ON THE COHOMOLOGICAL STRUCTURE OF GAUGE THEORIES

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ABSTRACT

We show that the structure of gauge theories gets particularly simple when space time is enlarged by adjunction of Grassmanian coordinates. The method applies in flat space as well as in curved space, with gravitational interactions.

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In some lectures given by the author in Cargese[1)], it was observed that the gauge theories involving local fields with p antisymmetrized Lorentz indices, $p \geqslant 1$, share generally the fascinating property that their algebraic, and possibly geometric, structure gets simplest when the flat space-time $\{x^\mu\}$ is enlarged by adjunction of a set of unphysical coordinates $\theta, \bar\theta$ into a space with local coordinates $\{x^\mu, \theta, \bar\theta\}$[2)]. A deeper understanding of the role of Bianchi identities in this description of gauge symmetries has been reached now. Furthermore, recent work has shown that one can naturally introduce the invariance under local change of coordinates in the formalism. As a matter fact, this provides a simple algebraic description of the symmetries of gravity and of gauge theories coupled to gravity[3)]. It has become more and more obvious that this formalism is a quite efficient tool for extracting the cohomological structure inherent to gauge invariant theories[4)]. In particular, a much deeper understanding of those questions related to anomalies has been reached[1,5)].

The θ and $\bar\theta$ unphysical coordinates are choosen as pure Grassmanians ($\theta^2 = \theta\bar\theta + \bar\theta\theta = \bar\theta^2 = 0$) with no spinorial charge. Therefore they violate the physical statistics, and this permits one to predict an effective dimension D-2 for the enlarged space $\{x^\mu, \theta, \bar\theta\}$. Notice that D-2 is the dimension of the tranverse space in which the physical parts of gauge fields of the D-dimensional Minkowski space-time $\{x^\mu\}$ are forced to live whenever physics is described by a gauge invariant Lagrangian.

In the enlarged space, the gauge fields are introduced as a set of extorior forms $\tilde{B}^{i}_{p_i}(x,\theta,\bar{\theta})$ $(p_i \geq 1)$, classified by the index i and possibly valued in given representations of Lie algebra. The value p=1 corresponds to the Yang-Mills case. Over the point $(x,\theta,\bar{\theta})$ the space P of differential forms is spanned by the basis of forms $\alpha^{q,\bar{q}}_{k-q-\bar{q}} = dx^{\mu_1}\wedge \ldots dx^{\mu_{k-q-\bar{q}}}\wedge (d\theta)^q \wedge (d\bar{\theta})^{\bar{q}}$, with $(d\theta)^q \wedge (d\bar{\theta})^{\bar{q}} \equiv \underbrace{d\theta \wedge \ldots \wedge d\theta}_{q} \wedge \underbrace{d\bar{\theta} \wedge \ldots \wedge d\bar{\theta}}_{\bar{q}}$. Notice that P is infinite dimensional because $(d\theta)^q \wedge (d\bar{\theta})^{\bar{q}}$ never vanishes whatever the values of q and $\bar{q}$ are. Indeed $d\theta$ and $d\bar{\theta}$ are exterior forms of Grassmanians, and thus are commuting objects. In local coordinates one can expand $\tilde{B}_p$ as

$$\tilde{B}_p = B_p(x,\theta,\bar{\theta}) + \sum_{g=1}^{p} \sum_{\substack{q+\bar{q}=g \\ q,\bar{q}\geq 0}} B^{q,\bar{q}}_{p-g}(x,\theta,\bar{\theta}) \qquad (1)$$

with $B^{q,\bar{q}}_{p-g}(x,\theta,\bar{\theta}) = \frac{1}{(p-g)!\,q!\,\bar{q}!} B^{q,\bar{q}}_{\mu_1\cdots\mu_g}(x,\theta,\bar{\theta})\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_g}\wedge(d\theta)^q\wedge(d\bar{\theta})^{\bar{q}}$.

We call B_p, which has no $d\theta$ or $d\bar{\theta}$ components the classical p-form gauge field and $B^{q,\bar{q}}_g$, $g = q+\bar{q} \geq 1$, the ghosts of order g of the p-form B_p. According to eq. (1), the spectrum of ghosts for a generalized p-form B_p has a pyramidal structure

$$\begin{array}{ccccccccc}
 & & & & B_p & & & & \\
 & & & B^{1,0}_{p-1} & & B^{0,1}_{p-1} & & & \\
 & & B^{2,0}_{p-2} & & B^{1,1}_{p-2} & & B^{0,2}_{p-2} & & \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & \\
B^{p,0}_0 & & B^{p-2,2}_0 & & \cdots\cdots & & B^{2,p-2}_0 & & B^{0,p}_0
\end{array} \qquad (2)$$

The fields which occur in ordinary field theory and are subject to the functional integration are identified as $B^{q,\bar{q}}_{\mu_1\cdots\mu_g}(x,0,0)$ $(d\theta)^q \wedge (d\bar{\theta})^{\bar{q}}$. They satisfy by definition the physical statistics when the ghost number $g = q+\bar{q}$ is even, and the unphysical one when g is odd. The commutation properties are identical to those given by the rules of exterior calculus in P, the grading being defined as the sum of the ghost number and Lorentz degree for any object. It follows that the effective number of degrees of freedom N_{phys} carried by all fields contained in $\tilde{B}_p$ is altogether

$$N_{phys} = \sum_{g=0}^{p} (-)^g (g+1) \binom{p}{D} = \binom{p}{D-2} \qquad (3)$$

This is exactly what is needed, since a classical p-form gauge field carries $\binom{p}{D}$ independent components, but describes only $\binom{p}{D-2}$ physical degrees of freedom in D-dimensional Minkowski space-time. This property is related to the above remark about the effective dimension D-2 of the space $\{x^{\mu},\theta,\bar{\theta}\}$. We have assumed that a field which can propagate independently counts negatively if its statistics is unphysical. This assumption is motivated by the rules of functional integration.

We shall in fact postulate that a p-form gauge field is described at the quantum level by all components, classical and ghost, contained in a generalized exterior p-form in P, eq. (2). All products of fields must be understood as exterior products in P, and the wedge product symbol will be generally omitted.

To determine the variations of all fields $B^{q,\bar{q}}_{p-q}(x,\theta,\bar{\theta})$ we introduce the exterior differential $\tilde{d}$ in P. $\tilde{d}$ splits into the "horizontal" component $d = dx^{\mu}\,\partial/\partial x^{\mu}$ and the "vertical" components $s = d\theta\,\partial/\partial\theta$ and $\bar{s} = d\bar{\theta}\,\partial/\partial\bar{\theta}$ Evaluated at $\theta = \bar{\theta} = 0$, the action of s and $\bar{s}$ on the classical and ghost components of $\tilde{B}_p$ is identified with that of the BRS and anti-BRS symmetry. In this way an infinitesimal gauge transformation is tantamount to a displacement along the unphysical direction θ or $\bar{\theta}$. The basic rule of exterior calculus is $\tilde{d}^2 = 0$. By expansion in ghost number, this is equivalent to $d^2 = 0$, $sd+ds = 0 = \bar{s}d+d\bar{s}$ and $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2 = 0$. The latter property corresponds in fact to the closure relation and Jacobi identity of the gauge symmetry associated with s and $\bar{s}$[6].

Using $\tilde{d}$ we can define the field strenghts of the "potential" $\tilde{B}^{i}_{p_i}$

$$\tilde{G}^{i}_{p_i+1} \equiv \tilde{d}\,\tilde{B}^{i}_{p_i} + R^{i}_{p_i+1}(\tilde{B},\tilde{G}) \tag{4}$$

$\tilde{d}\tilde{B}$ is the free part, i.e. the Maxwell part of the field strenght.

The interacting part R_{p+1} is a (p+1)-exterior form function of the potentials $\tilde{B}$ and their exterior derivatives, that is to say a function of $\tilde{B}$ and $\tilde{G}$. In the Yang-Mills case for instance, one has $\tilde{F} = \tilde{d}\tilde{A}+\tilde{A}\tilde{A} = \tilde{d}\tilde{A}+\frac{1}{2}[\tilde{A},\tilde{A}]$.

Given a set of gauge fields $\tilde{B}_p$, we need a criterion to determine the field strenghts $\tilde{G}_{p+1}$, i.e. the possible forms of the functions $R^{i}_{p_i+1}(\tilde{B},\tilde{G})$. In fact $\tilde{B}_p$ is to be interpreted as a potential with gauge degrees of freedom and therefore should not vary tensorially under s and $\bar{s}$ transformation. This implies in particular that the s transform of B_p must contain a non homogeneous part, the simplest example being $sA = -dc$ in electrodynamics. On the other hand, G_{p+1} must be eventually a physical quantity with no gauge degrees of freedom and must vary tensorially under s and $\bar{s}$, i.e. gauge, transformations. As will be shown shortly, this requirement and other consistency properties are all satisfied at once if R_{p+1} is such that the field strenghts G_{p+1} satisfy Bianchi identities.

$$\tilde{d}\,\tilde{G}^{i}_{p_i+1} = \sum_{j} d^{i,j}_{p_i-p_j}(\tilde{G},\tilde{B})\,\tilde{G}^{j}_{p_j} \tag{5}$$

The (p_i-p_j)-form d^{ij} is an exterior product of $\tilde{G}$ and $\tilde{B}$. Furthermore, one requires that the dependence of d^{ij} on B, at fixed G, is only through the 1-form (e.g. Yang-Mills) gauge fields contained in the set of fields $\tilde{B}$, and not more than linearly. From eq. (5) we can determine the possible forms of $R^{i}_{p_i+1}$ since the equations

$$\tilde{d}^2 = 0 \qquad \tilde{d}\tilde{G}^i = \sum_j d^{ij}\tilde{G}^j \tag{6a}$$

yield the algebraic constraints on R

$$\tilde{d} R^i_{p_i+1} = \sum_j d^{i,j}_{p_i-p_j}\left(\tilde{d}\hat{\tilde{B}}^j_{p_j} + R^j_{p_j+1}\right) \tag{6b}$$

Notice that $R(\hat{B},\hat{G})$ is defined modulo $\tilde{d}$ exact terms, $R_{p+1} \simeq R_{p+1} + \tilde{d}Z_p(\tilde{B},\tilde{G})$. The arbitrariness on Z_p corresponds in fact to mere redefinitions of $\tilde{B}_p$ into $\tilde{B}_p = \tilde{B}_p + Z_p(\tilde{B},\tilde{G})$.

Starting from a set of fields $\tilde{B}_p$, the last equation generally permits one to determine the admissible functions R_{p+1}, by inspection over all possible dependences on $\tilde{B}$ and $\tilde{d}\tilde{B}$, or $\tilde{B}$ and $\tilde{G}$. If for instance one starts from a Yang-Mills field $\tilde{A}$ and a 2-form $\tilde{B}_2$ with no Yang-Mills charge, the general solutions for the admissible field strenghts are $\tilde{F} = \tilde{d}\hat{A} + \frac{\alpha}{2}[\hat{A},\hat{A}]$ and $\tilde{G}_3 = \tilde{d}\tilde{B}_2 + \lambda \mathrm{Tr}(\hat{A}\tilde{d}\hat{A} + \alpha\frac{2}{3}\hat{A}\hat{A}\hat{A})$, so that $\tilde{d}\tilde{F} = -\alpha[\hat{A},\tilde{F}]$ and $\tilde{d}\tilde{G} = \lambda \mathrm{Tr}(\hat{F}\hat{F})$. Notice that α and λ are arbitrary parameters, which represent the only freedom left in the determination of $\tilde{F}$ and $\tilde{G}$ satisfying eq. (5)[7].

Suppose now that we have determined the admissible field strenghts, eqs. (4,5), of a system of gauge fields $\tilde{B}p$. To enforce the physicality of these field strenghts in flat space-time, and to determine in fact the dependence in θ and $\bar{\theta}$ of all quanti-

ties, we impose the following horizontality conditions on the field strenghts, which, somehow, are of the Maurer-Cartan type

$$\tilde{G}_{p+1} = \tilde{G}_{\mu_1\ldots\mu_{p+1}} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{p+1}} \equiv G_p$$

$$\Longleftrightarrow \quad \tilde{G}^{q,\bar{q}}_{p-g} = 0 \quad , \; g = q+\bar{q} \geq 1 \tag{7}$$

Eq. (7) means that the field strenghts have no ghost components, in contrast with the gauge fields. This most simple requirement has many consequences, which eventually determine consistently the expression of the gauge symmetry on all fields.

(i) Once the constraint (7) is expanded in ghost number, this determines the action of s and $\bar{s}$ with $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2 = 0$. In fact the equations $G^{q,\bar{q}}_{p-g} = 0$ can be thought of as differential equations in θ, $\bar{\theta}$, and eqs.(5) are their integrability conditions, which enforce the nilpotency of s and $\bar{s}$. The usual BRS operators are identified with s and $\bar{s}$, evaluated at $\theta = \bar{\theta} = 0$.

(ii) The field strenghts vary "tensorially" under s and $\bar{s}$ transformations, which mean that sG_{p+1} ($\bar{s}G_{p+1}$) is generally independent of the space-time variations of primary ghosts $dB^{1,0}_{p-1}$ ($dB^{0,1}_{p-1}$).

(iii) Invariant Lagrangians in D-dimensions, $\mathcal{L}_D$, which can be written as exterior products, correspond to invariant d-exact form of rank D+1, $I_{D+1}(G) = d\,\mathcal{L}_D(B,G)$. Other solutions may exist of the type *GG.

(iv) The existence of invariant d-exact forms of rank D+2, $I_{D+2}(G) =$

$d\,\Delta_{D+1}$ (B,G) implies the possibility of anomalies in D-dimensional space-time defined as the solutions of the consistency equation $s\,\Delta^1_D + d\,\Delta^2_{D-1} = 0$.

The correspondence between the anomaly Δ^1_D and the invariant form $I_{D+2} = d\,\Delta_{D+1}$ is given by the following equation, true modulo s - and d - exact terms.

$$\Delta^1_D = B^{1,0}_{p-1} \frac{\delta}{\delta B_p}\Big|_G \Delta_{D+1}(B,G) \tag{8}$$

We will give now short demonstrations of the point (i) $\rightarrow$ (iv).

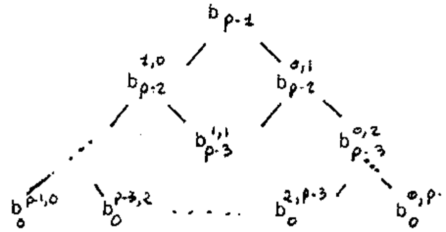
Point i: The action of s and $\bar{s}$ on the fields $\tilde{B}_p$ is given by expanding in ghost number eq. (7). For the fields which are on the edges of the pyramid (2), one has

$$s\,B^{g,0}_{p-g} = -d\,B^{g+1,0}_{p-g} - \left[R_{p+1}(\hat{B},G)\right]^{g,0}_{p-g} \tag{9a}$$

$$\bar{s}\,B^{0,g}_{p-g} = -d\,B^{0,g+1}_{p-g} - \left[R_{p+1}(\hat{B},G)\right]^{0,g}_{p-g} \tag{9b}$$

On the other hand, only $sB^{q,\bar{q}}_{p-g} + \bar{s}B^{q+1,\bar{q}-1}_{p-g}$ is determined for the fields inside the pyramid (2) with $q < g$, $\bar{q} < g$, $q+\bar{q} = g$. This degeneracy however is not essential, and can be raised by postulating an auxiliary generalized (p-1) form $\tilde{b}_{p-1}$. The field components in $\tilde{b}^i_{p_i-1}$ build up the following pyramid of auxiliary fields, which must be considered as fundamental fields, completing all fields components of $\tilde{B}^i_{p_i}$

$$\tilde{b}^{(i)}_{p_i-1} = b^{(i)}_{p_i-1} + \sum_{g=1}^{p_i} \sum_{\substack{q,\bar{q} \\ q+\bar{q}=g}} b^{(i)\,q,\bar{q}}_{p_i-1-g}$$



These fields generalize the well-known Stuckelberg-Kugo-Ojima auxiliary scalar field b of the genuine Yang Mills case. They can be interpreted as Lagrange multipliers for all relevant gauge fixing functions for the fields $B^{q,\bar{q}}_{p-g}$. We call the fields $b^{q,\bar{q}}_{p-1-g}$ auxiliary, because it is always possible to obtain gauge fixed and s and/or $\bar{s}$ invariant Lagrangians such that they have algebraic equations of motion. One defines $sB^{q,\bar{q}}_{p-g} = b^{q,\bar{q}-1}_{p-g}\, d\theta\, d\bar{\theta}$ ($q < g$, $\bar{q} < g$, $q+\bar{q} = g$) and $sb^{q,\bar{q}-1}_{p-g} = 0$, which determines in turn $\bar{s}B^{q+1,\bar{q}-1}_{p-g}$ and $\bar{s}b^{q,\bar{q}-1}_{p-g}$ with $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2 = 0$.

The gauge transformations of B_p are obtained from the expression of sB_p, by replacing all the primary ghosts $B^{1,0}_{p-1}$ by infinitesimal parameters ϵ_{p-1} with the opposite, i.e. physical, statistics

$$\delta B^i_{p_i} = -d\,\epsilon^i_{p_i-1} - \epsilon^j_{p_j-1} \frac{\delta}{\delta B^j_{p_j}}\Big|_G R^i_{p_i+1}(B,G) \tag{10}$$

It is simple to verify that eq. (7) effectively determines s and $\bar{s}$ with $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2 = 0$, which is necessary for consistency [6]. To prove this, one uses the Bianchi identity (5), written under both following equivalent forms, where we have set $\tilde{G} = G$

$$\tilde{d}\,\tilde{G}^{i}_{p_i+1} = \tilde{d}\,G^{i}_{p_i+1}$$

$$\tilde{d}\,\tilde{G}^{i}_{p_i+1} = \tilde{d}^{2}\,\tilde{B}^{i}_{p_i+1} + \sum d^{ij}_{p_i-p_j}(G,\tilde{B})\,G^{j}_{p_j+1} \qquad (11)$$

By comparing these two equations, and using the properties $d^2 = 0$, $sd+ds = 0$, $\bar{s}d+d\bar{s} = 0$, one finds that $\tilde{d}^{2}B_p$ must vanish as the only possible source of terms containing more than two $d\Theta$ and/or $d\bar{\Theta}$. The hypothesis that $d^{ij}(G,B)$ cannot depend on the gauge fields B at fixed G, except for a possible linear dependence on the 1-form gauge fields, is the clue of this demonstration. It follows that $s^2 = \bar{s}s+s\bar{s} = \bar{s}^{2} = 0$ on all the fields contained in $\tilde{B}p$.

<u>Point ii</u>: The Bianchi identity (5), expanded in ghost number, determines the action of s (and $\bar{s}$) on the field strenghts. One has

$$\tilde{d}\,\tilde{G}^{i}_{p_i+1} = (d+s+\bar{s})\,G^{i}_{p_i+1} = \sum_{j} d^{ij}_{p_i-p_j}(G, A^{(K)}+c^{(K)}+\bar{c}^{(K)})\,G^{j}_{p_j+1}$$

$$\Rightarrow s\,G^{i}_{p_i+1} = \sum_{j,K} c^{(K)}\,f^{(K)ij}\,G^{j}_{p_j+1}$$

$$f^{(K)ij} \equiv \frac{\delta}{\delta A^{(K)}}\Big|_{G}\, d^{ij}_{p_i-p_j}(G,A) \qquad (12)$$

In our notation $A^{(K)}$ stand for the 1-form gauge fields of the

system, and $c^{(K)}$ and $\bar{c}^{(K)}$ are the corresponding Faddeev-Popov scalar ghosts and anti-ghosts.

<u>Point iii</u>: Depending on the form of the matrix d^{ij} in eq. (6), it is often possible to write an s and $\bar{s}$ invariant Lagrangian in D dimension $\mathcal{L}_D(G,*G)$ function of G_p and their Hodge transform $(*G)_{D-p}$. The $*$ operation is defined as $*(Z_{\mu_1\ldots\mu_p}\,dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}) \equiv \epsilon^{\mu_1\ldots\mu_D}\,Z_{\mu_1\ldots\mu_p}\,dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_D}$. For the system $F = dA+\frac{\alpha}{2}[A,A]$, $G_3 = dB_2 + \lambda\,\mathrm{Tr}(AdA+\alpha\frac{2}{3}AAA)$, an invariant Lagrangian is for instance $\mathcal{L}_D = F{*}F + G{*}G$ independently of the value of D.

The existence of a Lagrangian in D-dimension is also guaranteed if one can construct a d-exact form of rank D+1 function of G. Indeed, the following algebraic identity, valid for any value of D,

$$I_{D+1}(G) = d\,\mathcal{L}_D(B,G) \qquad (13a)$$

implies obviously

$$I_{D+1}(\tilde{G}) = \tilde{d}\,\mathcal{L}_D(\tilde{B},\tilde{G}) \qquad (13b)$$

Thus, using the constraint (7), $\tilde{G} = G$, one gets $(d+s+\bar{s})\,\mathcal{L}_D(\tilde{B},G) = d\,\mathcal{L}_D(B,G)$. This equation can be expanded in ghost number, and yields the following equations

$$s\left[\mathcal{L}_D(\tilde{B},G)\right]^{g,0}_{D-g} + d\left[\mathcal{L}_D(\tilde{B},G)\right]^{g+1,0}_{D-g-1} = 0 \qquad (14)$$

For g = 0, eq. (14) indicates that $\mathcal{L}_D(B,G)$ is indeed an admissible Lagrangian since its action is s invariant, $s \int \mathcal{L}_D (B,G) = \int d\, [\mathcal{L}_D (\tilde{B},G)]^{1}_{D-1} = 0$. Observe that $\mathcal{L}_D$ is only defined up to a d-exact form (a pure divergency) or a s-exact term (a gauge fixing term). Cannonical gauge fixing terms, including the relevant ghost interactions, are of the form $s\bar{s}\, (\sum_{g \geq 0} \sum_{\bar{q} \leq q} B^{q,\bar{q}}_{p-g} * (B^{\bar{q},q}_{p-g}))$. In these gauges the equations of motion of the fields $\tilde{b}^{q\bar{q}}_{p-1-g}$ are algebraic. Other gauges exist in which the $b^{q,\bar{q}}_{p-1-g}$ propagate and can be interpreted as generalized Nielsen Kallosh ghosts.

Point iv: In the same way as some Lagrangians in D-dimensions are related to d-exact forms in D+1 dimensions, anomalies in D-dimensions, which can be written as exterior products, are related to d-exact forms in D+2 dimensions, $I_{D+2}(G) = d\, \Delta_{D+1}(B,G)$. Suppose the existence of such a form $I_{D+2}(B,G)$. One has identically

$$I_{D+2}(\tilde{G}) = \tilde{d}\, \Delta_{D+1}(\tilde{B}, \tilde{G}) \tag{15}$$

The BRS symmetry equation $\tilde{G} = G$ implies that $I_{D+2}(\tilde{G}) = I_{D+2}(G)$, and thus

$$(d + s + \bar{s})\, \Delta_{D+1}(\tilde{B}, G) = d\, \Delta_{D+1}(B, G) \tag{16a}$$

One has again a tower of equations

$$s\, [\Delta_{D+1}(\tilde{B},G)]^{g,0}_{D+1-g} + d[\Delta_{D+1}(\tilde{B},G)]^{g+1,0}_{D-g} = 0 \tag{16b}$$

among which one recognizes, for g = 1, the consistency equation for an anomaly in D-dimensional space-time. The corresponding solution $(\Delta_{D+1})^{1,0}_{D}$ is written in eq. (8), and the possible anomalous diagrams can be classified from the field decomposition in Δ^{1}_{D} [1].

These general predictions concerning Lagrangians and anomalies, are clearly related with the cohomology of the s and $\bar{s}$ operators for D-forms with ghost number 0 and 1[1]. It is only in the case of anomalies which can be written as exterior products in Yang Mills theories that these results have been rigorously verified[8]. As a matter of fact, all studied examples of theories with p-form gauge fields support the following general conjecture. The solutions Δ^{g}_{D} of the equation $s\, \Delta^{g}_{D} + d\, \Delta^{g+1}_{D-1} = 0$, $\Delta^{g}_{D} \neq s\, K^{g-1}_{D} + d\, K^{g}_{D-1}$, which are exterior products of the fields, can be determined for any value of g and D from the sole knowledge of a d-exact form I_{D+g+1} of degree D+g+1 such that

$$\Delta^{g}_{D} = [\Delta_{D+g}(\tilde{B}, G)]^{g}_{D} \tag{17a}$$

$$d\, \Delta_{D+g}(B,G) = I_{D+g+1}(B,G) \tag{17b}$$

$$\Delta_{g+D} \sim \Delta_{g+D} + d\, K_{D+g-1}$$

and the B and G dependences of the d-exact form I_{D+g+1} and of Δ_{D+g} are such that

$$\left(\frac{\delta}{\delta B_{p_1}} \cdots \frac{\delta}{\delta B_{p_q}} \right) \Big|_{G} \Delta_{D+g}(B,G) \neq 0$$

$$p_1 + \cdots + p_q = g$$

$$\left(\frac{\delta}{\delta B_{p_1}}\cdots\frac{\delta}{\delta B_{p_q}}\right)\Big|_G I_{D+G+1}(B,G) = 0$$

$$p_1+\dots+p_q = g \qquad (17c)$$

Clearly, eq. (17), if general, would amount to an easy algebraic determination of the cohomology of the operator s.

Gravity and curved space

Gravity, i.e. the gauge theory associated with invariance under local changes of coordinates, can be naturally introduced in the formalism as resulting from the structure of the Poincaré algebra.

We will assume for simplicity that no gauge field has a spinorial charge, and therefore we leave aside the case of supergravities, for which we known now to apply the method only in few examples[4].

In addition to the set of gauge fields $B^i_{p_i}$ described above, we introduce the vielbein $\tilde e^a = e^a_\mu dx^\mu + e^a_\theta d\theta + e^a_{\bar\theta} d\bar\theta$ and the spin-connection $\tilde\omega^{ab} = \omega^{ab}_\mu dx^\mu + \omega^{ab}_\theta d\theta + \omega^{ab}_{\bar\theta} d\bar\theta$. a and b stand for the indices of the Lorentz group in D-dimensions. The classical vielbein e^a_μ is assumed as invertible. Therefore, the part with ghost number 1 in $\tilde e$ must not be an independent ghost. Rather we impose the following constraint on $\tilde e$

$$\tilde e^a = e^a + i_\xi e^a + i_{\bar\xi} e^a = \exp(i_{\xi+\bar\xi})(e^a) \qquad (18)$$

ξ^μ and $\bar\xi^\mu$ are the ghost and antighost vector fields of the diffromorphism symmetry. The contraction operator i_φ along a vector field φ^μ is defined by $i_\varphi dx^\mu = \varphi^\mu$, $i_\varphi f = 0$ when f is a 0-form, and i_φ is graded by $g(\varphi)-1$, where $g(\varphi)$ is the ghost number of φ^μ.

Eq. (18) means $\tilde e^a_\mu = e^a_\mu + \xi^\mu e^a_\mu + \bar\xi^\mu e^a_\mu$. It determines in fact the correspondence between the algebra of diffeomorphisms, with ghosts ξ^μ, and the translation sector of the Poincare algebra, with ghosts $\eta^a = e^a_\mu \xi^\mu$. The field strenghts of $\tilde e$ and $\tilde\omega$, and their Bianchi identities are

$$\begin{aligned} \tilde T^a &= \hat d\, \tilde e^a + \tilde\omega^{ab}\tilde e^b & \qquad \hat d \hat T &= -\hat\omega \hat e + \hat R \hat e \\ \tilde R^{ab} &= \hat d\, \tilde\omega^{ab} + \tilde\omega^{ac}\tilde\omega^{cb} & \qquad \hat d \hat R &= -[\hat\omega, \hat R] \end{aligned} \qquad (19)$$

Eq. (18), and the Bianchi identity of $\hat T$, $\hat D\hat T = \hat R \hat e$, suggest that the notion of horizontality in curved space must be defined as

$$\begin{aligned} \tilde G_{p+1} &= (\exp i_{\xi+\bar\xi})\, G_{p+1} \\ \tilde R &= (\exp i_{\xi+\bar\xi})\, R \\ \tilde T &= (\exp i_{\xi+\bar\xi})\, T \end{aligned} \qquad (20)$$

This can be written also as $\tilde G_{p+1} = G_{a_1\dots a_{p+1}}(B,G,e,T,\omega,R)\, \tilde e^{a_1}\dots\tilde e^{a_{p+1}}$.

This definition of the horizontality, when the space has curvature, is in fact determined by the necessity of assigning a part with ghost number 1 to the vielbein. In turn, this provides a geometrical interpretation for the ghosts ξ^μ and $\bar{\xi}^\mu$ of diffeomosphisms.

By expanding the constraint on $\tilde{T}$ in ghost number, we obtain the action of s on the ghost of diffeomorphisms

$$
\begin{aligned}
s\,\xi^\mu &= \xi^\alpha \partial_\alpha \xi^\mu = \frac{1}{2}\{\xi,\xi\}^\mu \equiv \frac{1}{2} L_\xi \xi^\mu \\
\bar{s}\,\bar{\xi}^\mu &= \bar{\xi}^\alpha \partial_\alpha \bar{\xi}^\mu = \frac{1}{2}\{\bar{\xi},\bar{\xi}\}^\mu \equiv \frac{1}{2} L_{\bar{\xi}} \bar{\xi}^\mu
\end{aligned}
\tag{21}
$$

This is the expected transformation law for ξ , corresponding to the known diffeomorphism algebra[9]. The rest of BRS equations is deduced from eq. (20), and gives back the results of refs. 3,9).

The consistency of this construction of the BRS symmetry, i.e. the verification of $\tilde{d}^2 = 0$, is crystal clear by the following change of variables:[3,4]

$$
\begin{aligned}
&\hat{s} \equiv s - L_\xi \qquad \hat{\bar{s}} \equiv \bar{s} - L_{\bar{\xi}} \qquad \hat{d} = d + \hat{s} + \hat{\bar{s}} \\
&\hat{e} \equiv (\exp - i_{\xi+\bar{\xi}})\, \tilde{e} \\
&\hat{\omega} \equiv (\exp - i_{\xi+\bar{\xi}})\, \tilde{\omega} \\
&\hat{B}_p \equiv (\exp - i_{\xi+\bar{\xi}})\, \tilde{B}_p
\end{aligned}
\tag{22}
$$

$L_\xi \equiv i_\xi d - d i_\xi$ is the Lie derivative along the vector field ξ . One has the essential properties[4]

$$
\hat{d} = (\exp - i_{\xi+\bar{\xi}})\, \tilde{d}\, (\exp i_{\xi+\bar{\xi}}) \tag{23a}
$$

$$
\hat{B}\,\hat{C} = (\exp - i_{\xi+\bar{\xi}})\, \tilde{B}\,\tilde{C} \tag{23b}
$$

Notice that the redefinition (22) leads to the introduction of the modified ghosts $\hat{B}^{g,0}_{p-g} = B^{g,0}_{p-g} - i_\xi B^{g-1,0}_{p-g+1} + \dots + (-)^g \frac{i_\xi \cdots i_\xi}{g!} B_p$.

The horizontality constraints (20) on the field strenghts are simplest in terms of the new variables. Indeed, using the properties (23), one gets

$$
\begin{aligned}
\hat{\ } &= e \\
\hat{T} &\equiv \hat{d} e + \hat{\omega} e = T \\
\hat{R} &\equiv \hat{d} \hat{\omega} + \hat{\omega} \hat{\omega} = R \\
\hat{G}_{p+1} &\equiv \hat{d} \hat{B}_p + R_{p+1}(\hat{B}, \hat{G}, \hat{e}, \hat{\omega}, \hat{R}, \hat{T}) = G_{p+1}
\end{aligned}
\tag{24}
$$

We have therefore obtained a BRS structure completely similar to that of a gauge system in flat space, eq.7. The same analysis as in flat space, based on Bianchi identities, shows therefore that (24) determine $\hat{s}$ and $\hat{\bar{s}}$ with $\hat{d}^2 = 0$. Finally, eq. (23a), which implies trivially the equivalence $\hat{d}^2 = 0 \Leftrightarrow \tilde{d}^2 = 0$, insures that the full BRS operators $s = \hat{s} + L_\xi$ and $\bar{s} = \hat{\bar{s}} + L_{\bar{\xi}}$ are nilpotent, which proves finally the consistency of the geometrical determination of s and $\bar{s}$ from eqs. (20) or (24).

Notice that the disentangling of the diffeomorphisms from the internal symmetries, including the local Lorentz symmetry, has been achieved by the introduction of "hat" variables $\hat{\ }$ in eq.

(22), with the crutial properties (23).

As a consequence, the search of Lagrangians and anomalies in curved space is equivalent to a similar problem in flat space (points (iii) and (iv)) except that one must include a possible dependence on e, R, T for I_{D+1} and I_{D+2} in eqs. (13) and (15). The clue is that the Lagrangian and anomaly equations $\int s\, \mathcal{L}_D = 0$ and $\int s\, \Delta^1_D = 0$ are equivalent to $\int \hat{s}\, \mathcal{L}_D = 0$ and $\int \hat{s}\, \Delta^1_D = 0$ in D-dimensional space time, since $\int L_\xi (Z_D) = 0$ for any D-form which vanishes at the boundary. The equation that one must solve is thus of the flat-space type $\hat{s}\, \Delta^g_D(\hat{B},G) + d\, \Delta^{g+1}_{D-1}(\hat{B},G) = 0$.

As an example the Einstein Lagrangian and cosmological term correspond respectively to the (D+1)-forms $\epsilon_{a_1 a_2 \dots a_D} T^{a_1} e^{a_2} \dots e^{a_{D-2}} R^{a_{D-1} a_D} \sim d(\epsilon_{a_1 \dots a_D} e^{a_1} \dots e^{a_{D-2}} R^{a_{D-1} a_D})$ and $\epsilon_{a_1 \dots a_D} T^{a_1} e^{a_2} \dots e^{a_D} \sim d(\epsilon_{a_1 \dots a_D} e^{a_1} \dots e^{a_D})$. This result is similar to the one obtained in the group manifold approach of Regge et al.

To summarize, it appears that the algebraic structure of gauge symmetries in curved space is completely analogous to that in flat space. The invariance under diffeomorphisms is obtained from the translation sector of the Poincare algebra, owing to eq. (18). The correspondence between the gauge symmetries in curved space and in flat space is simply obtained by the redefinitions of fields and BRS operators from the changes of variables (22,23), $\hat{B} = \exp - i_{\xi+\bar{\xi}}\, \tilde{B}$, $\hat{d} = (\exp - i_{\xi+\bar{\xi}})\, \tilde{d}\, (\exp i_{\xi+\bar{\xi}})$. As a

consequence, the relevance of the enlarged space $\{x^\mu, \theta, \bar{\theta}\}$ for studying the properties of gauge symmetries is still existing in the presence of non vanishing curvature and torsion. Finally the possible gauge symmetries involving exterior p-form gauge fields can be classified from the algebraic determination of field strenghts satisfying Bianchi identities.

A byproduct of our analysis of gauge symmetries in curved space is that its points out the existence of a graded algebra of operators acting on the generalized ghost-classical gauge fields. This algebra is generated by the operators d, s, $\bar{s}$, i_ξ, $i_{\bar{\xi}}$ by mean of the commutator [, } which is graded by the sum g+L of the Lorenz degree L and ghost number g. Preliminary work[4] seems to indicate that this formalism is in fact also well adapted when local supersymmetry is present, and its aplication to the gauge fixing problem in curved space will be presented elsewhere.[10]

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