

The comments made in relation to the fermion and gauge metrics apply here. The classical field theoretic variant of the scalar field metric will be constant, but under renormalization in quantum field theory, it becomes scale dependent, effectively making the metric a function of the distance from point-like sources that tends asymptotically toward a constant, away from sources.

This translates into the renormalization factors associated with Higgs scaling.

1.6. Yukawa Sector

In the theoretical literature, this is the least well-developed part of the Standard Model, in terms of writing it as an instance of a general form. In general, the interaction between the fermions and scalar field is assumed to be given by a Lagrangian trilinear coupling of the form

$$\mathcal{L}_G = -\overline{\psi} G(\varphi)\psi .$$

In order to preserve gauge invariance and for the Lagrangian to remain Hermitean, the coupling $G(\varphi)$ must satisfy the following conditions

$$(\gamma^0 G(\varphi))^+ = \gamma^0 G(\varphi), \quad U^+\gamma^0 G(U\varphi)U = \gamma^0 G(\varphi) ,$$

under a unitary gauge transformation

$$\mathbf{A}_\mu \to U\mathbf{A}_\mu U^+ + U\partial_\mu U^+, \quad \varphi \to U\varphi, \quad \psi \to U\psi, \quad \overline{\psi} \to \overline{\psi}\gamma^0 U^+ \gamma^0 .$$

Writing the Higgs in polar form, we find that

$$\gamma^0 G(\varphi) = \gamma^0 G\left(U_\Phi \frac{H+v}{\sqrt{2}}\chi\right) = \frac{H+v}{\sqrt{2}} U_{\Phi\psi}\gamma^0 G(\chi) U_{\Phi\psi}^{\;+} .$$

Decomposing with respect to the $\{|\mathbf{r}\rangle, |\mathbf{l}\rangle\}$ basis

$$G(\chi) = G_{rr}|\mathbf{r}\rangle\langle\mathbf{r}| + G_{rl}|\mathbf{r}\rangle\langle\mathbf{l}| + G_{lr}|\mathbf{l}\rangle\langle\mathbf{r}| + G_{ll}|\mathbf{l}\rangle\langle\mathbf{l}| ,$$

$$U_{\Phi\psi} = |\mathbf{r}\rangle\langle\mathbf{r}| + U_\Phi |\mathbf{l}\rangle\langle\mathbf{l}|$$

we find

$$G(\varphi) = \frac{H+v}{\sqrt{2}}\gamma^0 U_{\Phi\psi}\gamma^0 G(\chi) U_{\Phi\psi}^{\;+} = \frac{H+v}{\sqrt{2}}\left(U_\Phi G_{rr}|\mathbf{r}\rangle\langle\mathbf{r}| + U_\Phi G_{rl} U_\Phi^{\;+}|\mathbf{r}\rangle\langle\mathbf{l}| + G_{lr}|\mathbf{l}\rangle\langle\mathbf{r}| + G_{ll}U_\Phi^{\;+}|\mathbf{l}\rangle\langle\mathbf{l}|\right).$$

By assumption, this is a trilinear coupling, which means the quadratic and quartic terms are not present. Therefore $G_{rl} = 0 = G_{lr}$. Furthermore, the Hermiticity condition implies that

$$G \equiv G_{rr} = G_{ll}^{\;+} .$$

Therefore, we may write

$$G(\varphi) = \Phi G|\mathbf{r}\rangle\langle\mathbf{r}| + G^+\Phi^+|\mathbf{l}\rangle\langle\mathbf{l}| .$$

Thus, the interaction Lagrangian is a Yukawa term

$$\mathcal{L}_G = -\overline{\psi}G(\varphi)\psi = -\overline{\psi}\left(\Phi G|\mathbf{r}\rangle\langle\mathbf{r}| + G^+\Phi^+|\mathbf{l}\rangle\langle\mathbf{l}|\right)\psi$$

with a coupling whose decomposition with respect to the 3-fold degeneracy is explicitly written as

$$G \equiv \sum_{m,n=1}^{3} G^{mn} |\mathbf{m}\rangle\langle\mathbf{n}|,$$

$$G^{mn} \equiv \sum_{m,n=1}^{F} |\mathbf{w}\rangle\langle\mathbf{w}| \otimes \left(N^{mn}|\mathbf{u}\rangle\langle\mathbf{u}| + E^{mn}|\mathbf{d}\rangle\langle\mathbf{d}|\right) + \left(|\mathbf{x}\rangle\langle\mathbf{x}| + |\mathbf{y}\rangle\langle\mathbf{y}| + |\mathbf{z}\rangle\langle\mathbf{z}|\right) \otimes \left(U^{mn}|\mathbf{u}\rangle\langle\mathbf{u}| + D^{mn}|\mathbf{d}\rangle\langle\mathbf{d}|\right).$$

This is the most general decomposition with respect to the remaining bases that has invariance under the $SU(3)_\Lambda$ sector.

By supposition,

$$N^{mn} = 0.$$

The term has been retained here, along with the right neutrino sector, for the sake of generality. In extended versions of the Standard Model, a neutrino mass has to be incorporated in some fashion. The simplest assumption is that the right-handed neutrino is, indeed, there, but simply unobservable because of its neutrality. It would interact with the Higgs and through gravity, but the Higgs is still unseen and the virtual masslessness of the neutrino would mean that its gravity would be difficult to see, as well.

1.7. Yang-Mills-Higgs Lagrangians

Combining these results, we find that the Standard Model is an instance of the general Yang-Mills-Higgs Lagrangian, which may be defined by

$$\mathfrak{L} = \varepsilon\left(\psi, \left(i\gamma^\mu\left(\partial_\mu + \mathbf{A}_\mu\right) - G(\varphi)\right)\psi\right) - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} k(\mathbf{F}_{\mu\nu}, \mathbf{F}_{\rho\sigma}) + \chi\left(\left(\partial_\mu + \mathbf{A}_\mu\right)\varphi, \left(\partial_\nu + \mathbf{A}_\nu\right)\varphi\right) - \lambda\left(\chi(\varphi,\varphi) - \frac{v^2}{2}\right)^2,$$

where

$$\mathbf{F}_{\mu\nu} \equiv \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + \left[\mathbf{A}_\mu, \mathbf{A}_\nu\right],$$

such that

$$\chi(U\varphi_1, U\varphi_2) = \chi(\varphi_1, \varphi_2),$$
$$(\gamma^0 G(\varphi))^+ = \gamma^0 G(\varphi), \quad U^+\gamma^0 G(U\varphi)U = \gamma^0 G(\varphi).$$

One important qualifier is worth noting here. In virtue of the neutrality of the right-neutrino sector, a more general coupling is allowed – thus ultimately leading to variants of the *See Saw* mechanism. Specifically, the action of the gauge group on the sector

$$|\mathbf{w}\rangle\langle\mathbf{w}| \otimes |\mathbf{l}\rangle\langle\mathbf{r}| \otimes |\mathbf{u}\rangle\langle\mathbf{u}| \to \overline{U}\left(|\mathbf{w}\rangle\langle\mathbf{w}| \otimes |\mathbf{l}\rangle\langle\mathbf{r}| \otimes |\mathbf{u}\rangle\langle\mathbf{u}|\right)U = |\mathbf{w}\rangle\langle\mathbf{w}| \otimes |\mathbf{l}\rangle\langle\mathbf{r}| \otimes |\mathbf{u}\rangle\langle\mathbf{u}|$$

is trivial, so that a mass term of the form

$$-\overline{\psi}\left(m_R |\mathbf{w}\rangle\langle\mathbf{w}| \otimes |\mathbf{l}\rangle\langle\mathbf{r}| \otimes |\mathbf{u}\rangle\langle\mathbf{u}|\right)\psi$$

may be inserted into the Lagrangian without violating gauge invariance. The Yukawa coupling will ultimately endow the both the left and right handed components with the same mass m_ν, thereby leading to a total neutrino mass matrix of the form

$$|\mathbf{w}\rangle\langle\mathbf{w}| \otimes |\mathbf{u}\rangle\langle\mathbf{u}| \otimes \left(m_\nu\left(|\mathbf{l}\rangle\langle\mathbf{l}| + |\mathbf{r}\rangle\langle\mathbf{r}|\right) + m_R |\mathbf{l}\rangle\langle\mathbf{r}|\right).$$

The nature of the neutrino spinor can then span the gap between pure Dirac to pure Majorana, and everywhere in between, depending on the relative size of m_R.

2. Breakdown of the Vacuum

The Standard Model hypothesizes that there is no fundamental mass. Instead, it arises through interaction with a universal scalar energy field, called the Higgs. In effect, the Higgs renders the vacuum as a dielectric medium, which impedes some of the components of the electroweak force. The only remaining component that the vacuum is transparent with respect to is the electromagnetic force, which is a combination of the W and B bosons. The combination is the only one that is parity-symmetric. The vacuum is not transparent with respect to the parity-assymmetric components of the electroweak force. The corresponding bosons therefore have a limited range which (via Yukawa's mass-range correspondence) effectively translates into a large mass. The photon remains massless. The resulting field equations (classically) are the Maxwell-

Proca equations. In effect, the massless components of the Higgs become the additional components of the respective Maxwell-Proca fields.

For the fermions, the effect of the Higgs is to alternate the fermion between left and right handed modes. The effective zig-zagging is precisely that which characterizes the Zitterbewegung of a massive Dirac particle. In effect, the fermion is traveling at light speed, but in such a jagged path because of this rapid alternation that its average motion is that of a massive particle. The strength of the particle's interaction with the Higgs determines its mass.

It is of interest to note that the hypothesis that the vacuum behaves as a dielectric medium not only originates with Maxwell, but is a central thesis of his entire treatment of classical electromagnetism. But he even went further and briefly discussed what, in modern language, are known as Abelian Yang-Mills field and the notion of a non-trivial electrogravitational unification via a mixing angle.

The potential

$$V(\varphi) = \lambda\left(\varphi^{+}\varphi - \frac{v^2}{2}\right)^2$$

has a minimum where

$$\varphi^{+}\varphi = \frac{v^2}{2}.$$

The Higgs has already been written in polar form

$$\varphi = U_{\Phi}\,\frac{H+v}{\sqrt{2}}\,\chi$$

This eliminates the three degrees of asymmetry of the vacuum – the Goldstone bosons, leaving behind the one remaining degree of freedom for the Higgs scalar field.

The minimum setting of the Higgs is only determined up to an overall gauge. Each setting defines a different vacuum state. It is assumed that the gauge degrees of freedom are defined such that for the vacuum state that defines this world,

$$\langle 0|\varphi|0\rangle = \frac{v\chi}{\sqrt{2}}.$$

As a consequence of this transformation, there will emerge 3 non-zero mass eigenmodes in the electroweak part of the boson spectrum. When massless, a boson has only 2 degrees of freedom; but in a massive state they have a third degree. The degrees of asymmetry become the respective 3rd degrees of freedom. The photon, however, remains massless with only its 2 helicity modes, while the Higgs remains unattached as a scalar field.

3. Mass Eigenstates

3.1. Boson Mass

Substituting the Higgs vacuum expectation $\langle 0|\varphi|0\rangle = v\chi/\sqrt{2}$ into the scalar part of the Lagrangian reveals the emergence of a mass matrix for the bosons,

$$g^{\mu\nu}\chi\big((\partial_\mu + \mathbf{A}_\mu)\varphi, (\partial_\nu + \mathbf{A}_\nu)\varphi\big) = \frac{v^2}{2}\, g^{\mu\nu}\chi(\mathbf{A}_\mu, \mathbf{A}_\nu) = \sum_{a,b} \mu_{ab}\, g^{\mu\nu} A_\mu^a A_\nu^b\ ,$$

where we write the boson field collectively as

$$\mathbf{A}_\mu = \sum_a A_\mu^a \mathbf{Y}_a\ ,$$

and define

$$\mu_{ab} = \mu(\mathbf{Y}_a, \mathbf{Y}_b) \equiv \frac{v^2}{2}\,\chi(\mathbf{Y}_a, \mathbf{Y}_b).$$

This is the square of the mass matrix. The only non-zero components are those associated with the $U(2)_{I,Y}$ electroweak sector, where we find that

$$\mu(g\mathbf{I}_i, g\mathbf{I}_j) = \left(\frac{vg}{2}\right)^2 \chi(\tau_i, \tau_j) = \delta_{ij}\left(\frac{vg}{2}\right)^2,$$
$$\mu(g'\mathbf{Y}, g\mathbf{I}_j) = \left(\frac{v}{2}\right)^2 gg' \cdot \chi(\tau_j, 1) = -\delta_{3j}\left(\frac{v}{2}\right)^2 gg',$$
$$\mu(g'\mathbf{Y}, g'\mathbf{Y}) = \left(\frac{vg'}{2}\right)^2 \chi(1,1) = \left(\frac{vg'}{2}\right)^2.$$

This leads to the following decomposition

$$\frac{v^2}{2} g^{\mu\nu}\chi(\mathbf{A}_\mu, \mathbf{A}_\nu) = \frac{1}{2} g^{\mu\nu}\left(\left(\frac{vg}{2}\right)^2 \left(W_\mu^1 W_\nu^1 + W_\mu^2 W_\nu^2 + W_\mu^3 W_\nu^3\right) - \frac{v^2 gg'}{2} W_\mu^i B_\nu + \left(\frac{vg'}{2}\right)^2 B_\mu B_\nu\right) \cdot$$
$$= g^{\mu\nu}\left(\left(\frac{vg}{2}\right)^2 \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}} \frac{W_\nu^1 + iW_\nu^2}{\sqrt{2}} + \left(\frac{vg}{2} W_\mu^3 - \frac{vg'}{2} B_\mu\right)\left(\frac{vg}{2} W_\nu^3 - \frac{vg'}{2} B_\nu\right)\right)$$

From this, we find the respective mass eigenstates and the associated eigenvalues,

$$W_\mu^\pm \equiv \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}}, \quad Z_\mu \equiv \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}}, \quad A_\mu \equiv \frac{gB_\mu + g'W_\mu^3}{\sqrt{g^2 + g'^2}},$$
$$M_W \equiv \frac{vg}{2}, \qquad M_Z = \frac{v\sqrt{g^2 + g'^2}}{2}, \qquad M_\gamma = 0.$$

The effective Lagrangian becomes

$$g^{\mu\nu}\left(M_W{}^2 W_\mu^+ W_\nu^- + \frac{M_Z{}^2}{2} Z_\mu Z_\nu\right).$$

If the Higgs is retained, written in polar form, this term becomes

$$g^{\mu\nu}\left(M_W{}^2 W_\mu^+ W_\nu^- + \frac{M_Z{}^2}{2} Z_\mu Z_\nu\right)\left(1 + \frac{H}{v}\right)^2.$$

The weak mixing angle, θ_W, is defined as the angle between 0 and 90 degrees for which $\tan\theta_W = g'/g$. Then the two neutral boson fields and the mass ratio of the two mass eigenvalues may be written

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W_\mu^3, \\ Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \end{aligned} \quad M_W = M_Z \cos\theta_W.$$

The constant g is ultimately related to the Fermi constant $G_F = 1.16639(2)\times 10^{-5}\,\mathrm{GeV}^{-2}$ by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W{}^2}.$$

The expression for the photon field A_μ yields a coupling $e = g\sin\theta_W = g'\cos\theta_W$ that is parity independent. Through this, one finds

$$M_W = \frac{1}{\sin\theta_W}\sqrt{\frac{\pi\alpha}{2G_F}},$$

in terms of the fine structure constant $\alpha \approx 1/137.036$. From this, one gets – as a first order estimate – the values

$$\sin^2\theta_W \approx 0.23, \quad v \cong 246\mathrm{GeV}, \quad M_W \cong 78\mathrm{GeV}, \quad M_Z \cong 89\mathrm{GeV}.$$

Higher order corrections refine these to values much more closer to their experimental values,

$$. \; M_W \cong 83\mathrm{GeV}, \quad M_Z \cong 91\mathrm{GeV}.$$

Of particular interest is that the trace of the first-order boson mass matrix is very nearly equal to the vacuum expectation value of the Higgs,

$$M_Z + 2M_W \approx v\,.$$

3.2. Higgs Mass

The Higgs self-potential also leads to a single massive eigenstate, along with the 3 massless eigenstates that get absorbed into the massive electroweak fields. The kinetic and potential parts of the Higgs sector become

$$g^{\mu\nu}\left(\partial_\mu \varphi\right)^+ \partial_\nu \varphi - \lambda\left(\varphi^+\varphi - \frac{v^2}{2}\right)^2 = g^{\mu\nu}\chi^+\partial_\mu \frac{v+H}{\sqrt{2}}\partial_\nu \frac{v+H}{\sqrt{2}}\chi - \lambda\left(\chi^+ \frac{(v+H)^2}{2}\chi - \frac{v^2}{2}\right)^2 .$$

The first term yields

$$g^{\mu\nu}\chi^+\partial_\mu \frac{v+H}{\sqrt{2}}\partial_\nu \frac{v+H}{\sqrt{2}}\chi = \frac{1}{2}g^{\mu\nu}\partial_\mu H \partial_\nu H\,,$$

as expected of a scalar field. The second term yields

$$-\lambda\left(\chi^+ \frac{(v+H)^2}{2}\chi - \frac{v^2}{2}\right)^2 = -\lambda\left(\frac{(v+H)^2 - v^2}{2}\right)^2 = -\lambda\left(vH + \frac{H^2}{2}\right)^2 = -\lambda v^2 H^2\left(1+\frac{H}{2v}\right)^2,$$

resulting in a total

$$\frac{1}{2}\left(g^{\mu\nu}\partial_\mu H\partial_\nu H - \left(m_H H\right)^2\left(1+\frac{H}{2v}\right)^2\right)$$

involving the appearance of a Higgs mass $m_H = \sqrt{2\lambda}v$. The lower bound $m_H > 60\mathrm{GeV}$ is currently known (*it may be higher at the time of writing*).

3.3. Fermion Mass

In the following, we will use the projection operators

$$L \equiv |\mathbf{w}\rangle\langle\mathbf{w}|, \quad B \equiv |\mathbf{x}\rangle\langle\mathbf{x}| + |\mathbf{y}\rangle\langle\mathbf{y}| + |\mathbf{z}\rangle\langle\mathbf{z}|, \quad U \equiv |\mathbf{u}\rangle\langle\mathbf{u}|, \quad D \equiv |\mathbf{d}\rangle\langle\mathbf{d}|\,.$$

The mass terms come out of the Yukawa sector, since this is the place where the Higgs mediates between the left-right Zitterbewegung of the fermion fields. Expanding the Higgs field we may write

$$\Phi \equiv U_\Phi \frac{v+H}{\sqrt{2}}\,.$$

The Yukawa term may then be reduced to

$$\mathcal{L}_G = -\overline{\psi}\left(\Phi G|\mathbf{r}\rangle\langle\mathbf{r}| + G^+\Phi^+|\mathbf{l}\rangle\langle\mathbf{l}|\right)\psi = -\overline{\psi}\left(m_\psi|\mathbf{r}\rangle\langle\mathbf{r}| + m_\psi{}^+|\mathbf{l}\rangle\langle\mathbf{l}|\right)\left(1+\frac{H}{v}\right)\psi$$

where

$$m_\psi \equiv \frac{v}{\sqrt{2}}U_\Phi G = \sum_{m,n=1}^{3} m_\psi{}^{mn}|\mathbf{m}\rangle\langle\mathbf{n}|\,,$$

Defining the projection operators

$$e \equiv |\mathbf{1}\rangle\langle\mathbf{1}|, \quad \mu \equiv |\mathbf{2}\rangle\langle\mathbf{2}|, \quad \tau \equiv |\mathbf{3}\rangle\langle\mathbf{3}|$$

the fermion mass matrix m_ψ may be diagonalized separately over its respective sectors

$$m_\psi = L_u{}^+ m_U R_u UB + L_d{}^+ m_D R_d DB + L_\nu{}^+ m_N R_\nu UL + L_e{}^+ m_E R_e DL,$$

in terms of 3×3 matrices unitary over generational space

$$L_u,\quad L_d,\quad L_\nu,\quad L_e,\quad R_u,\quad R_d,\quad R_\nu,\quad R_e\ .$$

and generation-diagonalized matrices

$$m_U = m_u e + m_c\mu + m_t\tau, \quad m_D = m_d e + m_s\mu + m_b\tau, \quad m_N = m_{\nu e}e + m_{\nu\mu}\mu + m_{\nu\tau}\tau, \quad m_E = m_e e + m_\mu\mu + m_\tau\tau$$

which are expressed in terms of the mass eigenvalues, with typical estimates given by

$$
\begin{aligned}
&m_u \approx 5.6 \pm 1.1\text{MeV}, \quad m_c \approx 1.35 \pm 0.05\text{GeV}, \quad m_s \approx 174 \pm 16\text{GeV},\\
&m_d \approx 9.9 \pm 1.1\text{MeV}, \quad m_s \approx 199 \pm 33\text{MeV}, \quad m_b \approx 4.7\text{GeV},\\
&m_{ve} \approx 0, \quad m_{v\mu} \approx 0, \quad m_{v\tau} \approx 0,\\
&m_e \approx .511\text{keV}, \quad m_\mu \approx 105\text{MeV}, \quad m_\tau \approx 2\text{GeV}.
\end{aligned}
$$

Since the Standard Model (originally) assumed that the right-neutrino sector was inert or non-existent, the neutrino mass eigenvalues were all assumed to be zero,

$$m_{ve} = m_{v\mu} = m_{v\tau} = 0 .$$

The other mass eigenvalues are free parameters. However, it is of interest to note that the trace of the lepton mass matrix satisfies an approximate identity similar to that gauge boson mass matrix, but with an interesting variation

$$Tr(m_L) = m_e + m_\mu + m_\tau \approx \alpha v ,$$

where α is the fine structure constant. It is approximately 1/137 of the Higgs vacuum expectation value.

The decomposition leads to the mass eigenstates of the fermions

$$\psi \to \psi_M \equiv V\psi,$$

where

$$V \equiv (L_u UB + L_d DB + L_v UL + L_e DL)|\mathbf{l}\rangle\langle\mathbf{l}| + (R_u UB + R_d DB + R_v UL + R_e DL)|\mathbf{r}\rangle\langle\mathbf{r}|$$

This leads to the expression of the Yukawa in the mass eigenspace

$$\mathfrak{L}_6 = -\overline{\psi_M}(m_U UB + m_D DB + m_N UL + m_E DL)\left(1 + \frac{H}{v}\right)\psi_M .$$

3.4. The Gauge Interactions and CKM Matrices

The transformation matrices do not simply go away. The conversion to the mass eigenstates affects the remainder of the Lagrangian involving fermions, where a residual of the transformation matrices will remain. Under the change to the mass eigenbasis, the fermion part of the Lagrangian becomes

$$\mathfrak{L}_\psi = \overline{\psi} i\gamma^\mu (\partial_\mu + \mathbf{A}_\mu)\psi = \overline{\psi_M}\,\overline{V^+} i\gamma^\mu (\partial_\mu + \mathbf{A}_\mu)V^+\psi_M = \overline{\psi_M} i\gamma^\mu V(\partial_\mu + \mathbf{A}_\mu)V^+\psi_M == \overline{\psi_M} i\gamma^\mu (\partial_\mu + V\mathbf{A}_\mu V^+)\psi_M$$

.

The transformation matrix V involves the projections $L, B, |\mathbf{l}\rangle\langle\mathbf{l}|, |\mathbf{r}\rangle\langle\mathbf{r}|, e, \mu, \tau$ which commute with the gauge generators, but also the projections U, D which commute with the generators for $U(1)_Y$ and $SU(3)_\Lambda$ but not those for $SU(2)_I$. The generator $\mathbf{I}_3$ commutes, but not $\mathbf{I}_1, \mathbf{I}_2$. Explicitly, for the corresponding matrices, we have

$$D\tau_i = \tau_1 U, \quad U\tau_i = \tau_i D \quad (i = 1,2) .$$

Thus,

$$V\tau_i V^+ = (L_d L_u^+ DB + DL + UB + L_v L_e^+ UL)\tau_i (L_u L_d^+ DB + DL + UB + L_e L_v^+ UL), \quad (i = 1,2) .$$

The matrices that emerge from this are the Kabibbo-Cobayashi-Maskawa (CKM) matrices,

$$V_Q \equiv L_d L_u^*, \quad U_L \equiv L_v L_e^* .$$

The lepton sector matrix U_L is referred to as the Maki-Nakagawa-Sakata (MNS) matrix, though we will refer to both collectively under the name CKM.

The result of the transformation is,

$$V\tau_i V^+ = (V_Q DB + DL + UB + U_L UL)\tau_i (V_Q DB + DL + UB + U_L UL)^* \quad (i = 1,2) .$$

which changes the gauge generators $\mathbf{I}_1, \mathbf{I}_2$ to

$$\mathbf{I}_i = \frac{\tau_{iM}}{2}|\mathbf{l}\rangle\langle\mathbf{l}|, \quad (i = 1,2,3) ,$$

where

$$\tau_{1M} = |\mathbf{u}\rangle_M \langle\mathbf{d}|_M + |\mathbf{d}\rangle_M \langle\mathbf{u}|_M, \quad \tau_{2M} = i(|\mathbf{d}\rangle_M \langle\mathbf{u}|_M - |\mathbf{u}\rangle_M \langle\mathbf{d}|_M), \quad \tau_{3M} = |\mathbf{u}\rangle_M \langle\mathbf{u}|_M - |\mathbf{d}\rangle_M \langle\mathbf{d}|_M ,$$

and the modified basis elements given by

$$|\mathbf{u}\rangle_M = (U_L L + B)|\mathbf{u}\rangle,\quad |\mathbf{d}\rangle_M = (L + V_Q B)|\mathbf{d}\rangle,\quad \langle\mathbf{u}|_M = \langle\mathbf{u}|(U_L{}^* L + B),\quad \langle\mathbf{d}|_M = \langle\mathbf{d}|(L + V_Q{}^* B).$$

If the neutrino has 0 mass, then L_ν is arbitrary and may be defined to be $L_\nu = L_e$, which will then reduce the leptonic CKM matrix, $U_L = I$. Otherwise, if a right-neutrino (and left anti-neutrino) sector is assumed, the matrix will be non-trivial.

Since, only the residual gauge invariance is apparent, the transformation between the charge and mass eigenstates may be considered to involve nothing more than these two matrices. By convention, one takes

$$L_u = I,\quad L_d = V_Q,\quad L_\nu = U_L,\quad L_e = I,$$
$$R_u = I,\quad R_d = I,\quad R_\nu = I,\quad R_e = I.$$

Explicitly, the transformation between charge and mass eigenstates for the left-handed components of the fields is then written as

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix} = \begin{pmatrix} u_M \\ c_M \\ t_M \end{pmatrix},\qquad \begin{pmatrix} d \\ s \\ b \end{pmatrix} = V_Q \begin{pmatrix} d_M \\ s_M \\ b_M \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}\begin{pmatrix} d_M \\ s_M \\ b_M \end{pmatrix},$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}\begin{pmatrix} \nu_{eM} \\ \nu_{\mu M} \\ \nu_{\tau M} \end{pmatrix},\qquad \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} e_M \\ \mu_M \\ \tau_M \end{pmatrix}.$$

The following estimates on the quark mass mixing matrix are (excluding the phase information),

$$V_Q = \begin{pmatrix} .9742-.9757 & .219-.226 & .002-.005 \\ .219-.225 & .9734-.9749 & .037-.043 \\ .004-.014 & .035-.043 & .9990-.9993 \end{pmatrix},$$

are cited in Tsun (arXiv:hep-th/0110256), who has proposed a theory accounting for the generational structure and mass mixing relations whose primary assertion is that the fermion mass matrices m_U, m_D, m_N and m_E are each of rank 1 and are all derivable from a common form by the running of a small set of parameters (3 of them). Experimental estimates for the lepton mixing matrix (again, excluding phase information) are cited as well:

$$U_L = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} * & 0.4-0.7 & 0.0-0.15 \\ * & * & 0.56-0.83 \\ * & * & * \end{pmatrix}.$$

The values derived theoretically are

$$V_Q = \begin{pmatrix} .9752 & .2215 & .0048 \\ .2211 & .9744 & .0401 \\ .0136 & .0381 & .9992 \end{pmatrix},\quad U_L = \begin{pmatrix} 0.97 & 0.24 & 0.07 \\ 0.22 & 0.71 & 0.66 \\ 0.11 & 0.66 & 0.74 \end{pmatrix},$$

which fits well, except the “solar neutrino angle” U_{e2}.

3.5. The Gauge Interactions with the Mass Eigenstate Boson Fields

The CKM matrices effectively become part of the gauge generators, as just shown. The effect is consonant with the reduction of the boson fields to mass eigenstates, which works in tandem with the reduction of the fermion fields. The CKM matrices are attached to the couplings of the $W, \overline{W}$ fields, while those of the A, Z fields remain unaffected. The former are, therefore, the only fields to mediate interactions between the different generations of mass eigenstates. It is only by these interactions that the multiplicity of generations seen is actually observed. This, of course, leads to an interesting question in its own right: 96 is a somewhat odd number for the total number of fermion states (32 per generation), while 128 would seem a whole lot more natural. Could there be a 4th generation that is sterile? While particle scattering experiments limit the size of the sector mediated by the CKM matrices to 3 generations, they have nothing directly to say about the existence of other CKM sectors not attached to the 3 known generations, or even sterile generations.

An interesting hypothesis in this regard is that the old flavor $SU(3)$ may not have been all that far off the mark. Perhaps the $3+1$ decomposition seen in the quark-lepton $SU(3)$ is complemented by a $3+1$ decomposition for the CKM sectors.

The reason 128 is significant is that it is a power of 2. The power of 2 structure already seen within a given generation is strongly suggestive of an underlying Clifford algebra basis. It is generally only these algebras, rather than simple or semi-simple Lie groups that lead to power of 2 patterns in the irreducible representations. Of the simple Lie groups, only $SO(10)$ has the capability of producing such a state space (it has a $\mathbf{16}$). A 32×32 matrix structure is naturally associated with the 11-dimensional Dirac algebra associated with $SO(10,1)$. However, to get 128 components requires 14 dimensions or 15.

4. Gravitational Extension

The above account cannot really be considered complete until the full effect of the gravitational field is brought in, as well. Though it is not strictly a part of the Standard Model, the fact remains that even in the absence of gravity (or in weak gravity) one would still like to resort to using non-Cartesian coordinates or even non-coordinate frames. Then there are a few notable differences, not the least of which is that an extra factor appears in the Lagrangian and participates in the various bilinear forms that we've encountered.

The approach adopted here is to treat gravity as a gauge theory for local Poincaré symmetry. This cannot be a Yang-Mills theory since the Poincaré group is not even semi-simple, much less compact. Others (notably Sardanashvily) have pointed out that since the fermions break the $GL(4)$ world symmetry down to $SO(3,1)$ in virtue their dependence on the Clifford bundle formed by the Dirac matrices, then gravity may best be regarded, instead, as a spontaneously broken symmetry, with the vielbein arising as the Goldstone-Higgs field associated with the symmetry breaking.

However, for the following, we will adopt the approach of treating the vielbein as the gauge field associated with the translation generators of the Poincaré group. Though the theory may not be a Yang-Mills gauge theory, it might yet be a generalized gauge theory in which the dual fields are only related functionally to the field strengths, subject to the requirement that the Lagrangian yield a variation of the form

$$\delta\mathfrak{L} = -\frac{1}{2}\mathfrak{G}^{\mu\nu}\cdot\delta\mathbf{F}_{\mu\nu} + \mathfrak{J}^{\mu}\cdot\delta\mathbf{A}_{\mu}$$

and that

$$[\mathfrak{G}^{\mu\nu}, \mathbf{F}_{\mu\nu}] = 0 .$$

This will still yield the field equations

$$\partial_\nu\mathfrak{G}^{\mu\nu} + [\mathbf{A}_\nu, \mathfrak{G}^{\mu\nu}] = \mathfrak{J}^{\mu} ,$$

and the force law

$$\mathfrak{K}_\nu = \mathfrak{J}^{\mu}\cdot\mathbf{F}_{\mu\nu}$$

will still be integrable into a conservation law

$$\mathfrak{K}_\nu = -\partial_\mu\mathfrak{T}^{\mu}_{\nu}$$

involving a stress tensor density

$$\mathfrak{T}^{\mu}_{\nu} = \mathfrak{G}^{\mu\rho}\cdot\mathbf{F}_{\rho\nu} - \delta^{\mu}_{\nu}\mathfrak{L} .$$

But the question of how to assign the dual fields is unresolved.

4.1. Local Spacetime Symmetry Group and Gravity

The full gauge group, in a suitable basis has additional generators for the local spacetime symmetry group

$$ISO(3,1):(\mathbf{p}_a, a=0,1,2,3);(\mathbf{s}_{ab}, a,b=0,1,2,3) .$$

Since $ISO(3,1)$ is not compact, nor even semi-simple, an adjoint-invariant metric over it reduces to 0

$$k(\mathbf{p}_a, v)=0=k(\mathbf{s}_{ab}, v) .$$

For the local spacetime symmetry group, the Lie algebra is given by

$$[\mathbf{s}_{ab},\mathbf{s}_{cd}]=i\hbar(\eta_{ad}\mathbf{s}_{bc}+\eta_{bc}\mathbf{s}_{ad}-\eta_{ac}\mathbf{s}_{bd}-\eta_{bd}\mathbf{s}_{ac}),\quad [\mathbf{s}_{ab},\mathbf{p}_c]=i\hbar(\eta_{bc}\mathbf{p}_a-\eta_{ac}\mathbf{p}_b),$$
$$[\mathbf{p}_a,\mathbf{s}_{cd}]=i\hbar(\eta_{ac}\mathbf{p}_d-\eta_{ad}\mathbf{p}_c),\qquad [\mathbf{p}_a,\mathbf{p}_c]=0.$$

This may be simplified by writing this in parametrized form in terms of an anti-symmetric matrix ω and vector α,

$$\mathbf{L}(\omega,\alpha)\equiv\frac{1}{2}\omega^{ab}\mathbf{s}_{ab}+\alpha^a\mathbf{p}_a ,$$

yielding the Lie bracket

$$[\mathbf{L}(\omega,\alpha),\mathbf{L}(\theta,\beta)]=\mathbf{L}(\omega\eta\theta-\theta\eta\omega,\omega\eta\beta-\theta\eta\alpha) .$$

There is an addition from the gravity field to the gauge field and the corresponding strength, given by

$$\mathbf{A}_\mu=\ldots+\frac{1}{2}\omega_\mu^{ab}\mathbf{s}_{ab}+e_\mu^a\mathbf{p}_a,\quad \mathbf{F}_{\mu\nu}=\ldots+\frac{1}{2}\theta_{\mu\nu}^{ab}\mathbf{s}_{ab}+\tau_{\mu\nu}^{a}\mathbf{p}_a .$$

The gravitational part only acts directly on the fermion sector. The $\mathbf{p}$ generators do not act directly anywhere, though it might be regarded as having already been included in the ∂_μ part of the covariant derivative operator by the representation $\mathbf{p}_a=ie_a^\mu\partial_\mu$, involving the inverse of the gauge field (more on this below). Extending this, one may define the charge operator by $\boldsymbol{\theta}_a=i\mathbf{Y}_a$, with the corresponding current $J_a^\mu=\overline{\psi}\gamma^\mu\boldsymbol{\theta}_a\psi$. Then the covariant derivative term becomes $iD=\mathbf{p}-\boldsymbol{\theta}$ which is just kinetic momentum.

For spin ½ Dirac fields, the Lorentz generators are just the spin operators,

$$\mathbf{s}_{ab}=\frac{i\hbar}{2}\gamma_{ab}$$

where

$$\gamma_{ab}=\frac{\gamma_a\gamma_b-\gamma_b\gamma_a}{2},\quad \gamma_a=\eta_{ab}\gamma^b .$$

In parametrized form the gravitational part of the field may thus be written

$$\mathbf{A}_\mu=\ldots+\mathbf{L}(\omega_\mu,e_\mu),\quad \mathbf{F}_{\mu\nu}=\ldots+\mathbf{L}(\theta_{\mu\nu},\tau_{\mu\nu}) ,$$

with

$$\theta_{\mu\nu}=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu+\omega_\mu\eta\omega_\nu-\omega_\nu\eta\omega_\mu,\quad \tau_{\mu\nu}=\partial_\mu e_\nu-\partial_\nu e_\mu+\omega_\mu\eta e_\nu-\omega_\nu\eta e_\mu .$$

4.2. The Gravitational-Gauge-Higgs Lagrangian

With these preliminaries set, the Lagrangian part of the action may be written out

$$\mathcal{L}=\frac{\varepsilon_{abcd}e^a\wedge e^b\wedge\theta^{cd}}{16\pi G}-\frac{1}{4}eg^{\mu\rho}g^{\nu\sigma}k(\mathbf{F}_{\mu\nu},\mathbf{F}_{\rho\sigma})+e\frac{\Lambda}{16\pi G}$$
$$+e\varepsilon\left(\psi,\left(i\gamma^a e_a^\mu\left(\partial_\mu+\mathbf{A}_\mu\right)-G(\varphi)\right)\psi\right)-eg^{\mu\nu}\chi\left(\left(\partial_\mu+\mathbf{A}_\mu\right)\varphi,\left(\partial_\nu+\mathbf{A}_\nu\right)\varphi\right)-e\lambda\left(\chi(\varphi,\varphi)-\frac{v^2}{2}\right)^2$$

where

$$e=\varepsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d .$$

An extra term involving the cosmological constant, Λ, which is part of the contribution from the gravitational field has been included.

One of the central results of the theory of principal bundles is that when the gauge metric, spacetime metric and gauge field are combined into a single metric

$$h(\partial_\mu, \partial_\nu) = g_{\mu\nu} + k(\mathbf{A}_\mu, \mathbf{A}_\nu); \quad h(\partial_\mu, \mathbf{Y}_b) = -k(\mathbf{A}_\mu, \mathbf{Y}_b); \quad h(\mathbf{Y}_a, \mathbf{Y}_b) = k(\mathbf{Y}_a, \mathbf{Y}_b) ,$$

or equivalently,

$$h(D_\mu, D_\nu) = g_{\mu\nu}; \quad h(D_\mu, \mathbf{Y}_b) = 0; \quad h(\mathbf{Y}_a, \mathbf{Y}_b) = k(\mathbf{Y}_a, \mathbf{Y}_b)$$

then the corresponding Einstein-Hilbert action will decompose into

$$\mathfrak{L}_{EH}(h) = \frac{\varepsilon_{abcd} e^a \wedge e^b \wedge \theta^{cd}}{16\pi G} - \frac{1}{4} e g^{\mu\rho} g^{\nu\sigma} k(\mathbf{F}_{\mu\nu}, \mathbf{F}_{\rho\sigma}) + \frac{1}{4} e k^{ad} f^{\ c}_{ab} f^{\ b}_{cd} ,$$

which provides a cosmological constant term. If the gauge metric is assumed to be variable (which amounts to assuming that the couplings are variable), extra terms corresponding to the derivative of the gauge metric appear – "dark energy" terms. Furthermore, the extra "cosmological constant" term also becomes variable. Without the variability, one is faced with the ***fine tuning problem*** of explaining the incredibly unlikely possibility of a extremely small, yet non-zero, cosmological constant arising out of the extra term.

Further details on these matters are not spelled out here.

The gauge field e^a_μ is assumed to be invertible as a matrix, with the inverse $e^{-1} = (e^\mu_a)$. The question of its invertibility is closely tied to the assumption of the specific form for the local spacetime group. In general, one can only say over a given n-dimensional manifold that the local symmetry group is $GL(n)$. The restriction to $SO(n-1,1)$, as opposed, say, to $ISO(n-1)$ or $SO(n)$ amounts to an implicit assumption of a certain degree of classical causal background into the underlying spacetime. We're assuming the *signature* of the metric is part of the background. The invertibility issue, therefore, may be regarded as a manifestation of the more general problem: the ***signature problem***. From the gauge field comes the metric and its dual,

$$g_{\mu\nu} = \eta_{ab} e^e_\mu e^b_\nu , \quad g^{\mu\nu} = \eta^{ab} e^\mu_a e^\nu_b .$$

Notable is that the dual metric or inverse e^{-1} only appears in the places where the various inner products or bilinear forms appear. As already seen, the inner product associated with the fermion involves the Dirac adjoint,

$$\overline{\psi} \equiv \psi^* \gamma^0 ,$$

and takes on the explicit form

$$\varepsilon(\alpha, \beta) \equiv \frac{\alpha^* \gamma^0 \beta + \alpha^T \gamma^0 \beta^{*T}}{2} = \frac{\overline{\alpha}\beta + \alpha^T \overline{\beta}^T}{2} .$$