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PERTURBATIVE GAUGE THEORIES†

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Abstract

We study the quantization of gauge theories by giving a central role to the BRS symmetry. We analyze as far as possible the case of a general gauge theory. Algebraic as well as geometrical properties of the BRS symmetry are discussed. The problems of anomalies, renormalizability, gauge independence and unitarity in Yang-Mills theories is carefully examined.

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I - INTRODUCTION

Gauge invariance plays a fundamental role in physics. The introduction of gauge theories, i.e. of theories invariant under local field transformations, is justified from the localized field concept as has been pointed out by Yang and Mills[1]. If one considers interactions which are invariant under a set of global transformations endowed with a group structure, the assumption that space-time is homogeneous makes it natural to explore the possibility of performing these transformations independently at different space time locations without any influence on observables. Following this principle, physicists attempt to associate with any given global symmetry of physical relevance a gauge invariant Lagrangian and/or a gauge covariant closed system of equations of motion. In that way general relativity is introduced as a theory which is associated with local Lorentz invariance[2] and the phenomenological electro-weak-strong interaction models as theories associated with local invariance under the action of a compact Lie group.[3] The latter theories are non trivial generalizations of electrodynamics and have also the important property of being renormalizable in four dimensional flat space-time[4,5,6]; they allow for consistent and unambiguous computations at any given finite order of pertrubation theory. Furthermore, the concept of gauge invariance has yielded the lattice gauge theory formulation which plays a fundamental role in the analysis of many non perturbative phenomena in statistical physics as well as in particle physics[7].

All gauge theories are plagued with problems which originate in the existence of zero modes within the classical Lagrangian and/or the classical equations of motion [8]. Let us recall the usual way a gauge theory is constructed. One starts from geometrical equations which express the action of a given gauge symmetry on a certain set of classical fields. Here "classical" means that the fields are assumed to satisfy the physical spin-statistics relation. All these fields transform tensorially under the action of the given global symmetry and the Lorentz symmetry. Then, one constructs a Lagrangian which is a gauge invariant local function of the classical fields. As soon as this function is specified one can identify some of the field components as non physical modes, i.e. as zero modes of the Lagrangian expanded to second order in the fields. The existence of such zero modes is indeed a direct consequence of the full gauge invariance of the Lagrangian. It turns out that, in general, one cannot express in a Lorentz covariant way the theory in terms of physical modes alone, that is to say solely in terms of modes orthogonal to zero modes. This impossibility leads to numerous difficulties. In particular, one encounters the following problem when one tries to quantize the theory. In order to define perturbation theory one must add a gauge fixing term to the Lagrangian to render invertible its quadratic field approximation. In this way the unphysical modes acquire a well defined propagator. But the general existence in the Lagrangian of interactions which depend on zero modes imply an effective propagation of these unphysical modes as virtual intermediary quantum states, and one may fear a violation of unitarity.

In order to handle these problems and make compatible the existence of zero modes and the physical requirement of unitarity, Dirac has built a method which applies to a large class of gauge theories within the framework of canonical quantization [8]. The method consists of making a prescription for quantizing constrained Hamiltonian systems and therefore also gauge theories which are very often examples of such systems. Then Faddeev and Popov have transposed Dirac's method to the functional integral formalism, thereby obtaining an explicitly Lorentz covariant quantization method for a gauge invariant Lagrangian [10,11,12]. The resulting formalism is by now very familiar to particle physicists. One obtains at the end a Lagrangian whose quadratic approximation degeneracy has been removed through the addition of a gauge fixing term. In addition the Lagrangian contains terms depending on new fields, known as ghosts, whose statistics is in general unphysical and whose role is to ensure that physics is gauge invariant in spite of the fact the full Lagrangian is not. Observe that in the Faddeev-Popov approach, the introduction of ghosts is merely a technical device to express the modification of the functional integral measure required to compensate the gauge dependence introduced by the gauge fixing term [10,13]. The great success of this method is that it has determined in the particular case of Yang-Mills theories with a linear gauge condition a Lagrangian

which is a correct starting point for perturbation theory. Thereby it has permitted a consistent construction of these theories [4,6]. If one wishes however to consider the prescription of Dirac-Faddeev-Popov as a general method for quantizing a gauge theory, one discovers that, even in perturbation theory, it contains loop-holes, some of which will be analysed later in this article. Moreover the construction of Faddeev and Popov is characterized by a succession of formal manipulations which contrast severly with the simplicity and beauty of the gauge invariance principle of the classical physics. Such complications may appear as unnatural and undesired when one goes from classical to quantum physics and our purpose here is to describe a technique which allows one to quantize gauge theories in a more direct way and to bypass the technical difficulties encountered with the use of the Faddeev-Popov method.

In order to realize this program we present in this article the quantization of perturbative gauge theories in a rather unorthodox way. As a matter of fact we shall abandon completely the method of Faddeev and Popov and adopt stricto sensu Feynman's original idea [14] that ghosts, i.e. fields with unphysical statistics, are truly necessary in all cases to compensate for the effects due to the quantum propagation of zero modes which are contained in any classical gauge field. Then we shall postulate that the fundamental fields for a gauge theory are classical fields <u>and</u> associated ghosts and that the full quantum Lagrangian is a local function depending a priori on these fields. In this approach it is clear that one must find a way to express the gauge symmetry in a form which involves both the classical and ghost fields. It turns out that the so-called BRS symmetry[6,15] originally discovered by Becchi, Rouet and Stora as a providential invariance of the familiar Faddeev-Popov Lagrangian in Yang Mills theories, can be extended to the case of a general gauge theory in a way which is independent of the notion of a Lagrangian. In fact, given any gauge symmetry, one can always build a set of transformation laws which act on the classical and ghost fields and represent in an intrinsic way the gauge symmetry at the quantum level[16]. As a consequence, one is lead to the natural strategy of determining as a first step the BRS symmetry, and only afterwards the full quantum Lagrangian as the most general Lagrangian function of ghosts and classical fields which is invariant under the BRS symmetry. This procedure naturally mirrors at the quantum level the strategy employed to determine the classical Lagrangian from the principle of gauge invariance [17].

At first sight the method may appear to be a mere formal improvement of the well known method of Faddeev and Popov. In fact it is not so. Not only does it provide a conceptual simplification in the formalism but it also permits in practice a straightforward derivation of the quantum Lagrangian in cases for which the usual method is unapplicable. Therefore in this new approach one can handle problems which would remain otherwise unsolved. In particular the method clarifies the role played

by those quartic ghost interactions[18] which are for instance necessary in Yang Mills theory to ensure renormalizability in certain gauges but are in contradictions with the Faddeev-Povov ansatz[17,19]. Going beyond the case of Yang-Mills theories, it can be applied to solve quantization problems such as those involving generalized p-form tensor gauge fields[20] for which the Faddeev-Popov method is unconsistent[21,22], and it has been recently generalized to incorporate the effects of a gravitational backgroudd[28,29].

In this article, Yang-Mills theories play the role of prototype. In their particular case since there is no possibility for disentangling the ghosts and unphysical modes in any gauge independent way, the intuitive idea of including the ghosts as a part of gauge field can be pushed further. Indeed, one can build a formalism where the classical and ghost field components are unified and put together as components of a single exterior 1-form. In order to do so one enlarges space-time by adjunction of new, but unphysical directions[17,23,24,25,26]. In this way the BRS equations can be geometrically interpreted: they express the vanishing of curvature components along the unphysical directions, and one finds that all field combinations with a physical (i.e. gauge independent) relevance are those which are left invariant under parallel transport along the unphysical directions[26]. Moreover this formalism allows for an interpretation of the gauge independence theorem of physics as a formal extension of Stokes theorem in the enlarged space and it makes much more transparent than in previous studies the discussion of the anomaly problem[26,27]. It also reduces the problem of classifying the possible anomalies to a problem similar to that of determining gauge invariant classical Lagrangians in higher dimension. It is important to note that the approach which one follows within this unified formalism is very different in nature from that of Faddeev and Popov. It somehow trivializes the quantization pattern by enlarging the phase space of dynamical variables to a larger space containing both real and Grassman variables; in contrast the approach of Dirac, i.e. of Faddeev Popov, consists of reducing the phase-space of classical dynamical variables down to a gauge dependent sub-space[8]. Quite interestingly this unified formalism has been recently generalized in curved space and can be applied to theories involving gravity and possibly local supersymmetry[28,29]. It appears therefore as quite general. On the other hand the mathematical status of this formalism, even in the simplest case of Yang Mills theories in flat space, is still unclear since a deeper understanding would require further knowledge of the concept of a fiber bundle made with Grassman variables in the "vertical" directions[30]. Nevertheless the efficiency and the aesthetic appeal of the unified formalism are suggestive enough to justify its presentation in this article, and we consider these qualities a further indication that the BRS symmetry is a fundamental concept and not just a convenient but mysterious invariance only useful for technical purposes.

To summarize we want to present the perturbative quantization of gauge theories by raising to the level of a principle the BRS invariance in its modernized form. At each step of the construction we shall analyse as far as possible the case of a general gauge symmetry. Concerning renormalization we shall give special attention to the problem of anomalies which is of a fundamental nature and is very often left aside when reviewing gauge theories.

In Section II we demonstrate the equivalence between the formulation of a general gauge symmetry either from the standard transformation laws of classical fields or from the BRS equations. Then specifying to the Yang-Mills case we display the unified formalism in wich the classical gauge fields and ghosts are assembled into a single exterior 1-form in an enlarged space-time.

In Section III, we present a general method for building BRS invariant Lagrangians function of classical and ghost fields. We demonstrate that such Lagrangians are in general gauge fixed. We apply the method to the Yang-Mills case in 4 dimension and give in that case the most general BRS and Lorentz invariant renormalizable Lagrangian. This Lagrangian contains in general quartic ghost interactions but we recover in a special case the Lagrangian given by the method of Faddeev and Popov.

In Section IV, we analyse the consequences of the BRS invariance for the Green functions evaluated in the tree approximation. We construct both Ward identities corresponding to BRS and anti-BRS invariances. Because we use in our formalism the auxiliary field that Stueckelberg introduced for QED and that Zinn-Justin[19], Kugo, Nakamishi, Ojima[31,32] have used for the particular Yang-Mills case, the Ward identities have no explicit dependence on any of the parameters which specify the Lagrangian. It turns out that the introduction of this auxiliary field in the formalism permits the restoration of the symmetry between the ghost and the anti-ghost and gives an interpretation to the BRS Slavnov operator as an intrinsic differential operator. We then display a general but heuristic proof that the BRS invariance of the Lagrangian leads to the gauge independence of physics. One may consider this proof as an a posteriori justification of the principle of BRS invariance. In the particular case of Yang Mills theories, we also display another heuristic argument which allows one to interpret the gauge independence of physics as a straightforward consequence of Stokes theorem in the enlarged space in which the classical and ghost fields are unified.

In Section V, we show a general method for inverting the Ward identities when they are satisfied by a local functional. In the case of a theory which is renormalizable by power counting and anomaly free, this method yields striking simplifications in the determination of the renormalized Lagrangian.

In Section VI, we analyse renormalization in Yang-Mills theories. We give special attention to the anomaly problem, and distinguish as far as possible between the problems which are of a purely algebraic nature and those which only concern power counting arguments. We show that the problem of classifying the anomalies, which seems a priori a pure quantum problem, can be reduced in fact to a classical physics problem. Finally we give a straightforward proof that Yang-Mills theories are multiplicatively renormalizable if one can use a regulator which preserves the BRS invariance.

In Section VII, we present a proof of the gauge independence and unitarity of the physical part of the renormalized S-matrix in spontaneously broken Yang-Mills theories. This proof generalizes the tree level arguments but necessitates the use of a regulator which preserves the BRS invariance. It is a refinement of the original demonstration of t'Hooft and Veltman.

II. CLASSICAL AND QUANTUM ASPECTS OF A GAUGE SYMMETRY

In sub-section (II-1) we review the usual definitions and properties of classical gauge transformations and BRS transformations for a Yang Mills system. In sub-section (II-2) we consider a general gauge theory with a "closed" algebra of infinitesimal gauge transformations satisfying the Jacobi identity. We demonstrate the existence of an associated extended BRS algebra and give an algorithm for constructing this algebra. In sub-section (II-3) we display the formalism which allows one to unify the Yang Mills ghosts and classical fields.

II.1 YANG-MILLS THEORY

II.1a Notations

Let G be a compact Lie Group of rank r with Lie Algebra g. Any group elements can be expanded in the neighbourhood of the identity as

$$g = 1 + i \sum_{a=1}^{r} \epsilon^a T_a + o(\epsilon^2)$$

where the ϵ^a are infinitesimal real parameters and the r generator Ta are squared matrices whose dimension depends on the choosen representation. The Ta satisfy the closure relation

$$[T_a , T_b] \equiv T_a T_b - T_b T_a = i f_{ab}^{c} T_c \qquad (2.1)$$

and the Jacobi identity

$$[T_a , [T_b , T_c]] + \text{cyclic perm.} = 0 \qquad (2.2)$$

true for any unitary representations. The numbers f_{ab}^{c} are the structure coefficients of the group and they characterize it locally.

In the adjoint representation the generators Ta are r x r matrices and their matrix elements $(Ta)^b c$ can be identified with the f_{ac}^{b} when the group is semi-simple. In a general representation R the Ta are p x p matrices with matrix elements $T_{a\,j}^{i}$ $(1 \leq i, j \leq p)$. The indices i(resp. j) label a basis of the corresponding representation space V(R). The closure and Jacobi relations (2.1) and (2.2) can be written as follows, directly in terms of $T_{a\,j}^{i}$ and f_{ac}^{b}.

$$T_{a\,k}^{i} T_{b\,j}^{k} - T_{b\,k}^{i} T_{a\,j}^{k} = f_{ab}^{c} T_{c\,j}^{i} \qquad (2.3a)$$

$$f_{ae}^{d} f_{bh}^{e} - f_{be}^{d} f_{ah}^{e} = f_{ab}^{c} f_{ch}^{d} \qquad (2.3b)$$

When the group is semi-simple the $T_{a\,j}^{i}$ and f_{ab}^{c} satisfy the trace condition $\sum_b f_{ab}^{b} = 0$ and $\sum_i T_{ai}^{i} = 0$. If it contains abelian factors the former relation remains true but not the latter one.

II.1.b Yang Mills classical gauge algebra

The classical dynamical variables in Yang-Mills theories are classified into two categories, matter fields Ψ and gauge fields A_μ.

The matter fields Ψ are either boson fields (Ψ_b) or Fermion fields (Ψ_F). They take their values in an arbitrary representation space V(R). Under infinitesimal gauge transformation Ψ transforms as

$$\delta_\epsilon \Psi = \epsilon \Psi \qquad (2.4a)$$

where the infinitesimal parameter ϵ takes its values in the Lie Algebra g and is a function of space-time variables x^{μ}. In eq. (2.4a) and throughout this article we use the very convenient notation which consist in assembling into a single matrix the set of components of any g-valued function. Then $\epsilon \equiv \epsilon^a T_a$ $(1 \leq a \leq r)$, and eq.(2.4a) reads in components as follows:

$$\delta_\epsilon \Psi^i = \epsilon^a T^i_{aj} \Psi^j \tag{2.4b}$$

The vector gauge fields A^a_μ $(1 \leq a \leq r)$ are g-valued and can be assembled into the matrix $A_\mu \equiv A^a_\mu T_a$. The infinitesimal gauge transformations of A are defined as

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon] \equiv (\partial_\mu \epsilon^c + \epsilon^a f^c_{ab} A^b_\mu) T_c \tag{2.5}$$

Because ϵ is a local function, none of field derivatives $\partial_\mu \Psi$ and $\partial_{[\mu} A_{\nu]}$ transform tensorially under gauge transformations. However, by introducing the covariant derivative

$$D_\mu \equiv \partial_\mu + A_\mu = \partial_\mu + A^a_\mu T_a \tag{2.6}$$

the following field strenghts can be built, $D_\mu \Psi = \partial_\mu \Psi + A_\mu \Psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu) T_a$ which transform in a tensorial way under the action of gauge transformations.

$$\delta_\epsilon (D_\mu \Psi^i) = \epsilon D_\mu \Psi^i = \epsilon^a T^i_{aj} (\partial_\mu \delta^j_K + T^j_{aK} A^a_\mu) \Psi^K$$

$$\delta_\epsilon (F_{\mu\nu}) = [\epsilon, F_{\mu\nu}] = \epsilon^b f^a_{bc} F^c_{\mu\nu} T_a \tag{2.7}$$

As a matter of fact, $D_\mu \Psi$ and $F_{\mu\nu}$ are the only functions of A_μ and Ψ which take their values in the same representations as Ψ and A respectively and are of the first degree in the field derivatives $\partial_\mu \Psi$ and $\partial_{[\mu} A_{\nu]}$ while transforming tensorially under the gauge transformations.

The g-valued field strenght $F_{\mu\nu}$ satisfies the following identity, which follows from the Jacobi identity in g and from $\partial_{[\mu}\partial_{\nu]} = 0$

$$[D_\mu, D_\nu] X = F_{\mu\nu} X \tag{2.8}$$

Eq. (2.8) is valid for any function X valued in any representation space of g. The Bianchi identity is also satisfied

$$D_{[\mu} F_{\nu\rho]} = 0 \tag{2.9}$$

Therefore, all local properties of classical Yang Mills transformations can be summarized as follows:

$$\delta_\epsilon A_\mu = D_\mu \epsilon \tag{2.10a}$$

$$\delta_\epsilon \Psi = \epsilon \Psi \tag{2.10b}$$

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']} \tag{2.10c}$$

$$[\delta_\epsilon, [\delta_{\epsilon'}, \delta_{\epsilon''}]] + \text{cyclic perm.} = 0 \tag{2.10d}$$

Eqs. (2.10a,b) define the gauge transformations on the gauge fields and matter fields. Eqs. (2.10c) and (2.10d) express respectively the closure and the Jacobi identity of the set of gauge transformations δ_ϵ which build up therefore an algebra. The latter equations apply to any tensorial combination of A_μ and Ψ.

Conversely, up to trivial overall rescalings of structure coefficients f^c_{ab}, the system of infinitesimal gauge transformations (2.10a,b) is the only one which can be parametrized by the infinitesimal parameters $\varepsilon(x)$ and their derivatives and acts on the fields A and Ψ while satisfying the closure and Bianchi relations (2.10c,d).

II.1.c Yang-Mills BRS algebra

Let $c = c^a(x)T_a$ and $\bar{c} = \bar{c}^a(x)T_a$ be g-valued scalar ghosts, i.e. anticommuting scalar boson fields. In Yang Mills theories, they were introduced by Feynman and B.S. de Witt[14] in order to compensate the unwanted effects due to unphysical modes contained in A_μ. The compensation between the ghosts and the classical zero modes requires that one introduces a symmetry between these fields which must be respected by all interactions. This symmetry, called BRS symmetry, is determined by two graded generators s and $\bar{s}$. By definition, the action of s and $\bar{s}$ on $A_\mu, c, \bar{c}$ is such that

$$\begin{array}{ll} s(A_\mu) = D_\mu c & \bar{s}(A_\mu) = D_\mu \bar{c} \\ s(\Psi) = -c\,\Psi & \bar{s}(\Psi) = -\bar{c}\,\Psi \\ s(c) = -\frac{1}{2}[c,c] & \bar{s}(\bar{c}) = -\frac{1}{2}[\bar{c},\bar{c}] \end{array}$$

$$s\,\bar{c} + \bar{s}\,c = -[\bar{c},c] \qquad (2,11a)$$

Eqs. (2.11a) leave $s(\bar{c})$ undetermined. Thus, in addition to A_μ, $c, \bar{c}$, one needs an auxiliary scalar commuting boson field $b = b^a T_a$ which takes its values in g. The action of s and $\bar{s}$ on $c, \bar{c}, b$ is

$$\begin{array}{ll} s(\bar{c}) = b & \bar{s}(c) = -[\bar{c},c] - b \\ s(b) = 0 & \bar{s}(b) = -[\bar{c},b] \end{array} \qquad (2.11b)$$

s and $\bar{s}$ are called BRS and anti-BRS operators respectively.

We shall postulate that A, c, $\bar{c}$, b, Ψ are the fundamental fields of the quantum Yang Mills theory. In d-dimensional space time, the field dimension of A, c, $\bar{c}$, Ψ_b is $d/2 - 1$ in mass units. It is $d/2 - 1/2$ for Ψ_F and $d/2$ for b. Besides, an additive ghost number is defined for all fields: it is zero for all classical fields A, Ψ_b, Ψ_F and for the auxiliary field b, $+1$ for c and -1 for $\bar{c}$. The ghost number of a product of fields is the sum of the ghost numbers of the fields. The field b deserves to be called auxiliary since it has dimension 2 in d=4 dimensions and thereby cannot acquire kinetic energy in any renormalizable Lagrangian. This means that the net number of degrees of freedom carried by the quartet of fields A^a_μ, c^a, $\bar{c}^a$, b^a is in fact d-2 (i.e. 1 degree of freedom for each space time component of A^a_μ, -1 for c^a or $\bar{c}^a$ owing to their unphysical statistics[14], and 0 for b^a).

The action of s and $\bar{s}$ on any function of fields is given by the graded Leibnitz rule:

$$\begin{array}{ll} s(XY) = (sX)\,Y \pm X\,sY & s\partial_\mu = \partial_\mu s \\ \bar{s}(XY) = (\bar{s}X)\,Y \pm X\,\bar{s}Y & \bar{s}\partial_\mu = \partial_\mu \bar{s} \end{array} \qquad (2.12)$$

where the minus sign occurs if X contains an odd number of ghosts and anti-ghosts. Therefore s and $\bar{s}$ can be truly interpreted as graded derivatives, the grading being identified as the Lorentz rank plus the ghost number[6,29]. One can trivially verify that the action of $s(\bar{s})$ upon any function of fields increases its dimension by one unit and increases (decreases) its ghost number by one unit.

Using the Jacobi and closure relations and also the composition law (2.12) one can check that on all elementary fields, and thus on any function of the fields, the s and $\bar{s}$ operators satisfy the nilpotency property

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0 \tag{2.13}$$

Conversely, supposing that s and $\bar{s}$ are graded derivatives carrying dimension 1 and ghost number 1 and -1, and that they satisfy the nilpotency eq. (2.13), one can show that the form of s and $\bar{s}$ is necessarily that given in eq.(2.11), up to overall rescaling factors. We shall demonstrate this important theorem in section (V.2) by using dimensional arguments.

Historically the s operator has been discovered by Becchi, Rouet and Stora[6,15] as leaving invariant the Faddeev-Popov Lagrangian, explaining thereby the Ward Identities used by t'Hooft and Veltman[4]. However, the original form of s given by Becchi, Rouet, Stora is slightly different from that given in eq.(2.11) because these authors didn't use the auxiliary field b[32]. Because of that, the geometrical nature of s was not obvious to understand. The anti-BRS operator $\bar{s}$ was discovered later , as an additional symmetry of Faddeev-Popov Lagrangian[18].

Under the form (2.11,12) the s and $\bar{s}$ operators are independent of the notion of a Lagrangian. We could now postulate that the quantum Yang-Mills theory is determined from the most general Yang Mills and Lorentz group scalar Lagrangian which is function of fields $A, c, \bar{c}, b, \Psi$ and is invariant under the

BRS transformations (2.11). Later on, this postulate will determine the Lagrangian without any ambiguity and a consistent perturbative construction will follow. In this method the nilpotency (2.13) of BRS operators s and $\bar{s}$ plays an essential role from a technical point of view. However, at this stage of our presentation, one might consider such a strategy as too formal. Thus, before applying it, we shall elucidate the link between the usual Yang-Mills symmetry definition (2.10) and the BRS equations (2.11). As a first step we shall demonstrate that the existence of nilpotent BRS and anti-BRS operators is a general feature shared by all gauge theories whose infinitesimal transformations build up a closed algebra with a Jacobi identity. Here the term "closed algebra" must be taken in a generalized sense, because the corresponding structure coefficients can be field dependent. Afterwards, going back to the Yang-Mills case, we shall give an interpretation of BRS operators s and $\bar{s}$ which allows one to identify them as exterior differential operators along unphysical directions.

II. 2-EXTENDED BRS SYMMETRY FOR A GENERAL GAUGE THEORY

In this sub-section we shall construct the nilpotent BRS algebra that one can associate with any given classical symmetry defined in the usual way by infinitesimal generalized gauge transformations of which Yang-Mills theories are a particular case[16].

Consider a set of classical fields (i.e. fields satisfying the physical spin-statistics relation) ϕ^i which are subject to the following gauge transformations

$$\delta_\epsilon \phi^i = R^i_\alpha(\phi)\, \epsilon^\alpha(x) \tag{2.14}$$

Here the indices i and α label internal symmetry as well as Lorentz symmetry indices. The $\epsilon^\alpha(x)$ are all the possibly local infinitesimal parameters which are necessary for defining a generalized infinitesimal gauge transformation. The "generators" R^i_α are in general functions of fields ϕ^i. For example, in the Yang-Mills case, the fields ϕ^i run over the set of gauge fields A^a_μ and matter fields ψ^i, and the generators R^i_α stand for $\partial_\mu \delta^a_b + f^a_{bc} A^c_\mu$ and $T^\kappa_{a\ell}\, \psi^\ell$.

The set of transformations (2.14) generates a possibly "field dependent algebra" by assumption. This means that one has on all fields

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi^i \equiv (\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\phi^i = R^i_\gamma f^\gamma_{\alpha\beta}\, \epsilon^\alpha_1 \epsilon^\beta_2 \tag{2.15}$$

where the structure coefficients $f^\gamma_{\alpha\beta}$ are in general functions of fields ϕ^i. In the Yang-Mills case the $f^\gamma_{\alpha\beta}$ are simply the structure coefficients of a compact Lie group, that is to say pure numbers.

For consistency, the relation (2.15) must be compatible with the Jacobi identity for commutators.

$$[\delta_{\epsilon_1}, [\delta_{\epsilon_2}, \delta_{\epsilon_3}]] + \text{cyclic perm.} = 0 \tag{2.16}$$

The above closure and Jacobi identities (2.15), (2.16) imply constraints on R^i_α and $f^\alpha_{\beta\gamma}$. The latter can be made explicit by using the definition (2.14) into eqs. (2.15), (2.16). At first sight, the relations that one obtains look

horrible. However, they can be reexpressed in an extremely simple way by using a very convenient trick to introduce new parameters $c^\alpha_1, \ldots, c^\alpha_m, \ldots$, each parameter c^α_m being in a one to one correspondence with ϵ^α, with the same covariance in α as for ϵ^α, but the opposite statistics. The commutation relations satisfied by the c^α_m variable are therefore as follows

$$c^\alpha_m\, c^\beta_{m'} = -(-)^{g(\alpha)g(\beta)}\, c^\beta_{m'}\, c^\alpha_m \tag{2.17}$$

where $g(\alpha)$ and $g(\beta)$ are the Lorentz gradings of ϵ^α and ϵ^β defined from the relation $\epsilon^\alpha \epsilon^\beta = (-)^{g(\alpha)g(\beta)} \epsilon^\beta \epsilon^\alpha$.

Using the definition (2.14), one can write the "closure" relation (2.15) as

$$R^i_{\alpha,j} R^j_\beta - (-)^{g(\alpha)g(\beta)} R^i_{\beta,j} R^j_\alpha = -R^i_\gamma f^\gamma_{\alpha\beta} \tag{2.18}$$

where $R^i_{\alpha,j} \equiv \frac{\delta}{\delta\phi^j} R^i_\alpha$. One can easily check by inspection over all possible cases that eq. (2.18) is equivalent to

$$R^i_{\beta,j} R^j_\alpha\, c^\alpha c^\beta = \frac{1}{2} R^i_\gamma f^\gamma_{\alpha\beta}\, c^\alpha c^\beta \tag{2.19}$$

where all gradings appearing in eq. (2.18) have been automatically taken into account.

In the same way the Jacobi identity can be simply written as

$$(f^\alpha_{\beta\gamma,i} R^i_\lambda - f^\alpha_{\rho\gamma} f^\rho_{\lambda\beta})\, c^\lambda c^\beta c^\gamma = 0 \tag{2.20a}$$

and also under the equivalent form when one uses two different variables c_1^α and c_2^α

$$\Big(-f^\alpha_{\mu\rho,i}\, R^i_\lambda - \tfrac{1}{2}(-)^{g(\rho)(g(\lambda)+g(\mu))} f^\alpha_{\lambda\mu,i}\, R^i_\rho + \tfrac{1}{2} f^\alpha_{\sigma\rho} f^\sigma_{\lambda\mu} - f^\alpha_{\lambda\sigma} f^\sigma_{\mu\rho}\Big)\, c_1^\lambda c_1^\mu c_2^\rho = 0 \qquad (2.20b)$$

One may note that $f^\alpha_{\beta\gamma}$ is always symmetric or antisymmetric in β, γ, but in general there is no symmetry betwen the upper index α and the lower ones β, γ. The trace relation $\sum_\alpha f^\alpha_{\alpha\beta} = 0$ is however very often realized, and will be shown to be a necessary condition in order to build an invariant functional integral measure for the fields.

Eqs. (2.19), (2.20) will shortly be shown to lead directly to the full nilpotency of the BRS symmetry that we shall now construct.

Let us consider the set of independent fields made from the classical fields $\phi^i(x)$ and the pairs of ghost fields $c^\alpha(x)$ and $\bar{c}^\alpha(x)$ whose tensorial properties and statistics are identical with those of variables c^α_m introduced just above. Therefore $c^\alpha(x)$ and $\bar{c}^\alpha(x)$ satisfy the same commutation relations as c^α_m and $c^\beta_{m'}$ in eq.(2.17). Then we define the extended BRS symmetry by the action of its generators s and $\bar{s}$ on the set of fields of $\phi, c, \bar{c}$ as follows

$$\begin{aligned} s(\phi^i) &= R^i_\alpha c^\alpha & \bar{s}(\phi^i) &= R^i_\alpha \bar{c}^\alpha \\ s(c^\alpha) &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma & \bar{s}(\bar{c}^\alpha) &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} \bar{c}^\beta \bar{c}^\gamma \end{aligned}$$

$$s(\bar{c}^\alpha) + \bar{s}(c^\alpha) = -f^\alpha_{\beta\gamma} c^\beta \bar{c}^\gamma \qquad (2\text{-}21\,a)$$

Furthermore, in order to determine the value of $s\bar{c}^\alpha$ in eq.(2.21a) we introduce an auxiliary field $b^\alpha(x)$. Thereby we can complete eq. (2.21a) by the following equations

$$\begin{aligned} s(\bar{c}^\alpha) &= b^\alpha & \bar{s}(c^\alpha) &= -b^\alpha - f^\alpha_{\beta\gamma} \bar{c}^\beta c^\gamma \\ s(b^\alpha) &= 0 & \bar{s}(b^\alpha) &= -f^\alpha_{\beta\gamma} \bar{c}^\beta b^\gamma - \tfrac{1}{2} R^i_\delta f^\alpha_{\beta\gamma,i}\, c^\delta \bar{c}^\beta c^\gamma \end{aligned} \qquad (2.21b)$$

The b^α field has the same covariance in α as the ghosts c^α and $\bar{c}^\alpha$. For consistency its statistics is the physical one

$$b^\alpha b^\beta = (-)^{g(\alpha)g(\beta)} b^\beta b^\alpha \qquad (2.22)$$

We wish to interpret s and $\bar{s}$ as linear differential operators graded by the ghost number (0 for the classical fields ϕ^i and b^α, 1 and -1 respectively for c^α and $\bar{c}^\alpha$). Thus, we define the following composition law for the action of s and $\bar{s}$ on arbitrary functions of fields $\phi, c, \bar{c}, b$

$$\begin{aligned} s(XY) &= (sX)Y \pm X\, sY & s\partial_\mu &= \partial_\mu s \\ \bar{s}(XY) &= (\bar{s}X)Y \pm X\, \bar{s}Y & \bar{s}\partial_\mu &= \partial_\mu \bar{s} \end{aligned} \qquad (2.23)$$

As in the Yang-Mills case the minus sign occurs if there is an odd number of ghosts c and $\bar{c}$ in X.

In eq. (2.21) $\bar{s}(b^\alpha)$ is equal to $-\bar{s}(f^\alpha_{\beta\gamma} \bar{c}^\beta c^\gamma)$. This can be verified by using the definitions of $\bar{s}\phi$, $\bar{s}c$, $\bar{s}\bar{c}$, the composition law (2.23) and the Jacobi identity (2.16).

We shall now demonstrate that the graded operators s and $\bar{s}$ (2.21) satisfy the following full nilpotency relation on all functions of fields ϕ, c, $\bar{c}$, b

$$s^2 = s\bar{s} + \bar{s}\,s = \bar{s}^2 = 0 \qquad (2.24)$$

as a consequence of "closure" property eq(2.15) and Jacobi relation (2.16).

As a matter of fact, one needs only to check eq. (2.24) when it acts on the elementary fiels ϕ, c, $\bar{c}$ and b owing to the composition law (2.23). Furthermore eq. (2.24) is automatically satisfied for b, provided it holds true for ϕ, c, $\bar{c}$. Let us prove this result as a first step.

Indeed sb=0 implies trivially $s^2 b = 0$. Now, since $\bar{s} b^\alpha = -\bar{s}(f^\alpha_{\beta\gamma}\bar{c}^\beta c^\gamma)$, it is obvious that $\bar{s}^2 b^\alpha = 0$ if $\bar{s}$ is nilpotent on the fields ϕ, c, $\bar{c}$. The verification of $(s\bar{s} + \bar{s}s)$ b = 0 follows from the following chain of equalities

$$\begin{aligned}(s\bar{s} + \bar{s}s) b^\alpha &= s\bar{s} b^\alpha = s\bar{s}(s\bar{c}^\alpha) \\ &= -s^2(\bar{s}\bar{c}^\alpha) = \frac{1}{2} s^2(f^\alpha_{\beta\gamma}\bar{c}^\beta\bar{c}^\gamma) = 0\end{aligned} \qquad (2.25)$$

where we have used sb = 0, $s\bar{c}$ = b and the relations $(s\bar{s} + \bar{s}s)\bar{c} = s^2\bar{c} = 0$.

The hard part of the problem thus reduces to proving eq. (2.24) when it acts on other fields than b. In fact the b field has no geometrical meaning since it has been introduced only to remove the degeneracy of equation $s\bar{c}^\alpha + \bar{s}c^\alpha = -f^\alpha_{\beta\gamma}\bar{c}^\beta c^\gamma$ without spoiling the relation $s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$. On the other hand the b field appears as necessary to connect the sectors with ghost numbers 1 and -1.

Now we shall prove the complete nilpotency relations for the fields ϕ, c, $\bar{c}$. We start from $s^2\phi$. Eqs. (2.21a) and (2.23) yield

$$s^2\phi^i = s(R^i_\alpha c^\alpha) = (R^i_{\alpha,j} R^j_\beta - \frac{1}{2} R^i_\gamma f^\gamma_{\beta\alpha}) c^\beta c^\alpha \qquad (2.26)$$

But the r.h.s term vanishes due to the gauge algebra closure relation under the form (2.19). One has therefore $s^2\phi = 0$. Consider now $s^2 c$. One has

$$s^2 c^\alpha = -\frac{1}{2} s(f^\alpha_{\beta\gamma} c^\beta c^\gamma) = -\frac{1}{2}(f^\alpha_{\beta\gamma,i} R^i_\lambda - f^\alpha_{\rho\gamma} f^\rho_{\lambda\beta}) c^\lambda c^\beta c^\gamma \qquad (2.27)$$

Therefore $s^2 c = 0$ due to the Jacobi identity under the form (2.20). Similarly one has $\bar{s}^2\phi = 0$, $\bar{s}^2\bar{c} = 0$. Let us proceed with $s^2\bar{c}$. It is trivially zero since $s\bar{c}$ = b and sb = 0. Furthermore, $\bar{s}^2 c = 0$ since we have defined $\bar{s}c^\alpha = -b^\alpha - f^\alpha_{\beta\gamma}\bar{c}^\beta c^\gamma$ and $\bar{s} b^\alpha = -\bar{s}(f^\alpha_{\beta\gamma}\bar{c}^\beta c^\gamma)$.

It remains to prove that all fields ϕ, c, $\bar{c}$ are annihilated by the action of $s\bar{s} + \bar{s}s$. A straightforward computation gives

$$\begin{aligned}s\bar{s}\phi^i &= R^i_{\alpha,j} R^j_\beta c^\beta \bar{c}^\alpha + R^i_\alpha b^\alpha \\ \bar{s}s\phi^i &= -(-)^{g(\alpha)g(\beta)} R^i_{\beta,j} R^j_\alpha c^\beta \bar{c}^\alpha \\ &\quad - R^i_\beta b^\beta - R^i_\beta f^\beta_{\alpha\gamma} \bar{c}^\alpha c^\beta\end{aligned} \qquad (2.28)$$

Consequently the closure relation (2.18) is equivalent to $(s\bar{s} + \bar{s}s)\phi = 0$. Next consider $(s\bar{s} + \bar{s}s)\bar{c}$. It can be written as follows

$$\begin{aligned}(s\bar{s} + \bar{s}s)\bar{c}^\alpha = (&-f^\alpha_{\mu\rho,i} R^i_\lambda - \frac{1}{2}(-)^{g(\rho)(g(\lambda)+g(\mu))} f^\alpha_{\lambda\mu,i} R^i_\rho \\ &+ \frac{1}{2} f^\alpha_{\sigma\rho} f^\sigma_{\lambda\mu} - f^\alpha_{\lambda\sigma} f^\sigma_{\gamma\rho})\bar{c}^\lambda \bar{c}^\mu c^\rho\end{aligned} \qquad (2.29)$$

and it vanishes owing to the Jacobi identity(2.20b). Analogously, $(s\bar{s} + \bar{s}s)c = 0$, and we have finally proven the nilpotency relation (2.24) in full generality.

The conclusion of our analysis is clear. Starting from a set of infinitesimal gauge transformation building up a possibly field dependent algebra and satisfying the Jacobi identity, a nilpotent associated BRS algebra can always be built. The nilpotency of its generators s and $\bar{s}$ is equivalent to the closure relation when s, $\bar{s}$ act on classical fields, while it is equivalent to the Jacobi identity when s, $\bar{s}$ act on ghosts and anti-ghosts. Moreover the action of s and $\bar{s}$ on classical fields, is identical to classical gauge transformations up to the change of infinitesimal gauge parameters into ghosts. It is therefore equivalent to define the gauge symmetry either by the usual classical gauge equations (2.14) completed by the closure and Jacobi relations (2.15), (2.16) or by the complete set of BRS equations (2.21) completed by the nilpotency constraint (2.24).

In field theory, the second formulation is most convenient since it introduces at once all fields, including the ghosts, which are necessary for having a proper counting of modes in intermediary states and ensuring eventually the unitarity. When we shall build a Lagrangian, the presence of the anti-ghost $\bar{c}$, and that of the b field, will appear as necessary in order to obtain a gauge fixed and BRS invariant Lagrangian with ghost number zero. The b field will then be interpreted as a Lagrange multiplier for the gauge condition.

In practice the BRS equations (2.21) are completely determined from the knowledge of $R^i_\alpha(\phi)$ and of structure coefficients $f^\gamma_{\alpha\beta}(\phi)$. However, when one tries to build a gauge symmetry, only the R^i_α are usually known and the practical determination of $f^\gamma_{\alpha\beta}$ may be difficult in certain cases.

For this reason we shall display now a simple and systematic way for obtaining the complete set of BRS equations when one knows the classical gauge transformations but not the $f^\alpha_{\beta\gamma}$. The method consists in replacing as a first step all infinitesimal parameter which determine a general infinitesimal gauge transformation by ghost fields. This provides the BRS equations for classical fields. Afterwards, one can compute step by step the BRS transformation of ghosts by imposing the nilpotency on all fields. If the starting classical transformations do correspond to a "closed" symmetry with a Jacobi identity, the process would end up after a finite number of steps and yields the whole set of BRS equations. In that way the $f^\alpha_{\beta\gamma}$ can be directly read from the transformation laws of ghosts. As a matter of fact, the method of imposing the nilpotency can be considered as a constructive method for building gauge transformations which are "closed" and satisfy a Jacobi relation (see ref(21) where we have used this technique to find gauge transformations involving non Abelian skew tensors). In order to see how this reverse construction works in practice, let us consider first the simplest Yang Mills case. Let us assume that we have inferred that the definition of a gauge transformation is $\delta_\epsilon A_\mu = D_\mu\epsilon \equiv \partial_\mu\epsilon + [A_\mu,\epsilon]$, but that we are too lazy to recognize that $\delta_\epsilon\delta_{\epsilon'} - \delta_{\epsilon'}\delta_\epsilon = \delta_{[\epsilon,\epsilon']}$, where $[X,Y]^a \equiv F^a_{bc}X^bY^c$.

The reverse method consists in substituting a ghost field $c^a(x)$ in place of $\epsilon^a(x)$. Then one obtains the BRS transform of A_μ

$$\delta A^a_\mu = D_\mu\epsilon^a \longrightarrow s A^a_\mu = D_\mu c^a \equiv \partial_\mu c^a + [A_\mu,c]^a \tag{2.30}$$

It is immediate to compute $s^2 A^a_\mu$, keeping sc^a as an unknown function

$$s^2 A^a_\mu = [sA_\mu, c]^a + D_\mu(sc^a) = [D_\mu c, c]^a + D_\mu(sc^a)$$
$$= D_\mu\left(sc^a + \tfrac{1}{2}[c,c]^a\right) \qquad (2.31)$$

Then the constraint $s^2 A_\mu = 0$ yields directly the value of sc^a, $sc^a = -\frac{1}{2}[c,c]^a$, and the structure coefficients of the algebra are thereby identified as the F^a_{bc} . One must also check $s^2 c^a = 0$ in order to be consistent. In fact this requirement implies that the F^a_{bc} satisfy a Jacobi identity, and are thus Lie algebra structure coefficients . Finally, after having determined sA_μ and sc, it is straightforward to complete the BRS equations under the form (2.11) by introducing the fields $\bar{c}$ and b.

As a much less trivial exercise, one can consider the following mixed Abelian and non Abelian gauge transformations whose role in N =1, d=10 supergravity has been pointed out by Chapline and Manton[33)]

$$\delta_{\epsilon,\epsilon_\mu} a_{\mu\nu} = \partial_{[\mu}\epsilon_{\nu]} + T_R(\partial_{[\mu} A_{\nu]}\,\epsilon)$$
$$\delta_{\epsilon,\epsilon_\mu} A_\mu = D_\mu \epsilon \qquad (2.32a)$$

Here $a_{\mu\nu}$ is an uncharged anti-symmetric tensor and A^a_μ a non Abelian vector Yang-Mills field . ϵ_μ and ϵ^a are respectively vector real and scalar g-valued infinitesimal parameters. In this example, the structure coefficients are non trivial and field dependent. In addition to the usual non Abelian scalar ghosts c and $\bar{c}$ of A_μ, the ghost spectrum involves two Lorentz

vector ghosts V_μ and $\bar{V}_\mu$ and three Lorentz scalar ghosts m,n,$\bar{m}$ with no Yang Mills charge which are associated with $a_{\mu\nu}$[22)].

The complete set of BRS equations corresponding to transformations (2.32) has been determined in ref.(22) by using the above mentionned reverse construction. For the sake of notational simplicity we shall only display here the expression of the BRS operator s that one obtains on the classical fields and ghosts V_μ, m and c

$$s\, a_{\mu\nu} = \partial_{[\mu} V_{\nu]} + T_R(c\, \partial_{[\mu} A_{\nu]}) \qquad s A_\mu = D_\mu c$$
$$s\, V_\mu = \partial_\mu m + T_R(c\, c\, A_\mu) \qquad s\, c = -\tfrac{1}{2}[c,c]$$
$$s\, m = -\tfrac{1}{3} T_R(c\,c\,c) \qquad (2.32b)$$

One has $s^2=0$ and one can check $[\delta_{\epsilon,\epsilon_\mu}, \delta_{\epsilon',\epsilon'_\mu}] = \delta_{\epsilon'',\epsilon''_\mu}$ with $\epsilon'' = [\epsilon,\epsilon']$ and $\epsilon_\mu = T_R(A_\mu[\epsilon,\epsilon'])$.

II.3 UNIFIED GHOST AND CLASSICAL FIELD FORMALISM

In the previous sub-section we have shown that the BRS equations for a general gauge symmetry can be thought of as an expression of the gauge symmetry when it acts on ghosts and classical fields, In this section we come back to the particular Yang Mills case. There, the generator of gauge transformations is the covariant derivative and can be interpreted geometrically as defining a parallel transport within physical space-time.

By enlarging space-time with additional, but non physical coordinates[17,23,24,25,26)], we shall in fact demonstrate that the BRS operators s and $\bar{s}$ can be given a sense as exterior differential opeators along these unphysical directions. In this formalism, the concept of space-time exterior derivative and BRS operator is unified and furthermore the ghosts seem to play the role of a connection along the unphysical directions. Generalizing the construction of the covariant derivative $D=d+A$ from the usual space-time exterior derivative d, we shall build from s and $\bar{s}$ the "covariant" BRS operators $S = s+c$ and $\bar{S} = \bar{s}+\bar{c}$ which should be interpreted as operators of parallel transport along the unphysical directions, i.e. as covariant derivatives along the unphysical direction. Intuitively enough, the functions of fields which are of a physical relevance turn out to be those which are left invariant under the action of S and $\bar{S}$[26]. The resulting formalism obviously enlightens the geometrical nature of the BRS transformations and transforms the concept of gauge invariance into that of invariance under displacement along unphysical directions[26]. From a practical point of view, it will also greatly simplify the resolution of equations which determine the possible form of the anomalies.

The main idea is therefore to enlarge the d-dimensional space-time $\{x^\mu\}$ up to an enlarged space P with local coordinates $\{x^\mu,\theta,\bar{\theta}\}$, by adding a pair of Grassmanian coordinates $\theta,\bar{\theta}$ at each point x^μ of M. Because the $\theta,\bar{\theta}$ coordinates are chosen Grassmanian, they should be counted negatively, and one may consider the effective dimension of P as d-2. This suggests that the P space can be used for a Lorentz covariant description of the transverse plane of dimension d-2 in which a massless gauge particle is forced to live, since it must propagate along the light cone.

It is natural to consider over each point $(x^\mu,\theta,\bar{\theta})$ the cotangent space $\mathcal{P}$ in order to introduce differential forms and

gauge fields. To introduce the Yang-Mills field and its ghosts one simply defines in $\mathcal{P}$ a general 1-form $\tilde{A}(x,\theta,\bar{\theta})$ which takes its values in the lie algebra g. In local coordinates, $\tilde{A}$ reads as

$$\tilde{A}(x,\theta,\bar{\theta}) = A_\mu(x,\theta,\bar{\theta})\,dx^\mu + A_\theta(x,\theta,\bar{\theta})\,d\theta + A_{\bar{\theta}}(x,\theta,\bar{\theta})\,d\bar{\theta} \tag{2.33}$$

and one identifies respectively the gauge field A, the ghost c and anti-ghost $\bar{c}$ as

$$A = A_\mu dx^\mu \qquad c = A_\theta d\theta \qquad \bar{c} = A_{\bar{\theta}}\, d\bar{\theta} \tag{2.34}$$

The differential operator $\tilde{d}$ in $\mathcal{P}$ is

$$\tilde{d} = d + s + \bar{s} \tag{2.35a}$$

with

$$d = dx^\mu \frac{\partial}{\partial x^\mu} \qquad s = d\theta \frac{\partial}{\partial \theta} \qquad \bar{s} = d\bar{\theta}\frac{\partial}{\partial\bar{\theta}} \tag{2.35b}$$

With these definitions, the nilpotency of s and $\bar{s}$, $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2 = 0$, and the commutation relations $sd+ds = 0 = \bar{s}d+d\bar{s}$ are automatically satisfied from the fundamental exterior calculus rule $\tilde{d}^2=0$. The anti-commutation properties of the ghosts are also automatically enforced: they are exterior 1-forms by definition and therefore anti-commute between themselves, and with the gauge field 1-form $A = A_\mu dx^\mu$.

One can then build exterior products of the basic fields $A, c, \bar{c}$. The ghost number of fields and products of fields can be naturally defined by counting algebraically the number of $d\theta$ and $d\bar{\theta}$ forms which are contained in the fields with a weight 1 for $d\theta$ and -1 for $d\bar{\theta}$. The sum of the Lorentz rank and the ghost number of any object determines its commutation properties. Observe that all fields are functions of $\theta, \bar{\theta}$ in addition to the x dependence. The link with the usual fields and equations occuring in field theory will be made by setting $\theta = \bar{\theta} = 0$, after having performed all the relevant derivations with respect to the variables θ and $\bar{\theta}$.

One defines from $\tilde{A}$ the g-valued exterior 2-form field strength $\tilde{F}$

$$\tilde{F} = \tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = \tilde{d}\tilde{A} + \tilde{A}_\wedge \tilde{A} \qquad (2.36)$$

where $\wedge$ is the wedge product in $\mathcal{S}$. By expansion of $\tilde{F}$ in local coordinates, one gets six components on the basis of 2-forms $dx^\mu_\wedge dx^\nu$, $dx^\mu_\wedge d\theta, \ldots, d\theta_\wedge d\bar{\theta}, d\bar{\theta}_\wedge d\bar{\theta}$ (one should observe that $(d\theta)^a{}_\wedge (d\bar{\theta})^b \neq 0$ for all values of the integers a and b owing to the grassmanian properties of θ and $\bar{\theta}$).

$\tilde{F}$ satisfies the Bianchi identity

$$\tilde{D}\tilde{F} = 0 \iff \tilde{d}\tilde{F} = -[\tilde{A}, \tilde{F}] \qquad (2.37)$$

as a consequence of the relation $\tilde{d}^2 = 0$ and of the Jacobi relation for the Lie Brackett $[\ ,\ ]$. Here we have defined

$$\tilde{D} = \tilde{d} + \tilde{A} \qquad (2.38a)$$

and it is relevant to expand $\tilde{D}$ in ghost number

$$\tilde{D} \equiv D + S + \bar{S}$$

$$D \equiv d + A \qquad S \equiv s + c \qquad \bar{S} = \bar{s} + \bar{c} \qquad (2.38b)$$

The similarity between D,S and $\bar{S}$ appears clearly by writing their action on any quantity X taking its values in g or in any representation space V(R) of g. One has $DX^i = dX^i + T_{\alpha j}^{i} A^\alpha X^j$, $SX^i = sX^i + T_{\alpha j}^{i} c^\alpha X^j$ and $\bar{S}X^i = \bar{s}X^i + T_{\alpha j}^{i} \bar{c}^\alpha X^j$ where the matrix elements $T_{\alpha j}^{i}$ have been defined in section (II.2). An intuitive interpretation of the S and $\bar{S}$ operators will be given shortly.

The important observation is that the BRS equations (2.11a) are simply recovered by imposing the following "horizontality" condition for $\tilde{F}$

$$\tilde{F} = \frac{1}{2}\tilde{F}_{\mu\nu}\, dx^\mu{}_\wedge dx^\nu \equiv F = dA + \frac{1}{2}[A, A]. \qquad (2.39a)$$

Indeed, expanding eq. (2.39a) in ghost number, one gets 5 equations which express the vanishing of the components of $\tilde{F}$ along the unphysical (i.e. vertical) directions

$$\tilde{F}_{\mu\theta}\, dx^\mu{}_\wedge d\theta = sA + Dc = 0 \qquad \tilde{F}_{\mu\bar{\theta}}\, dx^\mu{}_\wedge d\bar{\theta} = \bar{s}A + D\bar{c} = 0$$

$$\tilde{F}_{\theta\theta}\, d\theta_\wedge d\theta = sc + \frac{1}{2}[c, c] = 0 \qquad F_{\bar{\theta}\bar{\theta}}\, d\bar{\theta}_\wedge d\bar{\theta} = \bar{s}\bar{c} + \frac{1}{2}[\bar{c}, \bar{c}] = 0$$

$$F_{\theta\bar{\theta}}\, d\theta_\wedge d\bar{\theta} = s\bar{c} + \bar{s}c + [\bar{c}, c] = 0 \qquad (2.39b)$$

and one can easily verify that these equations have an identical structure as eq.(2.11a).

It is not difficult to include in this formalism the matter fields. Such fields are 0-forms and thus have no ghosts. Their generalization in $\mathcal{G}$ is thus $\tilde{\Psi}(x,\theta,\bar{\theta}) \equiv \Psi(x,\theta,\bar{\theta})$ with a field strength $\hat{G}_1 = \hat{D}\hat{\Psi}$. The BRS equations (2.11b) for ψ can be recovered by imposing the following horizontality condition for $\tilde{G}_1$

$$\hat{G}_1 \equiv \hat{D}\hat{\Psi} = (D+S+\bar{S})\Psi = \hat{G}_\mu dx^\mu = G = D\Psi \tag{2.40a}$$

as can been seen by a straightforward expansion in ghost number

$$\hat{G}_\theta d\theta = s\Psi + c\Psi = 0 \qquad \hat{G}_{\bar{\theta}} d\bar{\theta} = \bar{s}\Psi + \bar{c}\Psi = 0 \tag{2.41b}$$

The BRS transformations of field strengths F and G_1 can be computed directly from the Bianchi identities $\hat{\tilde{D}}\hat{\tilde{F}}=0$ and $\hat{\tilde{D}}\hat{\tilde{G}}_1 = \tilde{F}\Psi$. Indeed, combining these identities with the constraints $\tilde{F} = F$ and $\tilde{G} = G$, one gets by expansion in ghost number

$$\hat{D}\hat{F} = DF = 0 \Rightarrow SF = 0 = \bar{S}F = 0 \Leftrightarrow \begin{cases} sF = -[c,F] \\ \bar{s}F = -[\bar{c},F] \end{cases}$$

$$\hat{D}\hat{G}_1 = \hat{D}G_1 = F\Psi \Rightarrow SG_1 = 0 = \bar{S}G_1 = 0 \Leftrightarrow \begin{cases} sG_1 = -cG_1 \\ \bar{s}G_1 = -\bar{c}G_1 \end{cases} \tag{2.42b}$$

It is striking that those objects which are of a physical relevance as true observable such as the matter fields Ψ and the field strengths F and G_1 are left invariant by the action of S and $\bar{S}$. It is in fact suggestive enough to interpret S and $\bar{S}$ as the covariant derivative operators along the unphysical directions θ and $\bar{\theta}$. Then one can understand the principle of BRS

invariance for determining the admissible Langrangians as expressing the invariance of physics under parallel transport along the unphysical direction $\theta,\bar{\theta}$.

Finally the commutation relations among the operators $D = d+A$, $S = s+c$ and $\bar{S} = \bar{s}+\bar{c}$ are trivial to establish in this formalism. Indeed, one has the identities

$$\tilde{F} = \tilde{D}\tilde{D} \qquad\qquad F = DD \tag{2.43}$$

from the Jacobi identity and from $\hat{d}^2 = 0$ and $d^2 = 0$. Then applying the BRS constraint $\tilde{F} = F$, eq.(2.39), one gets

$$\tilde{D}\tilde{D} = DD \tag{2.44a}$$

which gives by expansion in ghost number the commutation relations

$$S^2 = S\bar{S} + \bar{S}S = \bar{S}^2 = 0\,, \quad SD + DS = 0 = \bar{S}D + D\bar{S} = 0 \tag{2.44b}$$

These properties of $S,\bar{S},D$ are most useful for solving the equations which determine the anomaly[26]. They can also be verified directly from the transformation laws(2.39b) of all fields under S and $\bar{S}$.

This possibility of unifying the classical and ghost fields in the enlarged space and of determining in such a simple way the BRS symmetry equations can be generalized for other gauge theories than the Yang Mills one. In ref 26), it is shown how to find generally the BRS symmetry in the case of flat space theories involving p-forms gauge fields of rank higher than one, i.e. antisymmetric tensor gauge fields $a_{[\mu_1 \cdots \mu_p]}$ [20], and in references 28,29) it is shown how to incorporate the invariance under local diffemorphisms, i.e. the gravitational effects. As a particular example of this generalization, let us show how to obtain straighforwardly the BRS equations (2.32b); one simply sets the horizontality constraints $\tilde{F} = F$ and $\tilde{G}_3 = G_3$ for the Yang Mills field strengths $\tilde{F} = \tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A},\tilde{A}]$ and the 3-form field strength $\tilde{G}_3 = \tilde{d}\tilde{B}_2 + \mathrm{Tr}(\tilde{A}\tilde{d}\tilde{A} + \frac{2}{3}\tilde{A}\tilde{A}\tilde{A})$, where the ghosts of $a_{\mu\nu}$ have been unified into the gauge ghost generalized 2-form $\tilde{B}_2 = a_{\mu\nu} dx^\mu_\wedge dx^\nu + V_\mu dx^\mu + \bar{V}_\mu dx^\mu + m + m + \bar{m}$ 22).

The elegance and efficiency, of the unified formalism are certainly consequences of a geometrical structure which is still unknown. One may hope that a superfield formalism exists in which the full theory, and in particular the Lagrangian and the Feynman rules, would only be expressed in terms of a single generalized gauge field containing classical and ghost components, leading thereby to a true unification of the ghost and classical fields in field theory. Unfortunately no one has been able yet to build such a superfield formalism, in spite of several attempts[34]. The main problem is of course a deeper understanding of a quantum field theory with a dependence on the unphysical variables in $\theta, \bar{\theta}$.

As a hint for a possible geometrical interpretation of the unified formalism, note that the enlarged space-time $\{x^\mu, \theta, \bar{\theta}\}$ is the same for any given gauge theory which involves p-form gauge fields[26]. In that sense, the $\theta, \bar{\theta}$ variables seem to have a pure

kinematic interpretation, and it is tempting to interpret them as allowing for a covariant description of the restricted transverse space with d-2 dimension in which on-shell gauge particles do live. Tentatively, one may also understand the constraints of the type $\tilde{F}=F$ and $\tilde{G} = G$ as a set of differential equations in θ and $\bar{\theta}$, since one has by definition $s = d\theta\partial/\theta$ and $\bar{s} = d\bar{\theta}\partial/\bar{\theta}$. Such differential equations mean that the dependence of all fields on the unphysical coordinates θ and $\bar{\theta}$ can always be integrated out from their value at some point, say $\theta = \bar{\theta} = 0$. The integrability conditions of these equations, which mean that one must have $\tilde{d}^2=0$ for consistency, and thus $s^2 = s\bar{s}+\bar{s}s = \bar{s}^2=0$, are then equivalent to Bianchi identities[26]. As a matter of fact, the necessity of Bianchi identities determines the possible forms of the field strengths[26,29]. Furthermore, the rule that a field strength must generally be set as horizontal, simply expresses that it is a physical quantity which can have no $\theta, \bar{\theta}$ dependence and thus no component along the unphysical directions. Besides, for the Yang Mills field strength $\tilde{F}$, the constraint $\tilde{F} = F$ can also be interpreted as expressing the physicality of the horizontal space $\{x^\mu\}$ as compared to the unphysicality of the "vertical" space $\{\theta, \bar{\theta}\}$[23,30]. We believe that deeper knowledges on the notion of a fibber bundle with Grassmanian vertical coordinates should clarify these yet unprecise notions.

III- QUANTUM LAGRANGIAN

III.1 - INTRODUCTION

The usual starting point for building a quantum theory associated with a gauge symmetry is the introduction of a gauge invariant Lagrangian which only depends on classical fields. In general, gauge invariance implies the presence of zero modes in the field - quadratic approximation of the Lagrangian and thereby does not permit the construction of propagators defining perturbation theory from the classical Lagrangian alone. Then, for building a well defined perturbative expansion, one adds to the Lagrangian a gauge fixing term while one modifies the functional integral measure by a compensating Jacobian (the so-called Faddeev-Popov determinant) for ensuring formally that physics be gauge invariant. This approach is due to Faddeev and Popov, and is based on Dirac's quantization method for constrained classical systems of which gauge theories are very often examples[0,8,10]. There, the ghosts seem to play only a formal role: one only introduces them as a convenient set of anti-commuting variables which allows one to exponentiate the Faddeev-Popov determinant and obtain a polynomial Lagrangian best suited for perturbation theory. The Faddeev-Popov Lagrangian is bilinear in the ghosts fields by construction. Only at the end of this construction does one discover the BRS symmetry with a freshment surprise as an invariance which survives in a mysterious way the breaking of gauge invariance and the formal manipulations leading to the Faddeev-Popov Lagrangian,

but which is nevertheless fundamental for proving that physics is gauge independent despite the Lagrangian is not.

However, the Faddeev-Popov quantization method contains loop holes (even in perturbation theory) which are often left aside thanks to the comforting sensation that one may feel after having demonstrated that it leads to a consistent theory in 4 dimensional Yang Mills theories when one chooses a linear gauge condition.

The first loop hole that we shall mention is the unability of the method for explaining the possible presence in the Lagrangian of interaction terms which are quartic in the ghost fields. The latter cannot be introduced in the Faddeev-Popov method since the exponentiation of a determinant can generate only terms quadratic in the ghosts. But on the other hand these quartic interactions can be introduced in a way compatible with the BRS symmetry, and thus they are a priori admissible[17,18]. Moreover, a one loop computation shows explicitly that Yang-Mills theories do require quartic ghost interactions in order to be renormalizable if a non-linear gauge condition is chosen. The possibility of setting equal to zero the quartic ghost interactions in linear gauges appears then as a lucky accident of perturbation theory, which necessitates in fact to be justified by a rather technical proof (see Appendix B).

There are other loop holes in different gauge theories tnan Yang-Mills one. In supergravity, quartic ghost interactions have been shown to be necessary for insuring (formally) the S-matrix unitarity[12,35]. Moreover, theories which contain non abelian antisymmetric tensor gauge fields $a_{[\mu_1 \ldots \mu_p]}$ with a Lorentz rank $p > 1$ cannot be quantized within the Faddeev-Popov procedure. Indeed, the ghost spectrum of such theories is made generally of an odd number of ghosts[21,22,26,36] in contradiction with the prediction of Faddeev-Popov ansatz where the ghosts are introduced always into pairs.

In each of these cases, all technical difficulties can be by-passed by abandoning the Faddeev-Popov prescription and by determining the quantum Lagrangian from the requirement of its BRS invariance, the latter symmetry being determined geometrically, in a way independent of the notion of a Lagrangian as shown in SectionII. In that way, the role of quartic ghost interactions gets clarified[16,17] and one also finds the correct ghost spectrum in theories involving antisymmetric tensors[21,22,26,29].

As a consequence, since on the one hand the BRS equations appear as a way for defining the gauge symmetry which involves at once all fields necessary at the quantum level, and since on the other hand it allows one to raise ambiguities which are inherent to the Faddeev-Popov method, it should appear natural to postulate BRS invariance as a general principle for building the quantum theory associated with any given gauge invariance.

According to this principle, we shall present a method for building the quantum Lagrangian in a general gauge theory. Afterwards we shall specify to the case of Yang-Mills theories and construct the most general group and Lorentz scalar Lagrangian which is BRS invariant and renormalizable by power counting.

III.2 · GENERAL GAUGE THEORY

Consider the general gauge theory of Section II.2. The problem is the construction of a gauge fixed action with ghost number zero which is a local function of fields ϕ, c, $\bar{c}$ and b, and invariant under the s and $\bar{s}$ transformations (2.21)[16].

Let us introduce the following Lagrangian

$$\mathcal{L}_Q(\Phi) = \mathcal{L}_{cl}(\phi) + \alpha\, s\bar{s}(P(\Phi)) + \bar{\beta}\, s(\bar{K}(\Phi)) + \beta\, \bar{s}(K(\Phi)) \tag{3.1}$$

where $\mathcal{L}_{cl}(\phi)$ only depends on the classical fields ϕ and is such that the action $\int d^dx\, \mathcal{L}_{cl}(\phi)$ is gauge invariant . P,K, $\bar{K}$ are local polynomials of all fields $\Phi = (\phi, c, \bar{c}, b)$, with ghost number 0,1, -1 respectively.

Since on the one hand s and $\bar{s}$ do act on classical fields as gauge transformations and since on the other hand they are fully nilpotent, $s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$, it is obvious that $\mathcal{L}_Q$ is s invariant if $\beta = 0$, $\bar{s}$ invariant if $\bar{\beta} = 0$ and simultaneously s and $\bar{s}$ invariant if $\beta = \bar{\beta} = 0$. In order $\mathcal{L}_Q$ to be hermitian, i.e. invariant under the exchange $c \Leftrightarrow -\bar{c}$, one must have in general that $\beta = \bar{\beta} = 0$. Conversely there is yet no proof that the expression (3.1) is the most general solution of equations $\int d^dx\, s\mathcal{L}_Q = 0$ and/or $\int d^dx\, \bar{s}\mathcal{L}_Q = 0$. Such a result, if true, would correspond to a beautiful theorem of cohomology theory for the s and $\bar{s}$ operators. It has been verified in Yang-Mills theories for d = 2,3,4 by a brute force inspection over all possible field polynomials[17]. It is only conjectured true for d >4.

The propagators which stem from $\mathcal{L}_{cl}$ are not all defined because the quadratic field approximation of $\mathcal{L}_{cl}(\phi)$ is degenerated, i.e. contains zero modes associated to the gauge invariance

$$\delta_\epsilon \phi^i = R^i_\alpha(\phi)\, \epsilon^\alpha(x) \tag{3.2}$$

On the other hand the propagators can be made invertible if a gauge fixing term of the following type

$$\frac{1}{2}\left(R^i_\alpha\, O_{ij}\, \phi^j \right)^2 \tag{3.3}$$

is added to $\mathcal{L}_{cl}$. Here Oij is a tensor built from numbers and derivatives.

We shall show as a first step that expression (3.1) precisely allows for a gauge fixing term such as the one in (3.3) when $\beta=\bar{\beta}=0$. Indeed, consider the following s and $\bar{s}$ invariant Lagrangian where we have made explicit a part of P in eq. (3.1)

$$\mathcal{L}_Q = \mathcal{L}_{cl} + \frac{1}{2} s\bar{s}\left(\phi^i O_{ij}\phi^j + \bar{c}^\alpha c^\alpha\right) \tag{3.4}$$

Using eqs. (2.21) one gets

$$\frac{1}{2} s\bar{s}\left(\phi^i O_{ij}\phi^j + \bar{c}^\alpha c^\alpha\right) = -\frac{1}{2}(b^\alpha)^2 + b^\alpha R^i_\alpha O_{ij}\phi^j + \ldots \tag{3.5}$$

where the dots contain no b field dependence, but ghost interactions, including quartic ones which arise from the expansion of $(\bar{s}c^\alpha)(s\bar{c}^\alpha)$. Thus, by eliminating the b field through its algebraic equation of motion $b^\alpha = R^i_\alpha O_{ij}\phi^j$, one recovers the gauge fixing term (3.3).

Consequently, in any gauge theory one can obtain a gauge fixed Lagrangian which is simultaneously s and $\bar{s}$ invariant but quartic ghost interactions arise in general.

Next, by relaxing the $\bar{s}$ invariance constraint (i.e. $\bar{\beta} \neq 0$), we shall show that one can reproduce the same result as that which is given by the procedure of Faddeev and Popov. Consider a gauge function $G_\alpha(\phi)$, for instance $G_\alpha = O_{ij} R^i_\alpha \phi^j$. The Lagrangian that one would obtain in the Faddeev and Popov way is

$$\mathcal{L}_{FP} = \mathcal{L}_{cl} - \frac{1}{2}(G_\alpha)^2 - \bar{c}^\alpha \frac{\delta G_\alpha}{\delta \phi^i} R^i_\beta c^\beta \tag{3.6}$$

On the other hand, among all s invariant Lagrangians of the type (3.1), let us consider the following one

$$\mathcal{L}_Q = \mathcal{L}_{cl} + s\left(\bar{c}^\alpha G_\alpha + \frac{1}{2}\bar{c}^\alpha b^\alpha\right) \tag{3.7}$$

which can be written as follows by using the definition (2.21) of s

$$\mathcal{L}_Q = \mathcal{L}_{cl} + \frac{1}{2}(b^\alpha)^2 + b^\alpha G_\alpha - \bar{c}^\alpha s(G_\alpha) , \tag{3.8}$$

Then, expression (3.8) gives back $\mathcal{L}_{FP}$ in (3.6) after elimination of the b field by its equation of motion. Eq. (3.7) shows that the Faddeev-Popov Lagrangian is in general not $\bar{s}$ invariant.

Furthermore, note that other terms, which are s invariant and thus a priori admissible, can be introduced such as $s\bar{s}\,(c^\alpha \bar{c}^\alpha)$ or $s\,(F_{\alpha\beta\gamma}\,\bar{c}^\alpha c^\beta \bar{c}^\gamma)$. As an example, radiative correction implies the introduction of terms of the latter type in Yang-Mills theories with a non linear gauge condition. Such terms are in contradiction with the prediction of Faddeev and Popov eq. (3.6). In the Yang-Mills case the structure coefficients $f^c_{\alpha b}$ are simplest, and one has $(b^a)^2 = s(\bar{c}^a b^a) = -\bar{s}(c^a b^a)$ and thus $s(b^2) = \bar{s}\,(b^2) = 0$. One should note however that in the general case the term $(b^\alpha)^2$ is trivially s invariant but not $\bar{s}$ invariant, as can be checked from the definition (2.21) of $\bar{s}\,b^\alpha$. Therefore, the only way in which one can obtain in general a term $(b^\alpha)^2$ while preserving both s and $\bar{s}$ symmetries is through the term $s\bar{s}\,(\bar{c}^\alpha c^\alpha)$, which implies the occurence of quartic ghost interactions.

III.3- YANG MILLS THEORY

In the Yang Mills case the fundamental fields are A, $c, \bar{c}, \psi$ and b.

We consider the renormalizable case, when space-time has dimension 4. In appendix A we show that the most general Lagrangian which is a local function of A, $c, \bar{c}$, b, and satisfies the following conditions:

(i) it is of dimension 4,

(ii) it is a Lorentz and Yang Mills group scalar,

(iii) it is fully BRS invariant under eqs. (2.11),

can be written under the canonical form[17]

$$\mathcal{L}_Q = \mathcal{L}_{c\ell} + s\bar{s}\,(\lambda_A A_\mu^2 + \lambda_c \bar{c}c + \lambda_B\ v\cdot\Psi_B) + \lambda_b b^2 \qquad (3.9a)$$

with

$$\mathcal{L}_{c\ell} = -\tfrac{1}{4}\alpha_A (F_{\mu\nu})^2 + \tilde{\alpha}_A\,\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \alpha_F\,\bar{\Psi}_F \not{D}\,\Psi_F - \tfrac{1}{2}\alpha_B (D_\mu \Psi_B)^2 + V(\Psi_B) \qquad (3.9b)$$

The proof is straightforward and works by inspection over all possible field monomials. Here $\alpha_A, \tilde{\alpha}_A, \alpha_F, \alpha_B, \lambda_A, \lambda_c, \lambda_B, \lambda_b$ are real arbitrary coefficients and all group indices are omitted. The potential $V(\Psi_B)$ is a Yang Mills group scalar with dimension 4, and v denotes the vacuum expectation value of scalar matter field Ψ_B. We have scaled away the gauge coupling constant g in the definition of structure coefficients f^b_{ac} and of matrix elements T^b_{ai}. The g dependence can be made explicit by a rescaling of all fields $\Phi \to g^{-1}\Phi$.

One should notice that b^2 is s and $\bar{s}$ invariant, (since $b^2 = s(\bar{c}b) = -\bar{s}(cb)$) but there is no function of fields from which one could obtain b^2 by action of $s\bar{s}$. As a consequence, when $\lambda_b \neq 0$ the Lagrangian (3.1) is not symmetric under the exchange $c \leftrightarrow -\bar{c}$.

$\mathcal{L}_{c\ell}$ is clearly distinguishable from the rest of $\mathcal{L}_Q$ since it is the most general s and $\bar{s}$ invariant Lagrangian which cannot be obtained from the action of s or $\bar{s}$ on any polynomial fields.

From now on we drop the second term in eq.(3.9b) which is a topological invariant, and is irrelevant in perturbative theory. The rescaling of parameters α and λ_A into field redefinitions is not a restriction. We can therefore rewrite $\mathcal{L}_Q$ under the following generic form

$$\mathcal{L}_Q = \mathcal{L}_{c\ell} - s\bar{s}\,(\tfrac{1}{2}(A_\mu)^2 - \lambda_c \bar{c}c - \lambda_B\, v\cdot\Psi_B) + \tfrac{1}{2}\lambda_b b^2$$

$$\mathcal{L}_{c\ell} = -\tfrac{1}{4}(F_{\mu\nu})^2 - \tfrac{1}{2}(D_\mu \Psi_B)^2 - \bar{\Psi}_F \not{D}\,\Psi_F + V(\Psi_B) \qquad (3.10)$$

where λ_c, λ_B, λ_b are independent real gauge parameters. Using eqs. (2.11), we may expend the $s\bar{s}$ terms as

$$\frac{1}{2} s\bar{s}(A_\mu^2) = A_\mu \partial^\mu b - \partial_\mu \bar{c} D^\mu c$$
$$s\bar{s}(\bar{c}c) = b^2 + b[\bar{c},c] + \frac{1}{2}[\bar{c},c]^2$$
$$s\bar{s}(v\cdot\Psi_B) = b\,[v,\Psi_B] + (\bar{c}v)(c\Psi_B) \qquad (3.11)$$

and express $\mathcal{L}_Q$ under a more conventional form

$$\mathcal{L}_Q = \mathcal{L}_{cl} + \frac{1}{2}(\lambda_b+\lambda_c)\,b^2 + b\left(\partial A + \lambda_B[v,\Psi_B] + \frac{\lambda_c}{2}[\bar{c},c]\right)$$
$$+ \partial_\mu\bar{c}\,D^\mu c + \lambda_B(\bar{c}v)(c\Psi_B) + \frac{\lambda_c}{4}[\bar{c},c]^2 \qquad (3.12)$$

Here the g-valued quantity $[v,\Psi_B]$ is defined in components as $[v,\Psi_B]_a \equiv T_{ai}^{\delta}\, v_\delta\, \Psi_B^{\,i}$.

In expression (3.12), we can see that the b dependence is purely algebraic. Thereby b can be eliminated away by using its equation of motion. Thus, by redefining the gauge parameters as $\lambda = \lambda_b + \lambda_c$, $\alpha = \lambda_c/2(\lambda_b+\lambda_c)$ and $\beta = \lambda_B/(\lambda_b+\lambda_c)$ we find that $\mathcal{L}_Q$ in eq. (3.12) is equivalent to the following Lagrangian

$$\mathcal{L}_Q = \mathcal{L}_{cl} - \frac{\lambda}{2}\left(\frac{\partial A}{\lambda} + \alpha[\bar{c},c] + \frac{\beta}{2}[v,\Psi_B]\right)^2 + \partial_\mu\bar{c}\,D^\mu c$$
$$+ \beta\lambda(\bar{c}v)(c\Psi_B) + \frac{1}{4}\alpha\lambda[\bar{c},c]^2$$
$$= \mathcal{L}_{cl} - \frac{(\partial A)^2}{2\lambda} + \frac{\lambda}{2}\alpha(1-\alpha)[\bar{c},c]^2 + (1-\alpha)\partial^\mu\bar{c}\,D_\mu c + \alpha(D_\mu\bar{c})\partial_\mu c$$
$$-\frac{\beta}{2}\partial A[v,\Psi_B] - \frac{\lambda\beta^2}{8}[v,\Psi_B]^2 + \beta\lambda\left((\bar{c}v)(c\Psi_B) - \frac{\alpha}{2}[\bar{c},c][v,\Psi_B]\right) \qquad (3.13)$$

while the equation of motion of b is

$$b = -\frac{\partial A}{\lambda} - \alpha[\bar{c},c] - \frac{\beta}{2}[v,\Psi_B] \qquad (3.14)$$

If we set $\alpha = 0$ within expression (3.13) we recover the same Lagrangian as the one which one would associate to $\mathcal{L}_{cl}$ by using the Faddeev-Popov ansatz with the gauge function $\mathcal{G} = -\frac{\partial A}{\lambda} - \frac{\beta}{2}[v,\Psi_B]$ .This corresponds to a maximal breaking of the symmetry $c \longleftrightarrow -\bar{c}$. When $\alpha = 1$ one has the corresponding symmetrical situation under the exchange $c \longleftrightarrow -\bar{c}$. When $\alpha \neq 0$ or $\alpha \neq 1$ the Lagrangian contains 4-ghost interactions. Moreover the strength of interactions between the ghost, anti-ghost and classical gauge fields is modulated by α . Only in the case $\alpha = 1/2$(corresponding to $\lambda_b = 0$ in eq. (3.9)) can we interpret $\bar{c}$ as the anti-particule of c.

In the limit $\lambda \to 0$ one can recover the Landau gauge since one has formally

$$\lim_{\lambda\to 0} \exp -\frac{(\partial A)^2}{2\lambda} \sim \delta(\partial A) \qquad (3.15)$$

In this limit the 4-ghost interactions disappear and the α-dependence of the Lagrangian becomes spurious when $\mathcal{L}_Q$ is inserted in the functional integral formula. Indeed, when $\lambda \to 0$ the only dependence in α of the Lagrangian is through the term $(1-\alpha)\partial_\mu\bar{c}\,D^\mu c + \alpha(D_\mu\bar{c})\partial_\mu c$ which is equal to $(\partial_\mu\bar{c})(D_\mu c) + (D_\mu\bar{c})(\partial_\mu c) - \alpha\,\partial A[\bar{c},c]$ up to a pure divergency, and the last term of the last expression is irrelevant because of equation (3.15).

The t'Hooft gauges are recovered when $\alpha = 0$ and $\beta = 1$. Then the off-diagonal terms $\partial A\,[v,\Psi_B]$ cancel with those stemming from the classical term $-1/2\,(D_\mu\Psi_B)^2$ up to a pure divergency and the quadratic approximation in fields of the Lagrangian becomes diagonal.

In appendix B it is shown that the values $\alpha = 0$ and 1 of the gauge parameters are not renormalized by radiative corrections, at any given finite order in perturbation theory. This non renormalization theorem is essential for justifying the method of Faddeev and Popov but it is only valid if a linear gauge condition is used.

Both Lagrangians (3.10) and (3.12) are equivalent up to the b field elimination. It is obvious that expression (3.1) is aestically more appealing than (3.12). Note that the expression of the BRS symmetry of Lagrangian (3.12) gets modified as a consequence of the b field elimination. The "modified" BRS symmetry of Lagrangian (3.12) can be simply obtained by replacing b in eqs. (2.12) by its equation of motion (3.14).

$$
\begin{aligned}
s A_\mu &= D_\mu c & \bar{s} A_\mu &= D_\mu \bar{c} \\
s \psi &= c\,\psi & \bar{s}\psi &= \bar{c}\,\psi \\
s c &= -\tfrac{1}{2}[c,c] & \bar{s}\bar{c} &= -\tfrac{1}{2}[\bar{c},\bar{c}] \\
s \bar{c} &= -\frac{\partial A}{\lambda} - \alpha[\bar{c},c] - \frac{\beta}{2}[\upsilon,\psi_\beta] & \bar{s} c &= \frac{\partial A}{\lambda} - (1-\alpha)[\bar{c},c] + \frac{\beta}{2}[\upsilon,\psi_\beta]
\end{aligned} \tag{3.16}
$$

Under this form, the nilpotency of s and $\bar{s}$ is partially broken by the equations of motion of c and $\bar{c}$ in Lagrangian(3.13). Indeed, one can check from eqs. (3.16)

$$
\begin{aligned}
s^2 A_\mu &= (s\bar{s} + \bar{s}s) A_\mu = \bar{s}^2 A_\mu = 0 \\
s^2 \psi &= (s\bar{s} + \bar{s}s)\psi = \bar{s}^2 \psi = 0 \\
s^2 c &= 0 \qquad \bar{s}^2 \bar{c} = 0
\end{aligned} \tag{3.17a}
$$

but

$$
\begin{aligned}
s^2 \bar{c} &= (s\bar{s} + \bar{s}s)\, c = \frac{1}{\lambda}\frac{\delta \mathcal{L}_Q}{\delta \bar{c}} \neq 0 \\
\bar{s}^2 c &= (s\bar{s} + \bar{s}s)\, \bar{c} = \frac{1}{\lambda}\frac{\delta \mathcal{L}_Q}{\delta c} \neq 0
\end{aligned} \tag{3.17b}
$$

At this point the advantage of postulating the BRS symmetry appears quite clearly. Had we started from the Faddeev-Popov Lagrangian (eq.(3.9) with $\alpha = 0$), the discovery of the symmetry (3.16) would have seemed miraculous. Moreover, the BRS equations under the form (3.16) depend on the gauge parameters and it would make their geometrical meaning quite obscure.

In what follows we shall analyse the Ward identities corresponding to the BRS invariance of the quantum Lagrangian. For this purpose, it is most advantageous to use the b field since it allows one to obtain Ward identities which are independent of all parameters specifying the Lagrangian. Thereby the Ward identities acquire an intrinsic, i.e. model independent, significance and the discussion of the anomaly problem becomes simplified as compared to other analysis which would use the Faddeev-Popov Lagrangian (3.12) as a starting point.

In appendix A II we briefly discuss the possible forms of a Lagrangian for space-time dimensions other than 4.

IV WARD IDENTITIES AND TREE APPROXIMATION GAUGE INDEPENDENCE

We have shown how to obtain BRS invariant and gauge fixed Lagrangians in a general gauge theory. Such Lagrangians determine Feynman rules with invertible propagators and permit an unambiguous diagrammatic expansion of tree level approximated Green functions of fields $\Phi = (\phi, c, \bar{c}, b)$ and of their BRS transforms, the local operators $s\Phi$, $\bar{s}\Phi$, $s\bar{s}\Phi$. We shall analyse the Ward identities between these Green functions by using the most convenient functional integral formalism[38,39].

We shall demonstrate that gauge fixed and BRS invariant Lagrangians of the type (3.1) with different gauge parameters lead to gauge independent physics. The forthcoming demonstration can be used for any gauge theory in which the S-matrix can be defined from LSZ formula, but it applies only in the tree approximation. Besides, in the particular case of Yang Mills theories, we shall also point out that the gauge independence theorem can be formally recovered as a generalized form of Stokes theorem in the enlarged space-time that we have introduced in section II.3. In Section VII, after the analysis of the problem of anomalies, we shall show how to extend the proof of gauge invariance when one takes into account the radiative corrections in the Yang Mills theory.

IV.1 GENERAL GAUGE THEORY

IV.1.a Ward Identities

The BRS invariance of $\mathcal{L}_Q$ leads to Ward identities among the Green functions of fields and BRS transforms of fields evaluated in the tree approximation. The latter are local operators and in order to take into account their insertions into Green functions, one builds the following effective Lagrangian

$$\mathcal{L}_{eff}(\Phi) = \mathcal{L}_Q(\Phi) + \bar{v}_\Phi(x)\, s\,\Phi(x) + v_\Phi(x)\,\bar{s}\,\Phi(x) + w_\Phi(x)\, s\bar{s}\,\Phi(x) \tag{4.1}$$

From now on, we shall adopt as a generic notation that Φ stands for all the fields $\phi, c, \bar{c}, b$, while $\bar{v}_\Phi$, v_Φ and w_Φ stand respectively for the sources of the complete set of BRS operators $s\Phi$, $\bar{s}\Phi$ and $s\bar{s}\Phi$. In this notation, one must be careful in order to avoid double countings between fields and BRS operators (since $s\bar{c} \equiv b$) and introduce not too many sources (since $sb = 0$ and $s\bar{s}b = 0$). We therefore choose the following basis of sources for operators

$$\bar{v}_\Phi\, s\Phi + v_\Phi\,\bar{s}\Phi + w_\Phi\, s\bar{s}\Phi \equiv \sum_{\Phi=\phi,c,\bar{c},b} v_\Phi\,\bar{s}\,\Phi + \sum_{\Phi=\phi,c}\left(\bar{v}_\Phi\, s\Phi + w_\Phi\, s\bar{s}\,\Phi\right) \tag{4.2}$$

while the sources of all fields ϕ, $c, \bar{c}, b$ are denoted by $J = (J_\phi, J_c, J_{\bar{c}}, J_b)$

Let W and Γ be respectively the generating functionals of connected and 1PI (one particle irreducible) Green functions of fields and BRS operators. In the tree approximation, W is given by the functional integral formula[0,39)]

$$\exp i\, W(J, v_\Phi, \bar{v}_\Phi, w_\Phi) = \int [d\Phi]\, \exp i \int d^dx \left\{\mathcal{L}_{eff}(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) + \Phi(x)J(x)\right\} \tag{4.3}$$

and $\Gamma(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi)$ is obtained from W by Legendre transformation on the couple of variables (J, ϕ).

$$\Gamma(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) \equiv W(J, v_\Phi, \bar{v}_\Phi, w_\Phi) - \int d^dx\, \Phi(x)\, J(x)$$
$$\Phi(x) \equiv \frac{\delta W}{\delta J(x)} \tag{4.4}$$

In equation (4.3) the functional integral measure $[d\Phi]$ is defined as

$$[d\Phi] = [d\phi][dc][d\bar{c}][db] \tag{4.5}$$

The classical measure $[d\phi]$ is gauge invariant by construction, and therefore is also s and $\bar{s}$ invariant. Then it is important to observe that the full quantum measure $[d\Phi]$ is s and $\bar{s}$ invariant only if the structure coefficients $f^{\alpha}_{\beta\gamma}$ satisfy the trace constraint

$$\sum_{\alpha} f^{\alpha}_{\alpha\beta} = 0 \tag{4.6}$$

This can be verified by a (formal) computation of the Jacobian of transformations (2.21). From now on we shall assume that the gauge symmetry is such that eq.(4.6) holds true.

In the tree approximation all the Ward identities which follow from the BRS symmetry of the Lagrangian amount to the following functional identities

$$\int [d\Phi] \int dy \left(s\Phi(y) J(y) + s\bar{s}\Phi(y) v_{\Phi}(y) \right) \exp i \int d^d x \left(\mathcal{L}_{eff} + \Phi(x) J(x) \right) = 0 \tag{4.7a}$$

$$\int [d\Phi] \int dy \left(\bar{s}\Phi(y) J(y) - s\bar{s}\Phi(y) \bar{v}_{\Phi}(y) \right) \exp i \int d^d x \left(\mathcal{L}_{eff} + \Phi(x) J(x) \right) = 0 \tag{4.7b}$$

These functional identities can be rewritten equivalently as constraints on the generating functionals W and Γ

$$\mathcal{S} W = 0 = \bar{\mathcal{S}} W \tag{4.8}$$

$$\mathcal{B}_{\Gamma} \Gamma = 0 = \bar{\mathcal{B}}_{\Gamma} \Gamma \tag{4.9}$$

where the differential linear operators $\mathcal{S}$, $\bar{\mathcal{S}}$ and $\mathcal{B}_{\Gamma}$, $\bar{\mathcal{B}}_{\Gamma}$ are defined as

$$\mathcal{S} = \int dx \left\{ \sum_{\Phi=\phi,c} \left(J_{\Phi}(x) \frac{\delta}{\delta \bar{v}_{\Phi}(x)} + v_{\Phi}(x) \frac{\delta}{\delta w_{\Phi}(x)} \right) + J_{\bar{c}}(x) \frac{\delta}{\delta J_b(x)} + v_{\bar{c}}(x) \frac{\delta}{\delta v_b(x)} \right\}$$

$$\bar{\mathcal{S}} = \int dx \left\{ \sum_{\Phi=\phi,c,\bar{c},b} \left(J_{\Phi}(x) \frac{\delta}{\delta v_{\Phi}(x)} \right) - \sum_{\Phi=\phi,c} \left(\bar{v}_{\Phi}(x) \frac{\delta}{\delta w_{\Phi}(x)} \right) \right\} \tag{4.10a}$$

and

$$\mathcal{B}_{\Gamma} = \int dx \left\{ \frac{1}{2} \sum_{\Phi=\phi,c} \left(\frac{\delta\Gamma}{\delta\Phi(x)} \frac{\delta}{\delta \bar{v}_{\Phi}(x)} + \frac{\delta\Gamma}{\delta \bar{v}_{\Phi}(x)} \frac{\delta}{\delta\Phi(x)} \right) + \sum_{\Phi=\phi,c} v_{\Phi}(x) \frac{\delta}{\delta w_{\Phi}(x)} + b(x) \frac{\delta}{\delta \bar{c}(x)} + v_{\bar{c}}(x) \frac{\delta}{\delta v_b(x)} \right\}$$

$$\bar{\mathcal{B}}_{\Gamma} = \int dx \left\{ \frac{1}{2} \sum_{\Phi=\phi,c,\bar{c},b} \left(\frac{\delta\Gamma}{\delta\Phi(x)} \frac{\delta}{\delta v_{\Phi}(x)} + \frac{\delta\Gamma}{\delta v_{\Phi}(x)} \frac{\delta}{\delta\Phi(x)} \right) - \sum_{\Phi=\phi,c} \bar{v}_{\Phi}(x) \frac{\delta}{\delta w_{\Phi}(x)} \right\} \tag{4.10b}$$

The most direct way for verifying the basic identities (4.7) is by performing the changes of variables $\Phi \to \Phi + \bar{\eta}\, s\Phi$ (for (4.7a)) and $\Phi \to \Phi + \eta\, \bar{s}\Phi$ (for (4.7b)) into eq. (4.3) and by using the invariance of $\mathcal{L}_Q$ and $[d\Phi]$ under s and $\bar{s}$ transformation. The resulting identities can be expanded linearly in function of the anti-commuting parameters η and $\bar{\eta}$ and yield directly eqs. (4.7). Then one trivially gets eq.(4.8) from eq. (4.7), and finally eqs. (4.9) by Legendre transformation. Using the expansion (4.2) for the source terms is essential in order not to get double countings in eqs. (4.10).

In order to obtain from these functional equations the Ward Identities between Green functions of fields and BRS operators, one has simply to differentiate functionally eqs. (4.8) with respect to J, v_{Φ}, $\bar{v}_{\Phi}$, w_{Φ} for connected Green functions and eqs. (4.9) with respect to Φ, v_{Φ}, $\bar{v}_{\Phi}$, w_{Φ} for IPI Green functions. The reader will have certainly noticed the slight and traditional notational abuse that we have done by using the same notation for the sources $\Phi_{c\ell}$ of fields Φ in Γ and for the fields themselves. In fact Γ should be written as $\Gamma(\Phi_{c\ell}, v_{\Phi}, \bar{v}_{\Phi}, w_{\Phi})$ with $\Phi_{c\ell} \equiv \frac{\delta W}{\delta J}$.

The operators $\mathcal{S}, \bar{\mathcal{S}}$ and $\mathcal{B}_\Gamma$, $\bar{\mathcal{B}}_\Gamma$ are linear differential functional operators. Their definition has no explicit dependence on any of the parameters which specify the Lagrangian which we have started from. This property holds true as long as one keeps the b field as an independent field, and in particular as far as b is not eliminated by its equation of motion. Another important property of $\mathcal{S}, \bar{\mathcal{S}}$ and $\mathcal{B}_\Gamma, \bar{\mathcal{B}}_\Gamma$ is that these operators are nilpotent. The operators $\mathcal{S}$ and $\bar{\mathcal{S}}$ satisfy algebraically the following relation on any functional of sources possibly non local

$$\mathcal{S}^2 = \mathcal{S}\bar{\mathcal{S}} + \bar{\mathcal{S}}\mathcal{S} = \bar{\mathcal{S}}^2 = 0 \tag{4.11}$$

while for any functional Γ of $\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi$, the Γ dependent differential operators $\mathcal{B}_\Gamma, \bar{\mathcal{B}}_\Gamma$ satisfy algebraically

$$\mathcal{B}_\Gamma \mathcal{B}_\Gamma \Gamma = (\mathcal{B}_\Gamma \bar{\mathcal{B}}_\Gamma + \bar{\mathcal{B}}_\Gamma \mathcal{B}_\Gamma)\Gamma = \bar{\mathcal{B}}_\Gamma \bar{\mathcal{B}}_\Gamma \Gamma = 0 \tag{4.12}$$

From these properties, one can attribute an intrinsic meaning to $\mathcal{S}$ and $\bar{\mathcal{S}}$ (resp. $\mathcal{B}_\Gamma$ and $\bar{\mathcal{B}}_\Gamma$), independent of the notion of a Lagrangian, as characterizing the action of the underlying gauge symmetry on functionals of $v_\Phi, \bar{v}_\Phi, w_\Phi$ in a way analogous as s and $\bar{s}$ characterize it on the fields Φ.

In the case of a theory renormalizable by power counting, such as a Yang Mills theory in 4 dimensions, the renormalization programme will consist in building a renormalized generating functional Γ_R , equal to Γ in the tree approximation, and satisfying the Ward identities (4.9) order by order in perturbation theory[4,6]. The algebraic constraint (4.12) will play a fundamental role in this construction

by allowing one to analyse properly the anomaly problem[6,39,40]. Before entering into these refinements, we wish to remain in the framework of a general gauge theory, not necessarily renormalizable by power counting, and show the general link between the Ward identities, i.e. the BRS symmetry of the Lagrangian, and the formal gauge independence of physical S-matrix elements.

IV.1.b <u>Gauge independence in the tree approximation</u>

Let us consider the "physical" S-matrix elements defined in the tree approximation by applying the LSZ reduction formula to Green functions whose arguments corresponds to physical classical field components. We shall demonstrate that these matrix elements are gauge independent in the sense that their value does not depend on any change of the gauge parameters α, β(or $\bar{\beta}$) which factor out in the non classical part of Lagrangian (3.1). This definition of gauge independence corresponds to the usual one, since s(resp. $\bar{s}$) invariant Lagrangians of the type (3.1) with different gauge fixing terms can only differ by an amount $\epsilon s \bar{K}(\Phi)$ (resp. $\epsilon \bar{s} K(\Phi)$) where ϵ is a real number and $\bar{K}(\Phi)$ (resp. $K(\Phi)$) is a local field polynomial with ghost number -1 (resp. +1).

The key identity which permits one to prove gauge independence is the following one, where one assumes that the parameter ϵ is infinitesimal

$$\int [d\Phi] \exp i \int (\mathcal{L}_Q + \epsilon s \bar{K}(\Phi) + \Phi(x) J(x))\, dx$$

$$= \int [d\Phi] \exp i \int \left(\mathcal{L}_Q + (\Phi(x) + \epsilon s \Phi(x) \int dy\, \bar{K}(\Phi(y))\right) J(x))\, dx \tag{4.13}$$

The easiest way to check eq.(4.13) is by changing the integration variables $\Phi \to \Phi + \bar{\eta}\epsilon s\Phi$ into the r.h.s. of the following expression:

$$<\int dy\, K(\Phi(y))> \equiv \int [d\Phi] \int dy\, K(\Phi(y))\, \exp i \int (\mathcal{L}_Q + \Phi(x) J(x))\, dx \qquad (4.14)$$

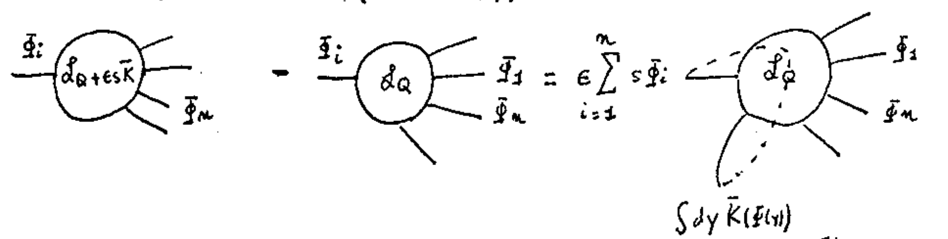
and by expanding the resulting identity linearly in $\bar{\eta}\epsilon$.

As a consequence of eq. (4.13) the variation of $\mathcal{L}_Q$ by the infinitesimal amount $\epsilon s \bar{K}(\Phi)$ is equivalent to the redefinition of the field source terms by the amount $\Phi(x)J(x) \to \Phi(x)J(x) + \epsilon s\Phi(x) \int dy\, \bar{K}(\Phi(y)) J(y)$.

Because of that, the value of a Green function $G_n := <T(\Phi_1, \cdots, \Phi_n)>$ where all the arguments $\Phi_p, 1 \leq p \leq n$, are chosen at different space-time locations, is not the same whether one computes G_n in the gauge determined from $\mathcal{L}_Q$ or from $\mathcal{L}_Q + \epsilon \bar{s} K$. Despite this fact we shall demonstrate that when all the arguments of G_n are classical fields set on shell by appication of LSZ reduction formula, the values of the corresponding physical S-matrix element are the same in both gauges.

Indeed, differentiate both sides of eq. (4.13) with respect to $J(x_1), \ldots, J(x_n)$. This permits one to identify the change of the Green function $<T(\Phi_1, \cdots, \Phi_i, \cdots \Phi_n)>$ when $\mathcal{L}_Q \to \mathcal{L}_Q + \epsilon s \bar{K}$ as the r.h.s. of the following diagrammatic equation

$$<T(\Phi_1, \cdots, \Phi_n)>^{\mathcal{L}_Q + \epsilon s \bar{K}} - <T(\Phi_1, \cdots, \Phi_n)>^{\mathcal{L}_Q} \equiv$$



(4.15)

Here the last diagram in the r.h.s. corresponds to the Green function obtained by substituting within $\langle(\Phi_1 \cdots \Phi_i \cdots \Phi_n)\rangle$ the operator $s\Phi_i(x_i) \int dy\, \bar{K}(\Phi(y))$ in place of Φ_i. Note that the ghost number + 1 which is carried by the operator $s\Phi_i$ is cancelled by that carried by the operator $\int dy\, \bar{K}(\Phi(y))$. This property is symbolized by the ghost line, drawn as a dashed line, in the r.h.s. ofeq.(4.15).

On the other hand, all arguments in G_n have been assumed as physical fields. As a consequence, the r.h.s. of eq.(4.15) vanishes when on-shell LSZ reduction formula projection operators are applied for obtaining the physical S-matrix elements. Indeed the LSZ operators cancel the s transforms of classical fields in all possible cases. It is so because the linear part of BRS transforms of classical gauge fields correspond to unphysical zero modes of the Lagrangian and are therefore orthogonal to physical sources by definition. Furthermore the remaining parts of the s transforms bf classical fields are made of local operators at least bilinear in fields. Thus they cannot develop a pole in perturbation theory. Thereby their IPI contribution to the r.h.s. of eq. (4.15) are also destroyed by the on shell LSZ operators. In the case of symmetry breaking, the s transform of matter fields may contain a part linear in fields and exhibit a pole, but this pole is always located at an unphysical ghost mass, and it is therefore also cancelled after application of the on-shell LSZ operators which by definition only extract poles located at the values of physical masses.

Therefore the above argument demonstrates that the Ward identity (4.13) implies the stability of any S-matrix element computed in the tree approximation either from the Lagrangian $\mathcal{L}_Q$ or from $\mathcal{L}_Q + \epsilon s \bar{K}$. The important point is that physics only depends on the part of the BRS invariant Lagrangian which is not s and/or $\bar{s}$ exact, i.e. which is not the s and/or $\bar{s}$ transform of any function of fields.

The generality of this proof which provides a direct link between the BRS invariance of the Lagrangian and the gauge independence of physics must be tempered however by the fact that infra-red problems can make completely formal the application of reduction formula. It is nevertheless an posteriori justification for postulating the BRS symmetry as the fundamental invariance at the quantum level.

IV.2 THE YANG MILLS CASE

IV.2.a Ward Identities

The Ward identities (4.8,(4.9) apply without modification to the Yang Mills case, where the classical fields stand for the gauge field A and the matter fields Ψ_F and Ψ_B. Since power counting theorems will be used later on, it is useful to give the dimensions in the relevant units of all sources of BRS operators. The source terms (4.2) of the effective Lagrangian read as follows

$$\begin{aligned} & v_A \bar{s} A + v_{\Psi_B} \bar{s} \Psi_B + v_{\Psi_F} \bar{s} \Psi_F + v_c \bar{s} c + v_{\bar{c}} \bar{s} \bar{c} + v_b \bar{s} b \\ & + \bar{v}_A s A + \bar{v}_{\Psi_B} s \Psi_B + \bar{v}_{\Psi_F} \bar{s} \Psi_F + \bar{v}_c s c \\ & + w_A s\bar{s} A + w_{\Psi_B} s\bar{s} \Psi_B + w_{\Psi_F} s\bar{s} \Psi_F + w_c s\bar{s} c \end{aligned} \tag{4.16}$$

The ghost number (G) and dimension in mass units (D) of fields and of sources of BRS transforms of fields are displayed within the following table where d stands for the dimension of space-time.

	v_A	v_{Ψ_B}	v_{Ψ_F}	v_c	$v_{\bar{c}}$	v_B
G	1	1	1	0	2	1
D	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2} - \frac{1}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2} - 1$

	$\bar{v}_A$	$\bar{v}_{\Psi_B}$	$\bar{v}_{\Psi_F}$	$\bar{v}_c$	w_A	w_{Ψ_B}	w_{Ψ_F}	w_c
G	-1	-1	-1	-2	0	0	0	-1
D	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2} - \frac{1}{2}$	$-\frac{d}{2}$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$	$\frac{d}{2} - \frac{3}{2}$	$\frac{d}{2} - 1$

As in the general case, the differential operators (4.10) do not depend explicitly on any of parameters which specify the Lagrangian (coupling constant, scalar field vacuum expectation value, gauge parameter). This property disappears as soon as one eliminates the auxiliary b field through its algebraic equation of motion. It is also important to observe that $\mathcal{D}, \bar{\mathcal{D}}, \mathcal{B}_\rho, \bar{\mathcal{B}}_\rho$ have no explicit dependence upon structure coefficients f^a_{bc}, nor on any numbers caracterizing a Yang Mills Algebra, but only on the Kronecker symbol δab.

IV.2.b Gauge independence in the tree approximation

The general argument of Section IV.1.b can be repeated. It will be improved later on in order to take into account the radiative corrections (Section VII).

In the remaining of this Section we shall give an alternative formal argument, using the $\{x^\mu, \theta, \bar{\theta}\}$ space formalism, to obtain in an heuristic way the gauge independence theorem as an extension of Stokes theorem.

Consider the unified classical ghost field formalism of Section II.3. Then all fields $\Phi = (A, c, \bar{c}, b, \Psi)$ depend on coordinates $\{x^\mu, \theta, \bar{\theta}\}$. Now, introduce the following action integrated over the whole space $\{x^\mu, \theta, \bar{\theta}\}$

$$\text{" } I \text{ "} = \int dx\, d\theta\, d\bar{\theta}\, \mu(\bar{\theta}, \theta)\, \mathcal{L}_Q\left(\Phi(x, \theta, \bar{\theta})\right) \tag{4.17}$$

where $\mu(\theta, \bar{\theta})$ is a suitable but yet unknown measure for the θ and $\bar{\theta}$ integration.

Since $\mathcal{L}_Q$ is invariant under s and $\bar{s}$ transformations, and since the s and $\bar{s}$ operators have been identified as derivatives along the θ and $\bar{\theta}$ directions, the folowing formal identity can be obtained by integration on θ and $\bar{\theta}$

$$\text{" } I \text{ "} = \Omega \int dx\, \mathcal{L}_Q\left(\Phi(x, 0, 0)\right) \tag{4.18}$$

where Ω stands for the "volume" of the $\{\theta, \bar{\theta}\}$ subspace (although not very well defined). On the other hand, one may define the functional integral measure on the enlarged space as follows

$$[d\Phi]_{\{x,\theta,\bar{\theta}\}} \sim \prod_{x,\theta,\bar{\theta}} \left(d\Phi(x,\theta,\bar{\theta})\right) \sim \prod_{\theta,\bar{\theta}} \left(\prod_{x} d\Phi(x,\theta,\bar{\theta})\right)$$
$$\sim \left([d\Phi]_{\{x\}\,\theta=\bar{\theta}=0}\right)^{\Omega} \tag{4.19}$$

Here we have used also the invariance of the measure under s and $\bar{s}$ transformations. One has therefore the following chain of equalities satisfied by the partition function which one may define within the enlarged space $\{x, \theta, \bar{\theta}\}$

$$\begin{aligned} \exp i\, W_{\{x,\theta,\bar{\theta}\}} &\equiv \int [d\Phi]_{\{x,\theta,\bar{\theta}\}} \exp i \int dx\, d\theta\, d\bar{\theta}\, \mu(\theta,\bar{\theta})\, \mathcal{L}_Q\left(\Phi(x,\theta,\bar{\theta})\right) \\ &= \left(\int [d\Phi]_{\{x\}} \exp i \int dx\, \mathcal{L}\left(\Phi(x,0,0)\right)\right)^{\Omega} \\ &\equiv \exp i\, \Omega\, W_{\{x\}} \end{aligned} \tag{4.20}$$

All these formal equalities suggest that the generating functional is the same in the enlarged space $\{x^\mu, \theta, \bar{\theta}\}$ and in the ordinary space (up to the inessential factor Ω). But the second term in eq. (4.20) shows that the Lagrangian is defined only up to a divergence in the space $\{x^\mu, \theta, \bar{\theta}\}$, and by recalling once more the interpretation of s and $\bar{s}$ as exterior differential operators along the θ and $\bar{\theta}$ directions, it follows that $\mathcal{L}_Q$ is defined only up to the addition of an exact s or $\bar{s}$ form. But this is exactly what means eq. (4.13)! Consequently the above chain of heuristic arguments suggests that one can interpret the gauge independence theorem as a generalized version of Stokes theorem! Although the proof is completely formal, it is certainly amusing. It is our own prejudice that a consistent quantization procedure in the $\{x^\mu, \theta, \bar{\theta}\}$ space is likely to exist, and should simplify in practice the techniques of computations.

V - WARD IDENTITY INVERSION METHOD FOR A LOCAL FUNCTIONAL

In anomaly free Yang Mills theories we shall demonstrate in Section VI that the obtention of the renormalized Lagrangian is tantamount to the determination of the most general local effective action I_R which is solution of Ward identities (4.9). By renormalized Lagrangian we mean a lagrangian which contains all the counterterms (expressed in function of a given regulator) which are necessary to generate an effective action finite at any given finite order of perturbative theory. The problem of inverting the Ward identities is a purely algebraic one[6]. In this section we shall present a straightforward method to analyse the case of a general gauge theory[16] and apply it for solving the Yang Mills case[17]

V.1 - GENERAL METHOD

We want to determine the most general local effective action $I_R[\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi]$ solution of Ward identities

$$\mathcal{B}_{I_R} I_R = 0 = \bar{\mathcal{B}}_{I_R} I_R \qquad (5.1)$$

where the operators $\mathcal{B}_{I_R}$ and $\bar{\mathcal{B}}_{I_R}$ are defined in eqs. (4.10). To solve eq.(5.1) we shall expand I_R in a Taylor series in v_Φ, $\bar{v}_\Phi$, w_Φ

$$I_R = \int dx \left(\mathcal{L}_R + v_\Phi(x)\, \bar{t}_\Phi + \bar{v}_\Phi(x)\, t_\Phi + w_\Phi(x)\, T_\Phi(x) + \dots \right) \qquad (5.2)$$

and determine the still unknown local polynomials of fields $\mathcal{L}_R(\Phi)$, $t_\Phi(\Phi)$, $\bar{t}_\Phi(\Phi)$, $T_\Phi(\Phi)$ from the constraint (5.1). In expansion (5.2) the dot points stand for terms at least quadratic in the sources v_Φ, $\bar{v}_\Phi$, w_Φ, which are in fact irrelevant for our purpose.

We introduce the following linear differential operators s_R and $\bar{s}_R$ which one can build from unknown functions t_Φ, $\bar{t}_\Phi$ and T_Φ

$$s_R = \int dx \left(\sum_{\Phi=\phi,c} t_\Phi(\Phi(x)) \frac{\delta}{\delta \Phi(x)} + b(x) \frac{\delta}{\delta \bar{c}(x)} \right)$$

$$\bar{s}_R = \int dx \left(\sum_{\Phi=\phi,c,\bar{c},b} \bar{t}_\Phi(\Phi(x)) \frac{\delta}{\delta \Phi(x)} \right) \qquad (5.3)$$

From this definition, one has $\bar{s}_R \Phi = \bar{t}_\Phi$ for all the fields $\Phi = \phi, c, \bar{c}, b$, $s_R \Phi = t_\Phi$ for the fields $\Phi = \phi, c$ and $s_R \bar{c} = b$, $s_R b = 0$. The operators s_R and $\bar{s}_R$ are graded by the ghost number in the same way as s and $\bar{s}$ in eq. (2.12).

Using eq.(5.2), (5.3) and (4.10b), we can now rewrite the Ward identity (5.1) under the form

$$s_R I_R + \int dx \left(\sum_{\Phi=\phi,c} v_\Phi(x) \frac{\delta I_R}{\delta w_\Phi(x)} + v_{\bar{c}}(x) \frac{\delta I_R}{\delta v_b(x)} \right) + \dots = 0$$

$$\bar{s}_R I_R - \int dx \left(\sum_{\Phi=\phi,c} \bar{v}_\Phi(x) \frac{\delta}{\delta w_\Phi(x)} \right) + \dots = 0 \qquad (5.4)$$

By using once more the expansion (5.2), one obtains finally the following identities which are satisfied for all values of $v_\Phi, \bar{v}_\Phi, w_\Phi$

$$\int dx \Big(s_R \mathcal{L}_R + \sum_{\Phi=\phi,c} \bar{v}_\Phi(x)\, s_R^2 \Phi(x) + v_\Phi(x) \left(s_R \bar{s}_R \Phi(x) + T_\Phi(\Phi(x)) \right)$$
$$+ w_\Phi(x)\, s_R T_\Phi(\Phi(x)) + v_{\bar{c}}(x) \left(s_R \bar{s}_R \bar{c}(x) + \bar{s}_R b(x) \right) + v_b(x)\, s_R \bar{s}_R b(x) \Big) = 0$$

$$\int dx \Big(\bar{s}_R \mathcal{L}_R + \sum_{\Phi=\phi,c,\bar{c},b} v_\Phi(x)\, \bar{s}_R^2\, \Phi(x) + \sum_{\Phi=\phi,c} \big\{ \bar{v}_\Phi(x) \big(\bar{s}_R s_R \Phi(x) - T_\Phi(x) \big)$$

$$+ w_\Phi(x)\, \bar{s}_R\, T_\Phi(x) \big\} \Big) = 0$$

$$(5.\ 5)$$

These equations imply that $s_R \mathcal{L}_R = 0 = \bar{s}_R \mathcal{L}_R$ and that the differential operators s_R, $\bar{s}_R$ are submitted to the constraints

$$
\begin{aligned}
s_R^2\, \phi &= (s_R \bar{s}_R + \bar{s}_R s_R)\, \phi = \bar{s}_R^2\, \phi = 0 \\
s_R^2\, c &= (s_R \bar{s}_R + \bar{s}_R s_R)\, c = \bar{s}_R^2\, \phi = 0 \\
\bar{s}_R^2\, \bar{c} &= 0 \\
\bar{s}_R^2\, b &= 0
\end{aligned}
$$

$$(5.\ 6a)$$

Besides, one has by definition (5. 3) of s_R

$$s_R \bar{c} = b \qquad\qquad s_R b = 0$$

$$(5.\ 6b)$$

and therefore also $s_R^2 \bar{c} = 0$ and $s_R^2 b = 0$. Eqs. (5. 6a) imply furthermore that $\bar{s}_R b = -s_R \bar{s}_R \bar{c}$ and since $b = s_R \bar{c}$ one has $(s_R \bar{s}_R + \bar{s}_R s_R)\bar{c} = 0$. At last, since $s_R \bar{s}_R b = 0$, one has $(s_R \bar{s}_R + \bar{s}_R s_R) b = 0$.

To summarize we have shown that the Taylor expansion in v_Φ, $\bar{v}_\Phi$, w_Φ of a local functional $I_R(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi)$ solution of eq.(5. 1) has necessarily the following form

$$I_R = \int dx \left(\mathcal{L}_R(\Phi) + v_\Phi\, \bar{s}_R \Phi + \bar{v}_\Phi\, s_R \Phi + w_\Phi\, s_R \bar{s}_R \Phi \right)$$

$$(5.\ 7)$$

where the differential linear operators s_R and $\bar{s}_R$ satisfy nilpotency relations on all the fields

$$s_R^2 = s_R \bar{s}_R + \bar{s}_R s_R = \bar{s}_R^2 = 0 \qquad (5.\ 8)$$

while $s_R \bar{c} = b$ and $s_R b = 0$, and $\mathcal{L}_R$ is a local function of fields $\phi, c, \bar{c}, b$ which is s_R and $\bar{s}_R$ invariant

$$s_R \mathcal{L}_R = 0 = \bar{s}_R \mathcal{L}_R \qquad (5.\ 9)$$

The simplicity of this general result is a consequence of the property that the operators $\mathcal{B}_r$ and $\bar{\mathcal{B}}_r$ are indeed intrinsic differential operators.

Other relations can be found from eq. (5. 5), but they are automatically and consistently satisfied from the already found nilpotency relations of s_R and $\bar{s}_R$. One gets for instance $T_\Phi = s_R \bar{s}_R \Phi = -\bar{s}_R s_R \Phi$ which is clearly compatible with $s_R T_\Phi = \bar{s}_R T_\Phi = 0$.

In the general case, it is not known if eq. (5. 8) determines uniquely s_R and $\bar{s}_R$ and if the most general form of s_R and $\bar{s}_R$ compatible with eq(5. 8) is alike that of s and $\bar{s}$ in eq. (2.21). We now restrict to the Yang Mills case in d=4 dimensions, for which one can use dimensionality arguments to determine s_R and $\bar{s}_R$.

V.2 YANG MILLS CASE

In the d=4 Yang Mills theory one can obtain the general form of the operators s_R and $\bar{s}_R$ acting on the fields $A, c, \bar{c}, b$ in such a way that $s_R \bar{c} = b$, $s_R b = 0$ and $s_R^2 = s_R \bar{s}_R + \bar{s}_R s_R = \bar{s}_R^2 = 0$. Using the property that all these fields have positive dimensions in mass units, that the structure constants are dimensionless and also that s_R and $\bar{s}_R$ are graded by the ghost number, we shall demonstrate that the action of s_R and $\bar{s}_R$ on all the fields must be of the following form

$$
\begin{aligned}
s_R A_\mu &= X\, D_{\mu_R} c & \bar{s}_R A_\mu &= \bar{X}\, D_{\mu_R} \bar{c} \\
s_R c &= -\frac{X}{2} [c, c]_R & \bar{s}_R \bar{c} &= -\frac{\bar{X}}{2} [\bar{c}, \bar{c}]_R \\
s_R \bar{c} &= X\, b_R & \bar{s}_R c &= \bar{X} \left(-\frac{X}{\bar{X}} b_R - [\bar{c}, c]_R\right) \\
s_R b_R &= 0 & \bar{s}_R b_R &= -\bar{X} [\bar{c}, b_R]_R
\end{aligned}
$$

and

$$
s_R \Psi = -X (c \Psi)_R \qquad \bar{s}_R \Psi = -\bar{X} (\bar{c} \Psi)_R \tag{5.10}
$$

Here X and $\bar{X}$ are arbitrary multiplicative factors which can be scaled away by redefinition of s_R and $\bar{s}_R$ and we have defined for conveniency $b_R = \frac{1}{X} b$. The symbols $(\ )_R$, $[\ ,\ ]_R$ will be made shortly fully explicit and mean that the Lie algebra generator matrix elements T^i_{aj} and the structure coefficients f^a_{bc} defined in eq. (2.1) must be renormalized by an arbitrary overall rescaling constant Z for each simple factor of the symmetry group.

In order to prove that the most general solution to eqs.(5. 8), (5. 6b) is effectively that given in eq.(5.10) we start from the most general possible definition of s_R and $\bar{s}_R$ which is compatible with dimensionality.

$$
\begin{aligned}
s_R A^a_\mu &= X (\partial_\mu c^a + F^a_{bc} A^b_\mu c^c) & \bar{s}_R A^a_\mu &= \bar{X} (\partial_\mu \bar{c}^a + \bar{F}^a_{bc} A^b_\mu \bar{c}^c) \\
s_R c^a &= -\frac{1}{2} G^a_{bc} c^b c^c & \bar{s}_R \bar{c}^a &= -\frac{1}{2} \bar{G}^a_{bc} \bar{c}^b \bar{c}^c \\
s_R \bar{c}^a &= \lambda K^a_{bc} \bar{c}^b c^c + \rho \partial_\mu A^a_\mu + \eta b^a
\end{aligned}
$$

and

$$
s_R \Psi^i = -M^i_{aj} \Psi^j \qquad \bar{s}_R \Psi^i = -\bar{M}^i_{aj} \bar{c}^a \Psi^j \tag{5.11}
$$

Let us recall that dimensionality means that $A_\mu, c, \bar{c}, b, \Psi_B, \Psi_F$ have respectively dimension in mass units 1,1,1,2,1,1/2 and ghost number 0,1,-1,0,0,0 while the action of operators ∂_μ, s_R, $\bar{s}_R$ increases respectively the dimension in mass units by 1,1,1 unit and the ghost number by 0,1,-1 unit.

$F, \bar{F}, G, \bar{G}, M, \bar{M}$ are yet unknown number matrices and X, $\bar{X}$, λ, ρ, η arbitrary real parameters. We shall determine them by imposing $s_R^2 = s_R \bar{s}_R + \bar{s}_R s_R = \bar{s}_R^2 = 0$.

We begin with the constraint $s_R^2 c = \bar{s}_R^2 \bar{c} = 0$. It implies that G and $\bar{G}$ satisfy the Jacobi identity (2.3a) and are therefore Lie Algebra generators. Yet their relative normalization is unknown. In order to proceed conveniently, we introduce the notation $(D_{\mu_R})^a_b = \partial_\mu \delta^a_b + F^a_{bc} A^c_\mu$ and $(\bar{D}_{\mu_R})^a_b = \partial_\mu \delta^a_b + \bar{F}^a_{bc} A^c_\mu$. Then the constraints $s_R^2 A_\mu = 0$ and $\bar{s}_R^2 A_\mu = 0$ imply respectively that $D_{\mu_R} (s_R c^a + \frac{X}{2} F^a_{bc} c^b c^c) = 0$ and $\bar{D}_{\mu_R} (\bar{s}_R \bar{c}^a + \frac{\bar{X}}{2} \bar{F}^a_{bc} \bar{c}^b \bar{c}^c) = 0$.

This leads to the matricial identities $G = XF$ and $\bar{G} = \bar{X}\bar{F}$. As a consequence F and $\bar{F}$ satisfy themselves the Jacobi identity. Thus we identify $F^a_{bc} = Z f^a_{bc}$ where the rescaling factor Z symbolizes an independent arbitrary rescaling factor for each simple component of the the Yang Mills group. Now, we consider the constraint $(s_R\bar{s}_R + \bar{s}_R s_R)A^a_\mu = 0$. Since one has

$$(s_R\bar{s}_R + \bar{s}_R s_R)A^a_\mu = \bar{D}_{\mu_R}(\bar{X} s \bar{c}^a) + D_{\mu_R}(X\bar{s}c^a)$$
$$+ X\bar{X}\left(F^a_{bc}(\bar{D}_{\mu_R}\bar{c}^b)c^c + \bar{F}^a_{bc}(D_{\mu_R}c^b)\bar{c}^c\right) \qquad (5.12)$$

and also $s_R\bar{c} = b$ and $\bar{s}_R c^a = \lambda K^a_{bc}\bar{c}^b c^c + \rho\partial^\mu A^a_\mu + \eta b^a$ one gets the constraints $\eta = -\frac{\bar{X}}{X}$, $F = \bar{F}$ in order to cancel the b and A_μ quadratic dependence in Eq.(5.12), and $\lambda K = -XF$ in order to cancel the ghost dependence.

Finally the constraint $\bar{s}^2_R c^a = 0$ gives $\bar{s}_R(-\frac{\bar{X}}{X}b^a - \bar{X}F^a_{bc}\bar{c}^b c^c) = 0$ which implies $\bar{s}_R b^a = -\frac{\bar{X}}{X}F^a_{bc}\bar{c}^b c^c$ by using the equations already found for $\bar{s}_R\bar{c}$ and the Jacobi identity satisfied by F^a_{bc}. At this stage we have determined the action of s_R and $\bar{s}_R$ on all the fields $A, c, \bar{c}$ and b. One may observe that we have exploited neither the constraint $s_R\bar{s}_R + \bar{s}_R s_R = 0$ for $c, \bar{c}, b$ nor $s^2_R b = 0$. In fact one can check that these constraints are now satisfied automatically from the already found relations.

In order to determine $s_R\Psi$ and $\bar{s}_R\Psi$, we impose $s^2_R\Psi = 0$ and $\bar{s}^2_R\Psi = 0$. Starting from the definition (5.11) and using the already determined values of $s_R c$ and $\bar{s}_R\bar{c}$, one gets

$$M^i_{ak}M^k_{bj} - M^i_{bk}M^k_{aj} = XZf^c_{ab}M^i_{cj}$$
$$\bar{M}^i_{ak}\bar{M}^k_{bj} - \bar{M}^i_{bk}\bar{M}^k_{aj} = \bar{X}Zf^c_{ab}\bar{M}^i_{cj} \qquad (5.13)$$

These equations imply that the matrices M and $\bar{M}$ satisfy the Jacobi identity, and thus that $M^i_{aj} = ZXT^i_{aj}$ and $\bar{M}^i_{aj} = Z\bar{X}T^i_{aj}$ where the T^i_{aj} are defined in eq.(2.3b). The relation $(s_R\bar{s}_R + \bar{s}_R s_R) = 0$ is then automatically satisfied.

Summarizing everything, we have obtained the following equations which determine the action of s_R and $\bar{s}_R$ on all the fields $A, c, \bar{c}, b, \psi$.

$$\begin{aligned}
s_R A^a_\mu &= X(\partial_\mu c^a + Zf^a_{bc}A^b_\mu c^c) & \bar{s}_R A^a_\mu &= \bar{X}(\partial_\mu\bar{c}^a + Zf^a_{bc}A^b_\mu\bar{c}^c)\\
s_R c^a &= -\tfrac{1}{2}XZf^a_{bc}c^b c^c & \bar{s}_R\bar{c}^a &= -\tfrac{1}{2}\bar{X}Zf^a_{bc}\bar{c}^b\bar{c}^c\\
s_R\bar{c}^a &= X\frac{b^a}{X} & \bar{s}_R c^a &= \bar{X}\left(-\frac{b^a}{X} - Zf^a_{bc}\bar{c}^b c^c\right)\\
s_R b^a &= 0 & \bar{s}_R b^a &= -\bar{X}Zf^a_{bc}\bar{c}^b b^c\\
s_R\Psi^i &= -XZT^i_{aj}\Psi^j & \bar{s}_R b^a &= -\bar{X}Zf^a_{bc}\bar{c}^b b^c
\end{aligned}$$

$$(5.14)$$

Eqs. (5.14) are the expected equations (5.10).

We have therefore proven that the nilpotency relation $s^2_R = s_R\bar{s}_R + \bar{s}_R s_R = \bar{s}^2_R = 0$ implies that s_R and $\bar{s}_R$ are identical to the bare BRS operators s and $\bar{s}$ in eqs.(2.11) up to arbitrary continuous deformations of Lie algebra structure coefficients f^a_{bc}.

From this, it follows that a general s_R and $\bar{s}_R$ invariant Lagrangian is identical to a general s and $\bar{s}$ invariant Lagrangian, as given in eq. (3.9), up to these dilatations of the structure coefficients, which amount in fact to arbitrary multiplicative renormalizations of the gauge coupling constant of each simple factor of the gauge group. The above results will turn out to be essential when studying the renormalization of Yang Mills theories, since we shall show that the determination of the renormalized action, when there is no anomaly, amounts to that of the most local action solution of Ward identities.

VI - YANG MILLS THEORY RENORMALIZATION

VI.1 - INTRODUCTION

The Lagrangian (3.10) is renormalizable by power counting in d=4 dimensions[0,43]. On the other hand, as shown in the last sections, the BRS invariance of the Lagrangian implies that a single coupling constant for each simple factor of the gauge group determines the strength of interactions between gauge and matter fields and of gauge bosons self-interactions. It is obvious that this unifying picture, which can be thought of as one of the basic motivations for using gauge theories in particle physics [1,2,3], should not be destroyed by radiative corrections. Moreover, the renormalization should respect the independence of physical quantities with respect to all parameters introduced for defining the unphysical longitudinal part of gauge boson propagators, and should be also compatible with the existence of a unitary scattering operator for external physical particles. Since the BRS symmetry guarantees in the tree approximation all these physical requirements, and since we have shown that the BRS symmetry is equivalent in content to the classical gauge symmetry, it should appear clear that renormalization must be done while preserving the Ward identities which express the BRS invariance in the tree approximation. By definition, an anomalous gauge theory correponds to a Lagrangian which cannot be renormalized with this requirement, although it is BRS invariant.

In this Section we shall expose the anomaly problem[6,44,45,46] and present a method to determine the most general expression of the possible anomalous vertices. This analysis permits the use of any scheme of regularization for the ultra-

violet divergences which occur in perturbation theory. In this way one can clearly define the notion of an anomaly, and reduce the field theory problem of classifying the anomalous vertices to the purely algebraic problem of classifying the solutions of the Wess and Zumino consistency conditions[6,44]. The latter problem turns out to be of a pure classical nature. Only in the case where all anomalies vanish shall we consider the possibility of a regulator preserving the BRS symmetry and show how the full renormalized Lagrangian can be determined, with a method which yields at once all possible relations between the renormalization constants and demonstrate the multiplicative renormalizability.

Our purpose it to give a general proof of the renormalizability of Yang-Mills theories in the most possible pedagogical way. All our arguments can, and should be verified by explicit one-loop computations. However we shall neither enter into these verifications nor into the general technics of diagrammatic computations, because it exists an extensive litterature available on the subject[0]. We will also not display proofs of the general results of the theory of renormalizability by power counting such as the Schwinger action principle that we shall use. We recommend ref. (43) for a rather satisfying approach to these theorems.

In a first perusal, the reader may feel disoriented because we start our study of gauge theory renormalization with the problem of anomalies. Anomalies are very often discussed only at the end of an extensive discussion about the possible existence of an ultraviolate regulator which would preserve the structure of Ward identities for the regularized but yet unrenormalized Green functions.

This is a rather misleading presentation in our opinion, because anomalies look in that way like an annoying but only technical problem which is not at the heart of the structure of gauge theories.

In fact it must be understood that anomalies are not an artefact due to a bad choice of a given scheme of regularization. Choosing a scheme rather than another one is an irrelevant question. Rather, anomalies represent a phenomenon which is a true product of the notion of a gauge theory, as can be checked in any soluble model, for instance in the two dimensional Schwinger model coupled to chiral fermions. The usual presentation of anomalies reflects of course the historical way they were discovered and gradually understood. The phenomenon was discovered by Schwinger in 1951 when he tried to build a σ-model with fermions coupled to axial currents. Before that, physicists had not realized that renormalization can conflict with symmetries apparent at the classical level, i.e. in the tree approximation. S. Adler, J.S. Bell and R. Jackiw, and W.A. Bardeen reaxamined thorougly the problem in the general context of non Abelien current algebra in 1969-1970[45,49]. Then, it became very clear that anomalies are a definitive obstruction to the implementation of a gauge symmetry in the presence of radiative corrections, although this obstruction can alter a sector of the theory or another one, depending on the way computations are done. As a matter of fact, ignoring all the theoretical refinments necessary for a full understanding of anomalies, J. Steinberger had done a very simple computation as early as in 1949 and shown an observable consequence of the existence of anomalies for the pion form factor[47]. The situation became again confused after the fun-

damental work of t'Hooft and Veltman in 1971 about the renormalization of vector like Yang-Mills theories[4]. The trouble was that these authors built a proof which relies essentially on the properties of a given regularization scheme, namely the dimensionnal one whose properties become less them obvious when chiral fermions are introduced, that is to say when anomalies can occur. Only in 1975, after they discovered the BRS symmetry, Becchi, Rouet and Stora elucidated completely the problem of anomalies in perturbative Yang-Mills theories. By using a method independent of the choice of the regularization, they gave the proper definition of an anomaly as a possible obstruction to the fulfilment of Ward identities. Then they found naturally a method for a systematic classification of anomalies, and proved rigorously that the renormalization program only makes sense when the coefficient of a well defined set of anomalous diagrams vanish at one loop. Besides, the classification of these diagrams only depend on the chosen gauge group and not on the choice of the model that one uses. Therefore, it should appear clear that one must analyze first the problem of anomalies when studying the renormalization of gauge theories. The algebraïc properties of the BRS symmetry that we have studied in detail, and in particular the results of section II and V will greatly simplify our analysis. Apart from technical details, important features concerning the anomalies will be made explicit. The generating functional of anomalies satisfies a purely geometrical equation which characterizes the gauge symmetry so that the existence of anomalies is a fact analogous to that of an invariant Lagrangian. These properties are true for any gauge theory and go beyond the properties of perturbation theory and renormalization, and one can say safely that the general consequences of the necessary existence of potential anomalies in gauge theories are not all determined yet.

VI.2 - ANOMALY PROBLEM

The problem consists in examining whether or not the Yang Mills theories can be renormalized consistently, while imposing order by order in perturbation theory all Ward Identities which exist in the tree approximation. It is important to observe that the answer to this problem is completely independent of the type of regularization that one may use in the renormalization process.

Two cases must be distinguished: (i) the massive case in which a scattering operator can be defined by using the L.S.Z. reduction formula [0], (ii) the massless case in which there is no particle interpretation of elementary fields (and thus no S-matrix) due to infra-red properties.

In case (i) for which all gauge particles are assumed as massive owing to a given suitable spontaneous symmetry breaking

mechanism, there is no freedom on the choice of the scheme of renormalization. All propagators must be renormalized on-shell, and the coupling constants must be defined as the value of vertices with external on-shell physical particles. Such a renormalization scheme is usually known as a physical scheme. Choosing other schemes of renormalization would lead in general to multi-pole expansions of propagators, which would invalidate the usual LSZ formalism. Once all parameters, or combinations of parameters, which determine the physical renormalization conditions, i.e. the value of the gauge coupling constant(s) and the location of masses of physical particles, have been isolated in $\mathcal{L}_Q$, the remaining ones are called unphysical by definition. The latter determine all necessary unphysical renormalization conditions, such as the location of the ghost propagator poles, or the value of 4-ghost interaction vertices. Then gauge independence means independence with respect to variations of unphysical parameters of the Lagrangian, the physical renormalization conditions being kept fixed [6]. In the zero loop approximation, this corresponds to the independence with respect to gauge paramerters α, β, λ in eq. (3.10).

In the massless case (ii), for instance QCD, the infra-red properties forbid the use of on-shell renormalization conditions for the elementary fields, and one must choose renormalization schemes with an arbitrary off-shell subtraction point $\mu \neq 0$ governed by the renormalization group equation.[4]

The forthcoming analysis applies equally well in both cases (i) and (ii) for determining the anomaly and constructing the renormalized Lagrangian. However, the proofs of gauge independence and unitarity that we shall display in Section VII are based on the existence of S-matrix elements of elementary fields and cannot be applied but formally to the massless case. In the case where there is a single massless gauge field corresponding to a U(1) factor in the gauge group, it is quite likely that the same mechanism as in pure QED permits to bypass the corresponding infra-red problem[0,4].

VI.3 - ANOMALY EQUATION

Consider either the case (i) or (ii) defined just above. Since the Lagrangian $\mathcal{L}_Q$ is gauge fixed and renormalizable by power counting, it determines a well defined set of Feynmann rules, and the generating functional $\Gamma_R(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi)$ of renormalized 1PI Green functions of fields and of their BRS transforms can be constructed order by order in perturbation theory in function of physical and unphysical parameters by imposing all relevant physical renormalization conditions which are necessary to render finite the theory[43]. Anomalous vertices are defined as those vertices which may appear at any given finite order in the loop expansion and forbid the fulfillment of Ward identities[6,43,46]. Therefore, by definition, there is no anomaly if Γ_R satisfies by definition the same Ward identity as Γ in eq. (4.9)

$$\mathcal{S}_{\Gamma_R} \Gamma_R = 0 = \bar{\mathcal{S}}_{\Gamma_R} \Gamma_R \qquad (6.1)$$

where $\mathcal{S}_\Gamma$ and $\bar{\mathcal{S}}_\Gamma$ are the linear differential operators (4.10b).

In the zero loop approximation, Γ_R is identical to the effective action $I = \int dx\, (\mathcal{L}_Q + \bar{v}_\Phi s\Phi + v_\Phi \bar{s}\Phi + w_\Phi s\bar{s}\Phi)$ and one has therefore $\Gamma_R = I + o(\hbar)$. As has been demonstrated in Section IV and V, the BRS invariance of $\mathcal{L}_Q$ is equivalent to the identity

$$\mathcal{S}_I I = 0 = \bar{\mathcal{S}}_I I \qquad (6.2)$$

Thus eq. (6.1) is verified in the zero-loop approximation. Suppose now that Γ_R has been built up to order n in the loop expansion, and that eq. (6.1) is satisfied up to order $(\hbar)^n$. This amounts to say that the n^{th}-order approximation of Γ_R can be expanded as

$$\Gamma_R^n = I + \hbar \Gamma^{(1)} + \dots + \hbar^n \Gamma^{(n)}$$

where the functionals $\Gamma^{(i)}$ $(1 \leq i \leq n)$ have been computed order by order with the given choice of physical renormalization conditions and are such that the following constraints hold

$$\mathcal{S}_{\Gamma_R^n} \Gamma_R^n = o(\hbar^{n+1}) = \bar{\mathcal{S}}_{\Gamma_R^n} \Gamma_R^n \qquad (6.3)$$

Since the theory is renormalizable by power counting, one can compute from Γ_R^n the radiative corrections at order $(\hbar)^{n+1}$ and build Γ_R^{n+1} while keeping the same physical renormalization conditions as for Γ_R^n. In this process, however, it is not guaranteed that the Ward identities will hold for Γ_R^{n+1}. As a matter of fact one expects that the Ward identities can be broken by terms of order $(\hbar)^{n+1}$ and one has in full generality

$$\mathcal{B}_{\Gamma_R^{n+1}} \Gamma_R^{n+1} = \hbar^{n+1} \int d^4x \; Q^+(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) + o(\hbar^{n+2})$$
$$\bar{\mathcal{B}}_{\Gamma_R^{n+1}} \Gamma_R^{n+1} = \hbar^{n+1} \int d^4x \; Q^-(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) + o(\hbar^{n+2}) \tag{6.4}$$

In these equations, the breaking terms $Q^\pm$ are <u>local</u> functionals of fields and BRS source operators as a consequence of Schwinger principle. The latter corresponds to a general theorem of renormalization theory [39]. Dimensionality requires that Q^+ and Q^- be Lorentz and Yang-Mills group scalars, with dimension 5 and ghost number 1 and -1.

The algebraic identities (4.12) $\mathcal{B}_\Gamma \mathcal{B}_\Gamma \Gamma = (\mathcal{B}_\Gamma \bar{\mathcal{B}}_\Gamma + \bar{\mathcal{B}}_\Gamma \mathcal{B}_\Gamma)\Gamma = \bar{\mathcal{B}}_\Gamma \bar{\mathcal{B}}_\Gamma \Gamma = 0$ hold true for any functional Γ and imply the following constraints on $Q^\pm$

$$\mathcal{B}_I \int d^4x \, Q^+ = \bar{\mathcal{B}}_I \int d^4x \, Q^+ + \mathcal{B}_I \int d^4x \, Q^- = \bar{\mathcal{B}}_I \int d^4x \, Q^- = 0 \tag{6.5}$$

since I is the lowest order approximation in $\hbar$ of Γ_R^{n+1} [26]. Eq. (6.5) generalizes the consistency conditions which were discovered originally by Wess and Zumino in the context of current algebra [40]. No other constraint on $Q^\pm$ exist than those discussed above.

Then, the problem that we face consists into determining whether it is possible to restore the Ward identities by absorbing the possible breaking terms $Q^\pm$ into redefinitions of local counterterms of order $\hbar^{n+1}$ within Γ_R^{n+1} while keeping the same physical renormalization conditions. Were it im-

possible, the recursive construction of Γ_R satisfying the Ward Identities would stop, and Q^+ and Q^- would represent an anomaly which spoils at order $\hbar^{n+1}$ the gauge theory physical requirements.

We shall first demonstrate the following theorem. The anomaly is spurious if $Q^\pm$ can be expressed as

$$Q^+ = \mathcal{B}_I \int d^4x \; Q^0(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) \qquad Q^- = \bar{\mathcal{B}}_I \int d^4x \; Q^0(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) \tag{6.6}$$

where Q^0 is a local functional of fields and BRS operator sources. Q^0 has dimension 4 in mass units and ghost number 0.

The proof goes as follows. Suppose that eq. (6.6) is true. Then one can build the following modified effective action

$$\tilde{\Gamma}_R^{n+1} = \Gamma_R^{n+1} - \hbar^{n+1} \int d^4x \; Q^0(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi) \tag{6.7}$$

$\tilde{\Gamma}_R^{n+1}$ satisfies by construction the Ward Identities up to order $\hbar^{n+1}$

$$\mathcal{B}_{\tilde{\Gamma}_R^{n+1}} \tilde{\Gamma}_R^{n+1} = o(\hbar^{n+2}) = \bar{\mathcal{B}}_{\tilde{\Gamma}_R^{n+1}} \tilde{\Gamma}_R^{n+1} \tag{6.8}$$

But, on the other hand, it is clear that, by adding to Γ_R^{n+1} the term $-\hbar^{n+1} \int d^4x \, Q^0$, one has generally modified the values of renormalization conditions by terms proportional to $\hbar^{n+1}$. However, we have still the freedom to modify $\tilde{\Gamma}_R^{n+1}$ by the further addition of a term $\hbar^{n+1} \delta I$, where δI is identical to the tree level effective action I, up to multiplicative rescalings of fields and parameters. Indeed, this correspondence between I and δI implies that $I+\hbar^{n+1}\delta I$ and I are also identical up to multiplicative factors and consequently that the local action $I+\hbar^{n+1}\delta I$ satisfies identically the Ward identities

$$\mathcal{B}_{I+\hbar^{n+1}\delta I}\,(I+\hbar^{n+1}\delta I) = 0 = \bar{\mathcal{B}}_{I+\hbar^{n+1}\delta I}\,(I+\hbar^{n+1}\delta I) \tag{6.9}$$

The later property follows from the fact that the operators $\mathcal{B}_I$ and $\bar{\mathcal{B}}_I$ have no explicit dependence on any of the parameters of which the local action I is function. Therefore all the actions I_R which are identical to I up to dilations of fields and parameters, and in particular $I_R = I + \hbar^{n+1}\delta I$, satisfy exactly the Ward Identities $\mathcal{B}_{I_R} I_R = 0, \bar{\mathcal{B}}_{I_R} I_R$.

Eq.(6.9) implies in turn that the effective action $\tilde{\tilde{\Gamma}}_R^{n+1} = \tilde{\Gamma}_R^{n+1} + \hbar^{n+1}\delta I$ still satisfies the Ward Identities up to order $\hbar^{n+1}$

$$\mathcal{B}_{\tilde{\tilde{\Gamma}}_R^{n+1}}\,\tilde{\tilde{\Gamma}}_R^{n+1} = O(\hbar^{n+2}) = \bar{\mathcal{B}}_{\tilde{\tilde{\Gamma}}_R^{n+1}}\,\tilde{\tilde{\Gamma}}_R^{n+1} \tag{6.10}$$

Finally, one can check by inspection over all possible cases that there are always more independent possible rescaling factors in the local action I than it is necessary for readjusting the physical renormalization conditions of $\tilde{\tilde{\Gamma}}_R^{n+1}$ as equal to those of Γ_R^{n} . Thus we have demonstrated that anomalous terms like those in eq. (6.6) are in fact spurious, and can be compensated

by suitable local redefinitions of counterterms.

We shall show now that the consistency conditions (6.5) allow one to reduce the most general possible anomalous term $Q^\pm$ into an expression which is only function of fields but not of BRS operator sources and corresponds to ABBJ anomalous vertices.

The method that one can follow is analogous to that we have already used in Section V for the sake of inverting the Ward Identities. It consists in Taylor expanding $Q^\pm$ in function of BRS sources

$$\int dx\, Q^\pm = \int dx\, \left(a^\pm(\Phi) + P^\pm(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi)\right) \tag{6.11}$$

where $a^\pm$ depends only on fields $\Phi=(A,c,\bar{c},b,\Psi)$. Then one applies the generalized Wess and Zumino constraint (6.5) to this expansion. As a result one gets a set of equations restricting the form of field polynomials which are coefficient of independent sources v_Φ , $\bar{v}_\Phi$, w_Φ . In ref[26,46] it is shown that these constraints imply the following form for $P^\pm$

$$\int dx\, P^+ = \mathcal{B}_I \int dx\, P^0 \qquad \int dx\, P^- = \bar{\mathcal{B}}_I \int dx\, P^0 \tag{6.12}$$

where P^0 is a local polynomial in Φ , v_Φ , $\bar{v}_\Phi$, w_Φ . The demonstration is straightforward but lengthy and we won't make it explicit it here. The interesting point, however, is that it reveals the cohomological nature of the anomaly problem. Indeed, after having performed elementary algebraic manipulations, one can show that the eqs. (6.5) for $Q^\pm$ can be reduced into local equations whose general form is as follows[26]

$$S\,G^+ = 0 \qquad \bar{S}\,G^+ + S\,G^- = 0 \qquad \bar{S}\,G^- = 0$$

$$S\,K_\mu - D_\mu K^+ = 0 = \bar{S}\,K_\mu - D_\mu K^- \tag{6.13}$$

where $D_\mu = \partial_\mu + A_\mu$ and $S = s + c$, $\bar{S} = \bar{s} + \bar{c}$ are the generalized covariant derivatives (2.35b) which satisfy the commutation relations (2.44)

$$S^2 = S\bar{S} + \bar{S}S = \bar{S}^2 = 0$$

$$S\,D_\mu - D_\mu S = 0 = \bar{S}\,D_\mu - D_\mu \bar{S} \tag{6.14}$$

One can then solve eq. (6.13) by inspection over all possible cases by using dimensionality [26,46]. The result is that the only possible solutions are the trivial ones

$$G^+ = S\,G^0 \qquad\qquad G^- = \bar{S}\,G^0$$

$$K^+ = S\,K^0 \qquad K_\mu = D_\mu K^0 \qquad K^- = \bar{S}\,K^0 \tag{6.15}$$

which yields eq. (6.12).

Eq. (6.12) has the form (6.6), and it implies that $P^\pm$ can be eliminated away by local modifications of counterterms. Therefore we can set $P^\pm = 0$ in eq. (6.11), and the problem of determining the anomaly is now reduced into that of finding the

local functions of fields $a^\pm(A, c, \bar{c}, b, \Psi)$ which are the general solutions of constraints (6.5) with dimension 5 and ghost number ± 1. The latter constraints read as

$$\mathcal{B}_I \int d^4x\, a^+(\Phi) = \bar{\mathcal{B}}_I \int d^4x\, a^+(\Phi) + \mathcal{B}_I \int d^4x\, a^-(\Phi) = \bar{\mathcal{B}}_I \int d^4x\, a^-(\Phi) = 0 \tag{6.16}$$

or, equivalently

$$\int d^4x\; s\,a^+ = \int d^4x\,(\bar{s}\,a^+ + s\,a^-) = \int d^4x\; \bar{s}\,a^- = 0 \tag{6.17}$$

Assuming that all fields vanish sufficiently fast at infinity, eq. (6.17) imply the classification of $a^\pm$ among two categories.

The first category is when the equations $s a^+ = \bar{s} a^+ + s a^- = \bar{s}\,a^- = 0$ are trivially satisfied

$$a^+ = s\,a^0 \qquad\qquad a^- = \bar{s}\,a^0$$

$$\rightsquigarrow\; Q^+ = \mathcal{B}_I \int d^4x\, a^0 \qquad Q^- = \bar{\mathcal{B}}_I \int d^4x\, a^0 \tag{6.18}$$

Such solutions are of the trivial type, eq. (6.6), and can be compensated by suitable local redefinitions of counterterms within the effective action.

The second category is when a^+ and a^- are not the s and $\bar{s}$ transform of a local polynomial but sa^+, $sa^- + \bar{s}a^+$ and $\bar{s}a^-$

are locally space-time derivatives of other fields polynomials. This amounts to the existence of Yang-Mills and Lorentz group scalar 3-forms b^+, b^0, b^- with

$$db^+ = d^4x\,(s\,a^+) \qquad d\,b^0 = d^4x\,(\bar{s}a^+ + s\,a^-) \qquad db^- = d^4x\,(\bar{s}\,a^-) \tag{6.19}$$

and there is no 3-form c^0 such that $b^+=sc^0$, $b^-=sc^0$. Then the Wess and Zumino consistency conditions (6.5) are satisfied (since all fields vanish at infinity), but eq. (6.6) does not hold true anymore, and there is no way for eliminating away $a^\pm$ by local redefinitions of counterterms in Γ_R . As a consequence, the most general form of the anomaly can be identified as the general non trivial solution of local equation (6.19) with dimension 5 and ghost number ± 1.

In what follows we shall use the unified formalism for solving eq. (6.19). There the s and $\bar{s}$ operators are identified as derivatives along the direction θ and $\bar{\theta}$. This method shows explicitly that the anomaly equations (6.19) are truly cohomology equations. Furthermore the formulation of the BRS symmetry in this formalism permits a straightforward determination of the solutions $a^\pm$ of eq. (6.19).

VI.4 - ADLER BARDEEN BELL JACKIW ANOMALY

We shall first show how to solve the anomaly equation (6.19). Only afterwards we shall give the correspondence between the obtained solutions and the possible anomalous vertices which may occur in field theory.

For conveniency, we shall absorb the space-time volume element d^dx into the definition of $a^\pm$. Thereby $a^\pm$ become differential d-forms and we rewrite the anomaly equation (6.19) as follows

$$s\,a^+ + d\,b^+ = 0 \qquad \bar{s}\,a^+ + s\,a^- + db^0 = 0 \qquad \bar{s}\,a^- + db^- = 0 \tag{6.20}$$

Under this form the space-time dimension is no more explicit. Because of relations $d^2=s^2=\bar{s}^2=sd+ds=\bar{s}d+d\bar{s}=s\bar{s}+\bar{s}s=0$, the solutions of eq. (6.20) are defined up to the following redefinitions

$$a^+ \to a^+ + s\,a^0 + d\,a^{+0} \qquad a^- \to a^- + \bar{s}\,a^0 + d\,a^{-0}$$
$$b^+ \to b^+ + db^{0+} + s\,a^{+0} \qquad b^- \to b^- + d\,b^{0-} + \bar{s}\,a^{-0}$$
$$b^0 \to b^0 + d\,b^{00} + \bar{s}\,a^{+0} + s\,a^{-0} \tag{6.21}$$

where the forms a^0, a^{+0}, a^{-0} and b^{00}, b^+, b^0 are arbitrary Yang-Mills and Lorentz scalar forms, respectively with Lorentz rank d,d,d and d-1, d-1,d-1 and with ghost number 0,1,-1 and 0, 1,-1. Such redefinitionscannot generate any new anomaly since a^0 can be eliminated away by redefinition of counterterms in Γ_R as shown in Section VI.3, and a^{+0} and a^{-0} vanish by space-time integration when one computes $Q^\pm$.

Solving eq. (6.20,21) amounts to a problem whose geometrical meaning appears more transparent in the unified formalism described in Section II.3. Indeed, if there are solutions to eq. (6.20), this means that a displacement in the x direction is equivalent to a displacement in the non physical direction θ , $\bar{\theta}$. Now, from a geometrical point of view, it is clear that such solutions must exist, because the s and $\bar{s}$ operators are in fact determined from the property that the field strength $\tilde{F}$, which can be thought of as a curvature, has vanishing components along the θ, $\bar{\theta}$ directions.

In practice the general solutions to eq. (6.20-21) must be found by inspection. They can be classified into two categories: the solutions such that b^+, b^-, b^0 vanish locally, and the others. The former solutions can only exist if the gauge group contains an Abelian factor. They read as follows

$$a^+ = c^{ABE} \mathcal{L}_{cl} d^d x \qquad a^- = \bar{c}^{ABE} \mathcal{L}_{cl} d^d x \tag{6.22}$$

where $\mathcal{L}_{cl}$ is a general s and $\bar{s}$ invariant Lagrangian, with $s\mathcal{L}_{cl} = 0 = \bar{s}\mathcal{L}_{cl}$ and c^{ABE} and $\bar{c}^{ABE}$ are respectively the ghost and anti-ghost of the Abelian gauge field, with $sc^{ABE} = 0$ and $\bar{s}\bar{c}^{ABE} = 0$. Note that the form of solutions in eq. (6.22) is analogous to that of solutions for the trace anomaly problem, if one replaces c^{ABE} by the ghost of dilatations. Explicit 1-loop computations suggest nevertheless that solutions as in eq. (6.22) cannot be generated in field theory, except if $\mathcal{L}_{cl}$ in eq. (6.22) is a pure derivative, $\mathcal{L}_{cl} = \partial_\mu K^\mu$, for instance $\mathcal{L}_{cl} d^4x = \mathrm{Tr}(F \wedge F)$.

We shall therefore draw our attention to the problem of solving eq. (6.20,21) when $b^\pm$ and b^0 do not vanish locally, or when one has an Abelian factor in the group and one can build $\mathcal{L}_{cl}$ with $\mathcal{L}_{cl} = \partial_\mu K^\mu$ and $s\mathcal{L}_{cl} = 0 = \bar{s}\mathcal{L}_{cl}$. As a result, we shall recover the general ABBJ formula.

As a primary observation, one can see that $a^\pm$, up to redefinitions (6.21), are independent of the matter fields Ψ. It is so because the s and $\bar{s}$ transform

of Ψ do not involve the space-time exterior differential d. Furthermore, a^+ (resp. a^-) must be independent of b and $\bar{c}$ (resp c). This can be shown by expanding a^+ (resp. a^-) as a general Taylor series in the fields b and $\bar{c}$ (resp. c) and their derivatives. Then, by substitution of this expansion in eq. (6.20), one finds that only a trivial dependence in these variables is permitted.

Therefore the problem restricts to finding solution a^+ and a^- which are respectively of the first degree in c (or any derivative of c) and $\bar{c}$ (or any derivative of $\bar{c}$), and only depend otherwise on the gauge field A_μ and its derivatives.

$$a^+ = c^a Z^a(A_\mu, \partial_\mu A_\nu, \ldots) \qquad a^- = \bar{c}^a Z^a(A_\mu, \partial_\mu A_\nu, \ldots) \tag{6.23}$$

Now, by using dimensionality in the cases d=2,4 one can show by inspection over all possibilities that the d- form Z^a can only depend on A_μ and its derivatives through exterior products of the 1-form $A = A_\mu dx^\mu$ and its exterior derivative dA. This implies that Z^a can be written as an exterior product of A and F. It is quite likely that this property should remain true for all values d>4, although this has not yet been generally verified. Assuming the validity of this property, one can then rewrite eq. (6.23) as

$$a^+ = c^a \frac{\delta}{\delta A^a}\Big|_F T_{d+1}(A,F) \qquad a^- = \bar{c}^a \frac{\delta}{\delta A^a}\Big|_F T_{d+1}(A,F) \tag{6.24}$$

where T_{d+1} is a Yang Mills and Lorentz scalar (d+1)-form made from exterior products of A and F and defined in a space-time higher dimension than d. But then, the equations (6.20,21) for $a^\pm$ can be interpreted as the components with ghost number 1,0,-1 of the following single equation which involves A, c, $\bar{c}$

under the combination $\tilde{A}=A+c+\bar{c}$

$$\tilde{d}\, T_{d+1}(\tilde{A}, F) = d\, T_{d+1}(A, F)$$
$$\tilde{T}_{d+1}(\tilde{A}, \tilde{F}) \neq \tilde{d}\, K_d(\tilde{A}, \tilde{F}) \qquad (6.25)$$

with $\tilde{d}=d+s+\bar{s}$. The unified formalism of Section (II 3), in which the ghost and gauge fields are precisely unified under the combination $\tilde{A}=A+c+\bar{c}$ and the operators d,s, and $\bar{s}$ into the operator $\tilde{d}=d+s+\bar{s}$ is therefore particularly well suited for solving the anomaly equation. By using the expression (2.39) of the BRS equations

$$\begin{aligned}\tilde{F} &= (d+s+\bar{s})(A+c+\bar{c}) + \tfrac{1}{2}[A+c+\bar{c},\ A+c+\bar{c}] \\ &= dA + \tfrac{1}{2}[A,A] \qquad (6.26) \\ &= F\end{aligned}$$

we shall in fact obtain in the most direct way the solutions to eq. (2.25) that is to say the relevant solutions to the anomaly equation (6.20,21) for even d. For odd dimension d, no solution exists, except for an exceptional case when the group contains at least two U(1) factors.

We shall use an identity which is due to Chern and plays a fundamental role in differential geometry. It holds true in any exterior algebra with a basis of exterior 1-forms denoted as $dy^{\alpha}_{\wedge\cdots\wedge}dy^{\alpha'}$ and the exterior differential operator as $\delta=dy^{\alpha}\frac{\partial}{\partial y^{\alpha}}$. It reads as follows[47]

$$\mathcal{P}^{inv.}_{\cdot p}(\underbrace{\mathcal{F},\ldots,\mathcal{F}}_{p}) = \delta \int_0^1 dt\, \mathcal{P}^{inv}_{\cdot p}(\mathcal{A}, \underbrace{\mathcal{F}_t,\ldots,\mathcal{F}_t}_{p-1}) \qquad (6.27a)$$

Here $\mathcal{F} = \delta\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}]$, and $\mathcal{F}_t = t\,\delta\mathcal{A} + \frac{t^2}{2}[\mathcal{A},\mathcal{A}]$. $\mathcal{A}$ is a general 1-form which takes its values in the Lie algebra g and reads in components as $\mathcal{A} = T_a \mathcal{A}^a_\alpha\, dy^\alpha$. The symbol $\mathcal{P}^{inv.}_{\cdot p}$ stands for any invariant symmetrical polynomials function of $\mathcal{A}$ through exterior products of p g-valued 2-form $\mathcal{F}$. Eq. (6.27 can be checked explicitely by performing the integration on the real variable t, and by using the Jacobi identity and the cyclic property of $\mathcal{P}^{inv}_{\cdot p}$. It can be also demonstrated from deeper methods of homotopy theory[47]. After having performed the t integration, one can express the r.h.s. of eq. (6.27a) as a polynomial in $\mathcal{A}$ and $\mathcal{F}$

$$\mathcal{P}^{inv}_{\cdot\cdot p}(\mathcal{F},\ldots,\mathcal{F}) = \delta\, T^0_{2p-1}(\mathcal{F},\mathcal{A}) \qquad (6.27b)$$

T^0_{2p-1} is benamed usually as Chern-Weil polynomial of rank p. Let us give as examples the cases p=2,3. If the Yang Mills group is simple, the only possibility for $\mathcal{P}^{inv}_{p}$ is the trace, and one gets

$$\begin{aligned}T^0_3(\mathcal{A},\mathcal{F}) &= T_R(\mathcal{A}_\wedge \mathcal{F} - \tfrac{1}{3}\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{A}) \\ T^0_5(\mathcal{A},\mathcal{F}) &= T_R(\mathcal{A}_\wedge\mathcal{F}_\wedge\mathcal{F} - \tfrac{1}{2}\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{F} \\ &\quad + \tfrac{1}{10}\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{A}_\wedge\mathcal{A})\end{aligned} \qquad (6.28)$$

One can check $T_R(\mathcal{F}_\wedge\mathcal{F}) = \delta T^0_3(\mathcal{A},\mathcal{F})$ and $T_R(\mathcal{F}_\wedge\mathcal{F}_\wedge\mathcal{F}_\wedge\mathcal{F}_\wedge\mathcal{F}) = \delta T^0_5(\mathcal{A},\mathcal{F})$.

The Chern formula is the key for solving eqs. (6.25). Indeed, since the BRS equations are equivalent to the geometrical constraint $F=\tilde{F}$, eq. (2.39), they imply in a trivial way the following identity

$$\mathcal{S}^{inv}_{p}(\underbrace{\tilde{F},\dots,\tilde{F}}_{p}) = \mathcal{S}^{inv}_{:p}(F,\dots,F) \qquad (6.29a)$$

which is valid for all values of p independently of the value of the dimension d of space-time. Then, by applying the Chern formula (6.27) to both sides of eq. (6.26) one gets

$$\tilde{d}\, T^{0}_{2p-1}(\hat{A},\tilde{F}) = d\, T^{0}_{2p-1}(A,F) \qquad (6.29b)$$

Eq. (6.29) means that the Chern-Weil polynominals of rank d/2+1 are solutions to eq. (6.25). These solutions may occur only in even dimension, since F is a 2-form and thus $\mathcal{S}^{inv}_{:p}$ (F) is necessarily of an even rank, and d=2p-2. In this way we have clearly seen the link between the solutions to Wess and Zumino consistency conditions in even d-dimensions and the topological invariant in d+2 dimensions, $\mathcal{S}^{inv.}_{d+2}(F) = d\, T_{d+1}(A,F)$. Observe that the possible anomalies in any even dimensions are thus related to the only admissible Lagrangians which can be written as exterior products of fields in odd dimensions that is to say topological mass terms[57] (see appendix A3).

By expansion of $T_{d+1}(\hat{A},F)$ in ghost number, one gets a pyramid of forms $T^{g-2\bar{g}}_{d+1-g}$, $0<\bar{g}\leq g$, $0<g\leq d+1$, which are classified according to their ghost numbers $g-2\bar{g}$ and Lorentz

rank d+1-g[26]. The terms with a ghost number larger than 1 satisfy also equations analogous to those obeyed by $a^{\pm}$, but it is still unknown whether they play a role in physics.

To illustrate these general properties, let us consider the cases of d=2 and 4 dimensions.

In the case of d=2 dimension, and for a simple group, the expression of T^{0}_{3} has been given in eq. (6.28), and formula (6.24) yields

$$a^{+} = \mathrm{Tr}\,(c\, dA) \qquad a^{-} = \mathrm{Tr}\,(\bar{c}\, dA) \qquad (6.30)$$

For d=4, the case of physical interest, one gets from eqs. (6.28) and (6.24)

$$a^{+} = \mathrm{Tr}\big(c\, d(AdA + \tfrac{1}{2}AAA)\big) \qquad a^{-} = \mathrm{Tr}\big(\bar{c}\, d(AdA+\tfrac{1}{2}AAA)\big) \qquad (6.31a)$$

or equivalently

$$\int a^{+} = \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\big(c\, \partial_\mu (A_\nu \partial_\rho A_\sigma + \tfrac{1}{2} A_\nu A_\rho A_\sigma)\big)$$
$$\int a^{-} = \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\big(\bar{c}\, \partial_\mu (A_\nu \partial_\rho A_\sigma + \tfrac{1}{2} A_\nu A_\rho A_\sigma)\big) \qquad (6.31b)$$

Using the Jacobi identity, the scrupulous reader may check the following identities

$$s a^{+} + d\, \mathrm{Tr}(A\, cc) = 0 \qquad \bar{s} a^{-} + d\, \mathrm{Tr}(A\, \bar{c}\bar{c}) = 0$$
$$\bar{s} a^{+} + s a^{-} + d\, \mathrm{Tr}\big(A[c,\bar{c}]\big) = 0 \qquad (6.32)$$

where $a^{\pm}$ is given by formula (6.30) and also

$$\begin{aligned} s\, a^{+} + \tfrac{1}{2} d\, Tr\left(c c A F + c [c, A]\, dA\right) &= 0 \\ \bar{s}\, a^{-} + \tfrac{1}{2} Tr\left(\bar{c}\bar{c} A F + \bar{c} [\bar{c}, A]\, dA\right) &= 0 \\ \bar{s}\, a^{+} + s\, a^{-} + \tfrac{1}{2} d\, Tr\left([\bar{c}, c] A F + \left([c, A]\bar{c} + [\bar{c}, A] c\right) dA\right) &= 0 \end{aligned} \tag{6.33}$$

where $a^{\pm}$ is given by formula (6.31). More details related to the full expansion of $T_{d+1}(\hat{A}, F)$ in ghost number can be found in ref. 26).

If the gauge group contains a $U(1)$ factor with ghost-gauge field $\tilde{A}^{ABE}$, one has in addition the following invariant in 6 dimension

$$\mathcal{P}_3^{inv.} = \tilde{F}^{ABE}\, Tr(\hat{F}\hat{F})$$

$$\rightsquigarrow \Delta_5(\hat{A}, \hat{F}) = \tilde{A}^{ABE}\, Tr(\hat{F}\hat{F}), \quad \mathcal{P}_3^{inv} = \tilde{d}\, \Delta_5 \tag{6.34}$$

and therefore the "Abelian" anomaly is

$$\begin{aligned} a^{+} &= c^{ABE}\, Tr(FF) = c^{ABE}\, d\left(A dA + \tfrac{2}{3} AAA\right) \\ a^{-} &= \bar{c}^{ABE}\, Tr(FF) = \bar{c}^{ABE}\, d\left(A dA + \tfrac{2}{3} AAA\right) \end{aligned} \tag{6.35}$$

In odd dimensions d, there is in general no solutions to the anomaly equations because invariant polynomials only exist in even dimensions. One has nevertheless an exceptional solution which occurs if the gauge group contains two Abelian factors, say A^{ABE} and A'^{ABE}. *Indeed*, the form

$$\Delta_{d+1} = \tilde{A}^{ABE}\, \tilde{A}'^{ABE}\, Q_{\frac{d-1}{2}}(F) \tag{6.36}$$

where the even form $Q_{\frac{d-1}{2}}$ is an invariant polynomial of all field strengths, satisfies eq. (6.25). The corresponding anomalies $a^{\pm}$, as defined by eq. (6.24), satisfy eq. (6.20,21), but an explicit computation demonstrates that such anomalies cannot be generated from 1-loop radiative corrections.

Let us now apply these results to field theory and determine the possible anomalous vertices in 4 dimensions from our classification of the solutions to the Wess and Zumino consistency conditions.

Suppose first that the gauge group has no U(1) factor. Then, by setting the solution (6.31) in eq. (6.4), the broken Ward identities can now be written under the following irreducible form

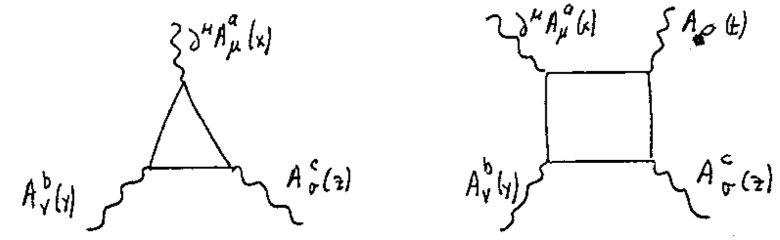
$$\begin{aligned} \mathcal{S}_{\Gamma_R^{n+1}} \Gamma_R^{n+1} &= \hbar^{n+1} \chi_{n+1} \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, Tr\left(c\, \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \tfrac{1}{2} A_\nu A_\rho A_\sigma\right)\right) \\ \bar{\mathcal{S}}_{\Gamma_R^{n+1}} \Gamma_R^{n+1} &= \hbar^{n+1} \chi_{n+1} \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, Tr\left(\bar{c}\, \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \tfrac{1}{2} A_\nu A_\rho A_\sigma\right)\right) \end{aligned} \tag{6.37}$$

in which only the coefficient χ_{n+1} depends on the chosen model, i.e. on the matter fields assignments in $\mathcal{L}_Q$, and represents the only freedom left after the determination of $a^{\pm}$ from eqs. (6.19). The presence of the antisymmetrization tensor $\epsilon^{\mu\nu\rho\sigma}$ demonstrates that it is necessary to have chiral couplings in the theory (i.e. chiral fermions) in order to have the possibility of a non spurious anomaly. Moreover, in order that the traces do not vanish in eq. (6.34), the Lie Algebra must contain a symmetric tensor d_{abc}. This shows for instance that no anomaly can exist in a SU(2) gauge theory for which there is no such symmetrical tensor.

The form of the anomaly is the same for all n in eq. (6.37), apart from the coefficient X_{n+1}. The anomaly can occur when going from zero loop to one loop. Thus, given any Yang Mills theory, one must compute the value of the single coefficient X_1. This can be done by applying upon eq. (6.34) either the functional operator $\delta^3/(\delta\bar{c}^a(x)\,\delta A^b_\nu(y)\,\delta A^c_\sigma(z))$ or $\delta^4/(\delta\bar{c}^a(x)\,\delta A^b_\nu(y)\,\delta A^c_\sigma(z)\,\delta A^d_\rho(t))$. In both cases it yields on the r.h.s. the coefficient X_1 times group and Lorentz coefficient. On the l.h.s., both operators act upon $\int d^4x\,(\delta\Gamma^1_R/\delta A^\mu_a(x))(\delta\Gamma^1_R/\delta\bar{v}_{A^a_\mu}(x))$, where $\bar{v}_{A^a_\mu}(x)$ is the source of operator $D_\mu c^a$, and yield respectively the 1PI longitudinal Green functions $\langle\partial^\mu A^a_\mu(x)\,A^b_\nu(y)\,A^c_\sigma(z)\rangle$ and $\langle\partial^\mu A^a_\mu(x)\,A^b_\nu(y)\,A^c_\sigma(z)\,A^d_\rho(t)\rangle$.[26] The latter vanish in the tree approximation because of the BRS symmetry of the Lagrangian. In a given model, and using any regularization which allows one to take care of ultra-violet divergencies, each of these Green functions can be explicitely evaluated by a one-loop computation, and it permits the determination of X_1. Obviously both computations determine the same value for X_1. Then we have recovered the usual definition of the anomaly as first given by Bardeen[45,49].

The only possibility for having $X_1=0$ is to assigning all the chiral fermions which couple to the gauge fields in suitable group representations. Indeed a trivial inspection shows that the only diagrams which can contribute to the anomalous part of 1PI diagrams $\langle\delta A_\mu, A_\nu, A_\rho\rangle$ and $\langle\delta A_\mu, A_\nu, A_\rho, A_\sigma\rangle$ must contain a continuous fermion line connecting the 3 and 4 extermal point, in order to generate the $\epsilon^{\mu\nu\rho\sigma}$ tensor present in the r.h.s. of eq. (6.37)

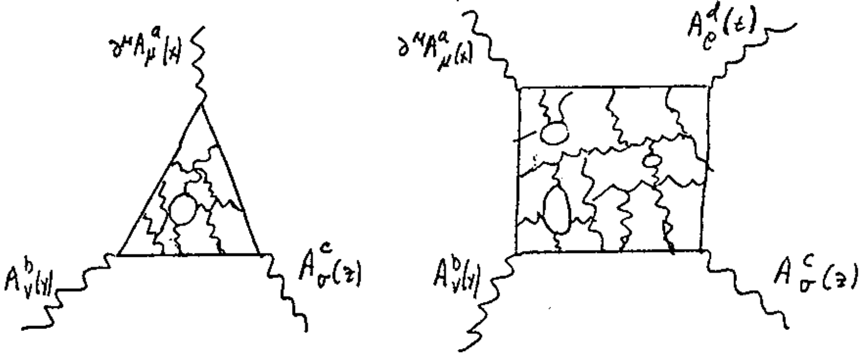
through γ^5 couplings. These diagrams are the well-known ABBJ triangle and quadrangle



(6.38)

and since they must vanish in order to have $X_1=0$, this yields a constraint between the number of fermions in the theory and values of group Casimir, depending on the representations in which the fermions are assigned [50]. The classification of potentially anomalous diagrams can be easily extended to the case of d=2p dimensions[26]. Indeed, using the general formula (6.25,29) one obtains as possible anomalous diagrams these polygons with $\frac{d}{2}\leq K\leq d-1$ external gauge fields A_μ and one longitudinal gauge field component $\partial^\mu A_\mu$ [26]. The practical computations of the corresponding coefficients has been done in ref. 48).

Now, supposing that $X_1=0$, what happens for X_n when $n>1$? There is in fact a non renormalization theorem for d=2,4 which states that X_n vanishes at any given finite order, provided $X_1=0$. This essential theorem have been rigorously demonstrated in ref. 51), in a regulator independent proof based on Callan-Szimanzik equation. This proof is rather involved, but the theorem can be also demonstrated in a more intuitive way, by using Bardeen's original heuristic argument which applies in the context of dimensional reguarlization[49]. Indeed, at any given finite order of perturbation theory, one can convince oneself that the diagrams which contribute to the anomalous part of $\langle \partial^\mu A_\mu, A_\nu, A_\rho \rangle$ and $\langle \partial^\mu A_\mu, A_\nu, A_\rho, A_\sigma \rangle$ have the same structure as the diagrams (6.38), except for all possible virtual interactions originated from the continuous fermion lines which connect the external point



(6.39)

The same group factors as those for the 1-loop diagrams (6.38) factor out when one evaluates the diagrams (6.39). Observe that this result has not been fully rigorously proven, but it can be heuristically verified if one uses the prescription of t'Hooft and Veltman[4] in order to define the γ^5 matrix within the dimensional regularizatioon[54]. Consequently the vanishing condition of χ_1 also implies that of χ_n for all n>1, which is exactly what states the non renormalization theorem.

If the gauge group contains several factors, one has as many

factors χ_1 as there are independent factors in the group. Furthermore, if one of these factors is abelian, one has the possibility of an anomaly corresponding to the mixed solution (6.35) for $a^\pm$, and the potentially anomalous diagrams are those stemming from the field decomposition of $\int c^{ABE}\, T_R(FF) = \int c^{ABE}\, T_R(d(AdA + \frac{2}{3}AAA))$, that is to say diagrams like those in eq. (6.38), but where $\partial^\mu A_\mu$ is replaced by $\partial^\mu_\mu A^{ABE}_\mu$. Such anomalies are known as Abelian anomalies .

If the theory is symmetric under a global symmetry in addition to the gauge symmetry, one may wish to couple the associated Noether currents. The latter are conserved in the tree approximation. However, it is not always possible to ensure the conservation of these currents at the renormalized level, and one must consider the possibility of having a current algebra anomaly. In ref. (26) it is shown how to analyze this problem in a formalism which is similar to the one for the pure gauge anomaly problem.

Finally let us stress that the link between the anomalies in d-dimension and the gauge invariant Lagrangians in d+1 dimensions is not a phenomenon specific to Yang Mills theories. Consider indeed a general gauge symmetry defined as in sections (II.2, III.2). The notion of an anomaly can be given a sense even for a nonrenormalizable theory. Indeed, starting from a BRS invariant Lagrangian as in section (III.2), one can always compute a finite 1-loop effective action by regularizing the one-loop divergences. Then, using only a finite number of local counterterms, one can obtain a renormalized 1-loop effective action satisfying all necessary physical renormalization conditions. Thus the 1-loop anomaly is defined exactly as in the Yang Mills case as the possible obstruction to the fulfillment of the Ward identity for the 1-loop renormalized effective action. The generating functional of the anomalies, i.e. the r.h.s. of the broken Ward identity, is therefore also defined for a general gauge theory as the general solution of the Wess and Zumino consistency condition of the considered BRS symmetry

$$\begin{cases} \mathcal{B}_{\Gamma^{1\text{-loop}}} \Gamma^{1\text{-loop}} = \hbar \int a^+ \\ s^2 = 0 \quad , \; a^+ \neq s a^0 \end{cases} \Rightarrow \int s a^+ = 0 \qquad (6.40a)$$

$$\Rightarrow \qquad s a^+ = d\, b^{++} \qquad (6.40b)$$

where a^+ is a d-form with ghost number 1, function of the fields, and s has been generally defined in eqs. (2.21). The solutions to the local equation (6.40b) should not depend generally on the value of space dimension d. Thus, to solve equation (6.40b) one may increase space-dimension to a large enough value. Then, by applying to both sides of eq. (6.40b) the differential operator d, one gets the following local equation, which is non trivial in (d+1) dimensional space-time

$$s\,(d\, a^+) = 0 \qquad a^+ \neq s\, a^0 \qquad (6.41)$$

But in most cases, because the s operator is nilpotent, $s^2=0$, this equation implies the existence of a (d+1)-exterior form

Δ_{d+1} with ghost number 0 such that one has

$$d\, a^+ = s\, \Delta_{d+1}(\phi) \qquad (6.42)$$

One might understand the passage from eq. (6.41) to eq. (6.42) as a generalization of the algebraic Poincare lemma. Note that Δ_{d+1} only depends on the classical fields ϕ because it has ghost number 0.

Eq. (6.42) simply means that $\Delta_{d+1}(\phi)$ is an admissible Lagrangian in d+1 dimension, since its integral over the (d+1) spacetime is s invariant, i.e. gauge invariant.

From this heuristic and formal proof it is quite suggestive that, for any general gauge theory, there exists a deep relationship between the anomalies in d-dimensions, which are usually considered as genuine quantum objects, and the gauge invariant classical Lagrangians which can be expressed as exterior forms in d+1 dimensions.

As last we would like to emphasize the advantages of using the unified gauge ghost field formalism to analyze the anomaly problem in Yang Mills theories. Not only does this provide in a most direct way the general anomaly formula, but it points out the geometrical nature of the problem. Indeed we have shown in a way which is completely independent of the notion of a Lagrangian that the existence of solutions to the Wess and Zumino consistency conditions is a straightforward consequence of the horizontality condition $\tilde{F}=F$ which is the expression of the BRS symmetry in the unified gauge-ghost field formalism. Furthermore, this geometrical method shares a number of characteristics with new technics which have been developed recently to explore possible nonperturbative consequences of the existence of anomalies[58]. The critical reader, however, will have noticed that we have not proven that formula (6.28) represents in fact the general solution of the cohomology eqs. (6.25). Only in d=2,4 dimension space-time a brute force inspection over all possibilities has verified the validity of this property. Although we are convinced that this result remains true for d>6, its general verification has not yet been completed in practice.

VI.5 BRS SYMMETRY PRESERVING REGULATOR AND MULTIPLICATIVE RENORMALIZATION

So far, we have proven that the Yang Mills theories in 4-dimensional flat-space can be renormalized at any given finite order of perturbation theory while preserving the form of the Ward identities from the tree level to the renormalized level, under the sole condition that the matter fields are arranged into group representations permitting the cancellations of the 1-loop overall coefficients of all the anomalies which possibly exist in the given Yang Mills group. We have conducted our analysis to demonstrate in the clearest way that the phenomenon of anomalies and their classification are deeply rooted within the gauge structure and not within the purely technical problem which consists into

choosing a given regularization scheme rather than another. On the other hand, after having established that the Ward identities can be enforced to all order of perturbation theory when the anomalies cancel in the 1-loop approximation, it is obvious that one must choose the type of regularization which is best suited for the sake of perturbative higher order computations, and in particular for the determination of the structure of counterterms in perturbative theory. In practice, the most convenient and efficient regularization scheme turns out to be the dimensional one. It consists into continuing the value of the space-time dimension from d=4 to $d=4-2\varepsilon$, with $\varepsilon>0$[4,53]. Then all divergencies which can appear in closed loops of Feynman diagrams when one passes from a given order of perturbation theory to the next one get automatically regularized under the form of a pole structure in ε. The only subtility that one must bypass in this scheme concerns the fermion loops and the choice of a consistent definition of the γ^5 matrix. The simplest, and perfectly adequate definition for γ^5 is that of t'Hooft and Veltman[4,54]: γ^5 is introduced as a genuine 4-dimensional matrix $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ which anti-commutes with the four matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ and commutes with the remaining "2ε" γ matrices. The important property of this scheme is that it allows one to compute, for any Yang Mills theory in which the anomalies vanish at one loop, regularized Green function which satisfy order by order in perturbative theory the Ward identities for their divergent part as well as for their finite part in ε[55].

In what follows, we shall be concerned with the determination of the renormalized Lagrangian of an anomaly free Yang Mills theory, that is to say the Lagrangian which one can build by adding to a bare Lagrangian $\mathcal{L}_Q$ as in eq. (3.13) all the local counterterms which are necessary in order to compensate the divergencies which appear order by order in the perturbation theory determined from $\mathcal{L}_Q$. We shall show that if one uses a regularization procedure which preserves the Ward identities, then the renormalized Lagrangian is identical to the bare one, up a multiplicative renormalization factor for each field belonging to an irreducible group representation and for each independent parameter of which the bare Lagrangian is function. In other words, we shall demonstrate that the use of a BRS symmetry preserving regulator leads to the property that the theory is multiplicatively renormalizable. Our method will be reminiscent of that followed by Zinn Justin in his Bonn lectures[19], but it uses the anti-BRS symmetry and it allows one to include in a straightforward way the gauges in which 4-ghost interactions are present in the Lagrangian. We shall also take into account those finite renormalization effects which are necessary in order to pass from the minimal renormalization scheme to the physical one to determine a meaningful S-matrix.

We shall in fact proceed by induction. The starting point is

the bare BRS invariant Lagrangian (3.10) or (3.13) which determines the complete set of Feynman rules for the fields $A, c, \bar{c}, b$ and the matter fields Ψ. All parameters in $\mathcal{L}_Q$ are real numbers independent of ε. One chooses to express all the parameters of which the classical part $\mathcal{L}_{cl}$ of $\mathcal{L}_Q$ depends as functions of the masses of physical particles (transverse components of gauge fields, fermions, and physical Higgs components) and of the dimensionless coupling constant(s) of the gauge interactions. All these parameters are called physical by definition. Then the remaining part of $\mathcal{L}_Q$, $\mathcal{L}_Q - \mathcal{L}_{cl}$, can be expressed in function of the physical parameters and of other parameters which are dimensionless, and called gauge parameters by definition.

By assumption $\mathcal{L}_Q$ is s and $\bar{s}$ invariant. Therefore, by defining as in the previous section $v_\Phi, \bar{v}_\Phi, w_\Phi$ the sources of the local operators $\bar{s}\Phi$, $s\Phi$, $s\bar{s}\Phi$, one may consider the effective action in the tree approximation $I = \int dx \left(\mathcal{L}_Q(\Phi) + v_\Phi \bar{s}\Phi + \bar{v}_\Phi s\Phi + w_\Phi s\bar{s}\Phi \right)$ which satisfies by construction the Ward identities (4.9)

$$\mathcal{S}_I I = 0 = \bar{\mathcal{S}}_I I \qquad (6.43)$$

Then, assuming that the matter fields belong to group representations such that the 1-loop anomaly vanishes, one can initiate the perturbation theory from the Feynman rules stemming from $\mathcal{L}_Q$, and compute the radiative corrections by using the dimensional regularization scheme and imposing all relevant physical renormalization conditions. Our recursion hypothesis, which is verified at the zero loop level, is that one can obtain a perturbative expansion up to n-loops of the renormalized generating functional of the 1PI Green functions, denoted as Γ_R^n, such that Γ_R^n satisfies the Ward identities order by order in $\hbar$ up to order n and that it corresponds to a renormalized local action $I_R^{(n)}$ which sums up all relevant local counterterms up to order n with a full ε dependence. It is also assumed that $I_R^{(n)}$ is identical to the bare local action I up to to a multiplicative renormalization factor $z^{(n)}$ for each parameter or field belonging to an irreducible representation of the gauge group. Each factor $z^{(n)}$ can be expressed as a formal Taylor serie in $\hbar$ which ends up at order n

$$Z^{(n)} = 1 + \sum_{\rho=1}^{n} \hbar^{\rho} c_{\rho}^{(n)}(\epsilon) \tag{6.44}$$

where each coefficient $c_{\rho}^{(n)}$ is itself a Laurent sery in the dimensional regulator ε, depending of all the parameters in $\mathcal{L}_Q$.

According to our hypothesis, $I_R^{(n)}$ determines a renormalized generating functional of 1PI Green functions $\Gamma_R^{(n)}(\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi, \epsilon)$ which is finite in the limit $\varepsilon \to 0$. The finiteness of $\Gamma_R^{(n)}$ is a consequence of the cancellation between the poles in ε arising from the loop integrations in the diagrams determined by the Feynman rules from I and the poles in ε arising from the $1-Z^{(n)}$ factors in $I_R^{(n)}$. By assumption also, the finite parts in ε of $Z^{(n)}$ factors are such that $\Gamma_R^{(n)}$ and I satisfy the same physical renormalization conditions. Furthermore, because I and $I_R^{(n)}$ are identical up to multiplicative renormalizations, $I_R^{(n)}$ satisfies

identically the Ward identity (4.9),

$$\mathcal{S}_{I_R^{(n)}} I_R^{(n)} = 0 = \bar{\mathcal{S}}_{I_R^{(n)}} I_R^{(n)} \tag{6.45}$$

and the hypothesis that $\Gamma_R^{(n)}$ satisfies the Ward identities up to order n means

$$\mathcal{S}_{\Gamma_R^{(n)}} \Gamma_R^{(n)} = o(\hbar^{n+1}) = \bar{\mathcal{S}}_{\Gamma_R^{(n)}} \Gamma_R^{(n)} \tag{6.46}$$

For what follows it is most convenient to define the Taylor expansion in $\hbar$ of $I_R^{(n)}$ and $\Gamma_R^{(n)}$

$$I_R^{(n)} = I + \hbar I_1 + \cdots + \hbar^n I_n \tag{6.47}$$

$$\Gamma_R^{(n)} = I + \hbar \Gamma_1 + \cdots + \hbar^n \Gamma_n \tag{6.48}$$

where the functions $I_1, \ldots I_i, \ldots I_n$ are local functional of $\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi$. Each term I_i, $1 \le i \le n$ is a Laurent serie in ε, with coefficients depending on all parameters in $\mathcal{L}_Q$, but contains no pole in ε with a degree higher than i. The functions $\Gamma_1, \ldots \Gamma_i, \ldots, \Gamma_n$ are nonlocal functionals of $\Phi, v_\Phi, \bar{v}_\Phi, w_\Phi$ but each term Γ_i is analytic in ε and is also a function of the parameters which specify $\mathcal{L}_Q$. We are now ready to come to the induction proof. From the above assumptions, we shall demonstrate that one can obtain a renormalized effective action Γ_R^{n+1} at order $\hbar^{n+1}$ by adding to I_R^n local counterterms of order $\hbar^{n+1}$, $I_R^n \to I_R^{n+1}$, in such a way that our recursion hypothesis still hold for I_R^{n+1} and Γ_R^{n+1}. We shall then determine the explicit form of I_R and thus of the renormalized Lagrangian $\mathcal{L}_R$ at any finite order n.

We consider the renormalized local action I_R^n with its full dependence in ε. One can use it and compute the regularized but yet unrenormalized 1PI Green functions up to n+1 loops. Thereby, one gets an effective action $\Gamma_{div.}^{n+1}$ which is analytic in ε up to terms of order $\hbar^n$ but contains poles in ε for the terms of order h^{n+1}: it is so because I_R^n only contains the relevant counterterms up to order n. Furthermore, since all the sub-divergencies up to order $\hbar^n$ have been subtracted owing to these counterterms in I_R^n, the divergences of order $\hbar^{n+1}$ in $\Gamma_{div.}^{n+1}$ are overall divergencies, and the coefficients of poles $1/\varepsilon, \ldots 1/\varepsilon^{n+1}$ are local polynomials in the fields Φ and BRS operator sources v_Φ, $\bar{v}_\Phi$, w_Φ. We shall denote as $\hbar^{n+1}\Gamma_{n+1}^{\infty}$ the part of $\Gamma_{div}^{n+1} - \Gamma_R^n$ which is solely function of poles in ε, and $\hbar^{n+1}\Gamma_{n+1}^{fin.}$ the remaining part which is analytic in ε. Thus we decompose $\Gamma_{div.}^{n+1}$ as follows

$$\Gamma_{div.}^{n+1} = \Gamma_R^n + \hbar^{n+1}\left(\Gamma_{n+1}^{\infty} + \Gamma_{n+1}^{fin.}\right) \tag{6.49}$$

Now there is no anomaly and we use a regulator which preserves the BRS invariance at the level of Feynman rules. Consequently, the regularized generating functional Γ_{div}^{n+1} satisfies the Ward identities up to order $\hbar^{n+1}$

$$\mathcal{B}_{\Gamma_{div.}^{n+1}} \Gamma_{div.}^{n+1} = O(\hbar^{n+2}) = \bar{\mathcal{B}}_{\Gamma_{div.}^{n+1}} \Gamma_{div.}^{n+1} \tag{6.50}$$

By isolating in this equation the pole parts proportional to $\hbar^{n+1}$ one gets

$$\mathcal{B}_{\Gamma_{n+1}^{\infty}}(I) + \mathcal{B}_I(\Gamma_{n+1}^{\infty}) = 0 = \bar{\mathcal{B}}_{\Gamma_{n+1}^{\infty}}(I) + \bar{\mathcal{B}}_I(\Gamma_{n+1}^{\infty}) \tag{6.51}$$

We can therefore construct the following local renormalized action of order n+1 in $\hbar$

$$\tilde{I}_R^{n+1} = I_R^n - \hbar^{n+1}\Gamma_{n+1}^{\infty} \tag{6.52}$$

It is then obvious that $\tilde{I}_R^{n+1}$ determines a generating functional $\tilde{\Gamma}_R^{n+1} = \Gamma_R^n + \hbar^{n+1}\Gamma_{n+1}^{fin.} + O(\hbar^{n+2})$ which is finite up to order $\hbar^{n+2}$. Furthermore $\tilde{I}_R^{n+1}$ and $\tilde{\Gamma}_R^{n+1}$ satisfy by construction Ward identities at least up to order $\hbar^{n+1}$

$$\mathcal{B}_{\tilde{\Gamma}_R^{n+1}} \tilde{\Gamma}_R^{n+1} = O(\hbar^{n+2}) = \bar{\mathcal{B}}_{\tilde{\Gamma}_R^{n+1}} \tilde{\Gamma}_R^{n+1} \tag{6.53}$$

$$\mathcal{B}_{\tilde{I}_R^{n+1}} \tilde{I}_R^{n+1} = O(\hbar^{n+2}) = \bar{\mathcal{B}}_{\tilde{I}_R^{n+1}} \tilde{I}_R^{n+1} \tag{6.54}$$

Indeed one can easily verify eqs. (6.53,54) as a direct consequence of eqs. (6.50), (6.51) and from the recursive hypothesis (6.45, 46). This construction of $\tilde{I}_R^{n+1}$ and $\tilde{\Gamma}_R^{n+1}$ corresponds to a minimal renormalization of $\Gamma_{div.}^{n+1}$. However, we have not yet completed the recursion proof since, in this construction, the renormalization conditions of $\tilde{\Gamma}_R^{n+1}$ differ a priori from those of Γ_R^n by finite terms of order $\hbar^{n+1}$. Thus we still have to prove that the renormalization conditions can be readjusted without spoiling the Ward identities (6.46). We shall proceed in a way analogous as we did in the anomaly problem. One has the freedom of adding to $\tilde{I}_R^{n+1}$ a term $\hbar^{n+1}\delta I$ where δI is identical to I, up to finite rescaling factors δZ for each field belonging to an irreducible representation of the Yang Mills group and each parameter. In that way $I_R^{n+1} = \tilde{I}_R^{n+1} + \hbar^{n+1}\delta I$ still leads to a generating functional $\Gamma_R^{n+1} = \tilde{\Gamma}_R^{n+1} + \hbar^{n+1}\delta I$ which is finite up to order $\hbar^{n+1}$ and the renormalization conditions of Γ_R^{n+1} can be arranged now as identical to those of Γ_R^{n} by choosing suitably the δZ factors. Furthermore the identity up to rescaling factors between I and $I + \hbar^{n+1}\delta I$ implies that the latter local action satisfies too

$$\mathcal{S}_{I+\hbar^{n+1}\delta I}\left(I + \hbar^{n+1}\delta I\right) = 0 = \bar{\mathcal{S}}_{I+\hbar^{n+1}\delta I}\left(I + \hbar^{n+1}\delta I\right) \tag{6.55}$$

It implies in turn

$$\mathcal{S}_{\Gamma_R^{n+1}}\,\Gamma_R^{n+1} = o\left(\hbar^{n+2}\right) = \bar{\mathcal{S}}_{\Gamma_R^{n+1}}\,\Gamma_R^{n+1} \tag{6.56}$$

$$\mathcal{S}_{I_R^{n+1}}\,I_R^{n+1} = o\left(\hbar^{n+2}\right) = \bar{\mathcal{S}}_{I_R^{n+1}}\,I_R^{n+1} \tag{6.57}$$

The final step consists now in demonstrating that the Ward identities are satisfied exactly for I_R^{n+1}, and not only up to order $\hbar^{n+1}$. As a matter of fact, this amounts to a trivial verification since we have shown in Section IV and V that the most general local function which is a Yang Mills and Lorentz group scalar and satisfies the power counting requirement and the Ward

identities (6.57) can be expressed as follows

$$I_R^{n+1} = \int dx \left(\mathcal{L}_R + \frac{1}{X}\,\bar{v}_\Phi\, s_R\Phi + \frac{1}{\bar{X}}\, v_\Phi\, \bar{s}_R\,\Phi + \frac{1}{X\bar{X}}\, s_R\,\bar{s}_R\,\Phi \right) \tag{6.58}$$

where the renormalized BRS operators s_R and $\bar{s}_R$ are defined as

$$\begin{aligned}
s_R A_\mu^a &= X D_{\mu_R} c^a & \bar{s}_R A_\mu^a &= \bar{X} D_{\mu_R} \bar{c}^a \\
s_R c^a &= -\frac{X}{2} f^a_{bc_R} c^b c^c & \bar{s}_R \bar{c}^a &= -\frac{\bar{X}}{2} f^a_{bc_R} \bar{c}^b \bar{c}^c \\
s_R \bar{c}^a &= b^a & \bar{s}_R c^a &= -b^a - \bar{X} f^a_{bc_R} \bar{c}^b c^c \\
s_R b^a &= 0 & \bar{s}_R b^a &= -\bar{X} f^a_{bc_R} \bar{c}^b b^c \\
s_R \Psi^i &= -X T^i_{a\delta_R} \Psi^\delta c^a & \bar{s}_R \Psi^i &= -\bar{X} T^i_{a\delta_R} \Psi^\delta \bar{c}^a
\end{aligned} \tag{6.59}$$

Here $(D_{\mu_R})^a_b \equiv \partial_\mu \delta^a_b + f^a_{bc_R} A^c_\mu$, $f^a_{bc_R} = \sqrt{Z_3}\, Z_g\, f^a_{bc}$ and $T^i_{a\delta_R} = \sqrt{Z_3}\, Z_g\, T^i_{a\delta}$. Eqs (and (6.59) are identical, apart from the notational change $b_R \to b$ $Z \to \sqrt{Z_3}\, Z_g$. In eq. (6.58) $\mathcal{L}_R$ is the most general Lorentz and Yang-Mills group scalar Lagrangian which is s_R and $\bar{s}_R$ invariant. As a consequence of the analysis done in Section III.3, this implies that $\mathcal{L}_R$ has necessarily the following form

$$\begin{aligned}
\mathcal{L}_R = &-\frac{1}{4} Z_3 \left(F_{\mu\nu_R}\right)^2 - \frac{1}{2} Z_B \left(D_{\mu_R} \Psi_B\right)^2 - Z_F\, \bar{\Psi}_F\, \not{D}_R\, \Psi_F + V_R(\Psi_B) \\
&- \frac{1}{2}\frac{\tilde{Z}_3}{X\bar{X}}\, s_R \bar{s}_R \left(A_\mu^2 - \lambda_c\, \bar{c}\, c - \lambda_B\, Z_{\lambda_B}\, v \cdot \Psi_B \right) \\
&- \frac{1}{2} \tilde{Z}_3\, Z_{\lambda_b}\, \lambda_b\, b^2
\end{aligned} \tag{6.60}$$

On the other hand, all the renormalization factors Z, X and $\bar{X}$ which may appear in I_R^{n+1} are not anymore arbitrary. They are constrained by the identity between I_R^{n+1} and I when $\hbar=0$ and by the facts that Γ_R^{n+1} is finite up to order $\hbar^{n+1}$ when $\epsilon\to 0$ and furthermore satisfies the choosen physical renormalization conditions. These constraints determine the Z factors as formal series in $\hbar$ which stop at order n+1, $Z^{(n+1)} = 1 + \sum_{p=1}^{n+1} \hbar^p c_p^{(n+1)}(\epsilon)$, where the $c_p^{(n+1)}$ are Laurent series in ε. Using the definitions (6.49) of s_R and $\bar{s}_R$, one can easily expand $\mathcal{L}_R$ in function of all fields. Then one can check that $\mathcal{L}_R$ is identical to the bare Lagrangian (3.10) or (3.13) up to multiplicative renormalization factors

$$
\begin{array}{ll}
A_\mu^a \to \sqrt{Z_3}\, A_\mu^a & b^a \to \dfrac{\tilde{Z}_3}{\sqrt{Z_3}} \dfrac{b^a}{X} \\
c^a \to \sqrt{\tilde{Z}_3}\, c^a & \Psi_B^i \to \sqrt{Z_B}\, \Psi_B^i \\
\bar{c}^a \to \sqrt{\tilde{Z}_3}\, \bar{c}^a & \Psi_F^i \to \sqrt{Z_F}\, \Psi_F^i \\
g \to Z_g\, g & \lambda_B \to Z_{\lambda_B} \lambda_B \\
\lambda_c \to Z_{\lambda_c} \lambda_c & \lambda_b \to Z_{\lambda_b} \lambda_b
\end{array}
\qquad (6.61)
$$

These results mean precisely that the theory is multiplicatively renormalizable and furthermore that I_R^{n+1} in eq. (6.58) satisfies identically the equations

$$
\mathcal{B}_{I_R^{n+1}} I_R^{n+1} = 0 = \bar{\mathcal{B}}_{I_R^{n+1}} I_R^{n+1} \qquad (6.62)
$$

and not only up to order $\hbar^{n+1}$, which concludes our induction proof.

We shall end up this section with several remarks concerning specific properties of Z factors. The Z factors, displayed in eq. (6.61), are in fact not all independent. Some of them satisfy mutual relations which are gauge dependent. In order to obtain these relations in the easiest way one can write down the following Ward identities[17)]

$$
0 = s_R \langle b^a, \bar{c}^b \rangle_R = \langle b^a, b^b \rangle_R
$$

$$
0 = \bar{s}_R \langle b^a, c^b \rangle_R = -\langle b^a, b^b \rangle_R + \langle [\bar{c}, c]_R^a, b^b \rangle_R \qquad (6.63)
$$

and then eliminate the b field by its equation of motion. For instance, in a massless theory with a linear gauge condition, eq. (6.62) gives the well-known relation $Z_\alpha Z_3 = 1$ between the gauge field wave function and gauge parameter renormalization constants. More interesting are the non renormalization theorems satisfied by some of the Z factors in certain gauges. The most striking example in our opinion is that of parameter λ which modulates the strength of quartic ghost interactions. One can show (see Appendix B) that if λ_c is chosen equal to 0 or 1, it does not get renormalized at any order of perturbation theory. This property is the hint which justifies the use of Faddeev and Popov method in linear gauges. It is also interesting to observe that 1-loop corrections do not renormalize λ_c even when $\lambda_c \neq 0, 1$, but it is not known if this property survives 2-loop corrections.[17)]

To conclude this section, observe that our formalism applies equally well for a Yang Mills group which is either simple, or factorizable in a product of simple factors as in the case of the Weinberg-Salam-Glashow model based on SU(2)xU(1). However, in the latter case, and for the sake of practical computations in perturbation theory, one must handle technical problems which are related to the mixing of fields with the same quantum numbers through the renormalization. These problems have been studied in ref. (19).

VII GAUGE INDEPENDENCE AND UNITARITY IN RENORMALIZED YANG-MILLS THEORIES

VII.1 - INTRODUCTION

We shall present in this section a proof of the gauge independence and unitarity of the physical part of S-matrix in renormalizable Yang-Mills theories. For the validity of this proof, it is necessary to use a regulator which preserves the BRS invariance when one computes the renormalized Green functions from which the S-matrix elements are defined by application of LSZ reduction formula. Alternative proofs can be found in the literature[6,43] which are independent of the type of regulator that one may use. The latter type of proofs only requires the fulfillment of Ward identities for the renormalized Green functions, a property that we have proven in Section VI under the sole condition that the one loop coefficient of the anomaly vanishes. These demonstrations are more general because they do not depend on the choice of regularization, but they are rather complicated from a practical point of view. The proof that we shall display for the gauge independence has the advantage of generalizing in a more direct way the intuitive arguments of the tree approximation which we have presented in Section IV and which apply formally equally well for any general gauge theory. Unitarity of the physical part of the S-matrix will be demonstrated as the result of mutual cancellations between the unphysical components of classical field and ghosts among the intermediary states of any renormalized matrix elements, in a way which generalizes the historical example of Feynman.[14]

The property of multiplicative renormalizability is essential in what follows. This is the reason why we must choose a regulator which preserves the BRS invariance. We thus assume that the perturbation theory has been determined order by order from a renormalized Lagrangian such as in eq.(6.66) where all relevant renormalization Z factors are functions of the dimensional regulator ε. Because the expansion in the loop number of the 1PI vertices is equivalent to that in the gauge coupling constant g, it is meaningful to express all Z factors as a Taylor serie in g. Each term of this serie is therefore itself a Laurent serie in ε such that all renormalized Green functions are finite order by order in perturbation of g and satisfy the Ward identities and the relevant physical renormalization conditions. The algorithm for determining the Z factors has been made explicit in the last section.

As before we denote all fields $A, c, \bar{c}, b, \psi$ of which $\mathcal{L}_R$ is function by the generic notation Φ, and we introduce the notation ζ for all parameters in the Lagrangian other than the gauge coupling constant g. We also denote as Z_Φ , Z_g and Z_ζ the renormalization factors of fields Φ and parameters g and ζ respectively. The ε dependence in the renormalized Lagrangian is only through the Z factors.

We then introduce the following rescaled quantities respectively associated with g, Φ, and ζ

$$g_0 = Z_g\, g \tag{7.1a}$$

$$\Phi_0 = Z_\Phi^{1/2}\, \Phi \tag{7.1b}$$

$$\zeta_0 = Z_\zeta\, \zeta \tag{7.1c}$$

At any finite order of perturbation theory, because $Z_g = 1 + O(g^2)$ is a formal Taylor serie in g, there is a one to one correspondence between g and g_0 and one can invert eq. (7.1a) and express g as a formal serie in g_0, with coefficients depending on ε. In an analogous way, one can express the parameters ζ as Taylor series in g_0, with coefficients depending on the rescaled parameter ζ_0 and on ε.

The definitions (7.1) provide two equivalent expressions for computing in perturbation theory the renormalized generating functional $W_R[J]$ of connected Green functions of fields Φ. Observe that W_R is defined as the inverse Legendre transform in the couple of variables (J,Φ) of the renormalized effective action Γ_R already determined in Section VI, by keeping the regulator ε different from zero and setting all the source of BRS transforms of fields equal to zero. One can then express W_R as a functional in J depending on ε (with $d=4-2\varepsilon$) whose explicit dependence on the parameters of the theory can be expressed either through the renormalized parameters ζ and g

$$\exp iW_R[J] = \int [d\Phi]\, \exp i\int d^dx \left(\mathcal{L}_Q(Z_\Phi^{1/2}\Phi, gZ_g, \zeta Z_\zeta) + \Phi(x)J(x)\right) \tag{7.2a}$$

or through the rescaled parameters g_0 and ζ

$$\exp i\, W_R[J] = \int [d\Phi] \exp i \int d^d x \left(\mathcal{L}_Q(\Phi, g_0, \zeta_0) + \frac{\Phi(x) J(x)}{Z_\Phi^{1/2}} \right) \qquad (7.2b)$$

In expression (7.2a) the Z factors must be understood as expressed in function of ζ, g, ε, while in eq. (7.2b) Z_ϕ must be expressed in function of g_0, ζ_0 and ε.

Indeed, one can verify that both expressions (7.2a) and (7.2b) are equal up to the irrelevant overall factor $\prod_\Phi (Z_\Phi)^{1/2}$ by changing the variables ϕ into $Z_\Phi^{-1/2}\Phi$ in eq. (7.2a).

Therefore keeping $\varepsilon \neq 0$, one can compute equivalently any renormalized Green function by functional differentiations with respect to J either of eq. (7.2a) or of eq. (7.2b). Eq. (7.2a) yields the Green functions as a formal Taylor serie in g, while eq. (7.2b) yields them as another formal Taylor serie in the rescaled coupling constant g_0. Obviously, by changing the variables $g_0 \to g Z_g$ and $\zeta_0 \to Z_\zeta \zeta$ within any Green functions computed from eq.(7.2b), one recovers its expression as given from eq. (7.2a). Eqs. (7.2) can be seen as the most compact way of expressing the rules of perturbation theory at any given finite order in the dimensional regularization scheme. It is only when $W_R[J]$ is expressed in function of the finite parameters g and ζ that it has an analytic expression in ε, and determines directly the renormalized Green functions in the limit $\varepsilon \to 0$. If one express $W_R[J]$ as a Taylor serie in g_0 whose coefficients are functions of rescaled parameters ζ_0 and ε, each coefficient of this Taylor serie contains explicit poles in ε.

Such poles are in fact necessary to cancel the poles which are contained implicitly within g_0 and ζ_0. Note that in this construction $W_R[J]$ satisfies the Ward identities for $\varepsilon \neq 0$ so that the Ward identities between Green functions can be written as identities between formal series expressed order by order either in perturbation of g or of g_0.

VII.2 - GAUGE INDEPENDENCE

The general expression for $\mathcal{L}_Q$ has been given in eq. (3.10), (3.13). Thus in eq. (7.2b) $\mathcal{L}_Q$ can be written as follows

$$\mathcal{L}_Q = \mathcal{L}_{cl}(A, \Psi, g_0, \mu_0, \lambda_0) + s\bar{s}\left(\frac{\alpha_0}{2} b^2 + \frac{\beta_0}{2} \bar{c}c + \gamma_0 V \cdot \Psi_B \right) + \frac{\delta_0}{2} b^2 \qquad (7.3)$$

where s and $\bar{s}$ are defined in eq. (2.1).

Let us introduce another notation. We shall use the generic symbol ρ_0 for all the rescaled gauge parameters α_0, β_0, γ_0, δ_0 in Lagrangian (7.3), and ν_0 for all the parameters other than g_0 and ρ_0. We define as a change of gauge any change of the bare gauge parameters ρ_0 at fixed value of the rescaled coupling constant g and at fixed values of physical masses, i.e. at fixed locations of the zeros of 1PI 2 point functions of physical field components. Clearly, a change of gauge corresponds to a modification of the Feynman rules and therefore also of the value of $W_R[J]$. Our purpose is to prove that such a change leaves nevertheless invariant the physical part of the renormalized S-matrix, at any given finite order of perturbation theory.

A change of gauge amounts to adding to the Lagrangian (7.3) a term $\delta\rho_0\, s(\bar{K})$ where K is a local polynomial of all fields ϕ with dimension 3 in mass unit and ghost number -1. As said previously, by using the Lagrangian $\mathcal{L}_Q^{\rho_0+\delta\rho_0}$ instead of $\mathcal{L}_Q^{\rho_0}$ one modifies the Feynman rules. Therefore, all the renormalization factors Z^{ρ_0} which are necessary to obtain a finite $W_R[J]$ in the theory defined from $\mathcal{L}_Q^{\rho_0}$ are a priori different from the corresponding renormalization factors $Z^{\rho_0+\delta\rho_0}$ in the theory defined from $\mathcal{L}_Q^{\rho_0+\delta\rho_0}$.

The renormalized generating functionals $W_R^{\rho_0}[J]$ and $W_R^{\rho_0+\delta\rho_0}[J]$ of these two theories can be expressed as follows

$$\exp i\, W_R^{\rho_0}[J] = \int [d\Phi] \exp i \int d^dx \left(\mathcal{L}_Q^{\rho_0} + \frac{J(x)\Phi(x)}{(Z_\Phi^{\rho_0})^{1/2}} \right) \tag{7.4a}$$

$$\exp i\, W_R^{\rho_0+\delta\rho_0}[J] = \int [d\Phi] \exp i \int d^dx \left(\mathcal{L}_Q^{\rho_0} + \delta\rho_0\, s(\bar{K}) + \frac{J(x)\Phi(x)}{(Z_\Phi^{\rho_0+\delta\rho_0})^{1/2}} \right) \tag{7.4b}$$

where $\mathcal{L}_Q^{\rho_0}$ is defined in eq. (7.3). By differentiating functionally eqs. (7.4a) and (7.4b) with respect to the field sources J, one finds the following expressions of the renormalized connected Green functions of fields in the gauges ρ_0 and $\rho_0+\delta\rho_0$ respectively

$$G_R^{\rho_0}(\Phi_1,\dots\Phi_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{Z_i^{\rho_0}}} \langle T(\Phi_1,\dots,\Phi_n)\rangle^{\mathcal{L}_Q^{\rho_0}} \tag{7.5a}$$

$$G_R^{\rho_0+\delta\rho_0}(\Phi_1,\dots\Phi_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{Z_i^{\rho_0+\delta\rho_0}}} \langle T(\Phi_1,\dots,\Phi_n)\rangle^{\mathcal{L}_Q^{\rho_0+\delta\rho_0}} \tag{7.5b}$$

Here the indices i=1,....,n label the different field argument of Green functions, and $\langle T(\Phi_1,\dots\Phi_n)\rangle^{\mathcal{L}_Q^{\rho_0}}$ means the "bare" regularized connected Green function of fields $\Phi_1, \dots, \Phi_n$ which one can compute from the Lagrangian $\mathcal{L}_Q^{\rho_0}(\Phi, g_0, \rho_0, \gamma_0)$, in regularized perturbative theory of the <u>rescaled coupling constant g_0</u>

$$\langle T(\phi_1\cdots\phi_n)\rangle^{\mathcal{L}_Q^{\rho_0}} \equiv \frac{\int [d\Phi]\, \phi_1\cdots\phi_n \exp i \int d^dx\, \mathcal{L}_Q^{\rho_0}(\Phi, g_0, \rho_0, \gamma_0)}{\int [d\Phi] \exp i \int d^dx\, \mathcal{L}_Q^{\rho_0}(\Phi, g_0, \rho_0, \gamma_0)} \tag{7.5c}$$

One must consider the possibility that the values of the renormalized coupling constant g in both theories defined from $\mathcal{L}_Q^{\rho_0}$ and $\mathcal{L}_Q^{\rho_0+\delta\rho_0}$ are different. Indeed g is defined from the value of a given physical on-shell vertex, and we have defined the change of gauge as the variation of bare gauge parameter $\rho_0 \to \rho_0+\delta\rho_0$ while keeping fixed the values of the bare coupling constant g_0 and of physical masses. Therefore the equality of coupling constants in both theories is equivalent to the property $\frac{\partial}{\partial\rho_0} Z_g \big|_{m_{phys},\, g_0} = 0$ which is not guaranteed a priori. As a matter of fact this nontrivial result holds true, but we shall only be able to prove it later. Only after that, we shall recognize that our definition of gauge independence is the one which is of physical interest.

We shall use as a basic tool the following Ward identities which allow one to relate the renormalized Green functions (7.5a) and (7.5b) respectively computed in both gauges ρ_0 and $\rho_0+\delta\rho_0$

$$\prod_i \left(Z_i^{\rho_0+\delta\rho_0}\right)^{-1/2} \langle T(\Phi_1,\cdots\Phi_n)\rangle^{\rho_0+\delta\rho_0} = \prod_i \left(Z_i^{\rho_0}\right)^{-1/2} \langle T(\Phi_1,\cdots\Phi_n)\rangle^{\rho_0}$$
$$+\prod_i \left(Z_i^{\rho_0}\right)^{-1/2} \Big\{ \sum_j \Big(-\frac{1}{2}\frac{\Delta Z_j^{\rho_0}}{Z_j^{\rho_0}} \langle T(\Phi_1,\cdots,\Phi_n)\rangle^{\rho_0}$$
$$-\delta\rho_0 \left(Z_j^{\rho_0}\right)^{-1/2} \langle T(\phi_1,\cdots, s\Phi_j \textstyle\int dy\, \overline{K}(\Phi(y)), \cdots \Phi_n)\rangle^{\rho_0} \Big) \Big\} \tag{7.6}$$

In this equation we have defined $\Delta_i^{\rho_0} = Z_i^{\rho_0+\delta\rho_0} - Z_i^{\rho_0}$. Equation (7.6) holds true up to terms linear in $\delta\rho_0$ and to all orders in g_0. It generalizes eq. (4.15) by taking into account the renormalizaion constants and is equivalent to the functional identity

$$\exp iW_R^{\rho_0+\delta\rho_0}[J] = \int [d\Phi] \left\{1 + i\delta\rho_0 \int d^dy\, s\overline{K}(\Phi(y))\right\} \exp i\int d^dx \Big(\mathcal{L}_Q^{\rho_0} + \frac{\Phi(x)J(x)}{(Z_\Phi^{\rho+\delta\rho})^{1/2}}\Big)$$
$$= \int [d\Phi] \left\{1 - \frac{i\,\delta\rho_0}{(Z_\Phi^{\rho_0})^{1/2}} \int d^dy\, s\Phi(y) J(y) \int dz\, \overline{K}(\Phi(z))\right\} \exp i\int d^dx \Big(\mathcal{L}_Q^{\rho_0} + \frac{\Phi(x)J(x)}{(Z_\Phi^{\rho_0})^{1/2}}\Big) \tag{7.7}$$

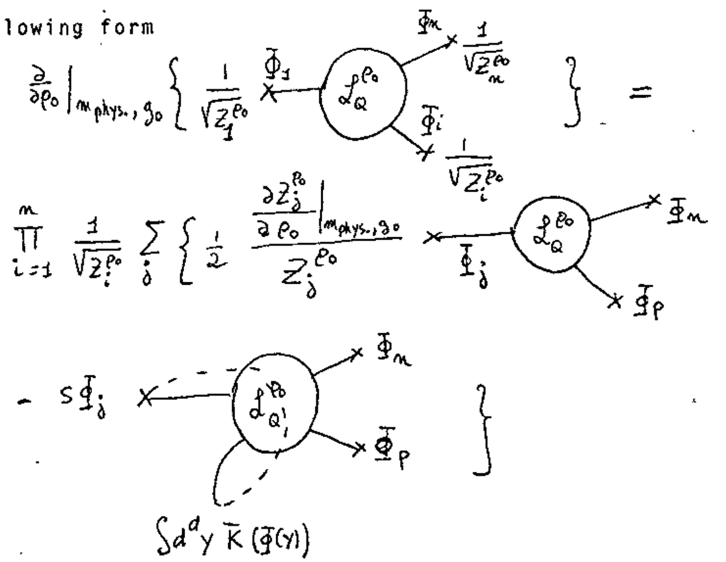
The first equalty in eq. (7.7) can be demonstrated by expanding eq. (7.4b) up to terms linear in $\delta\rho_0$. The last equality in eq. (7.7) can be verified by changing the variable $\Phi \to \Phi + \delta\rho_0\, \bar\eta\, s\Phi$ in the expression $\int [d\Phi]\, \overline{K}(\Phi) \exp i\int d^dx (\mathcal{L}_Q + J(x)\Phi(x))$ as we have done in Section IV in order to get eq. (4.13). All these manipulations make sense at any given finite order of perturbation theory, provided that ϵ is kept different from zero.

We shall demonstrate now that eq. (7.7) implies

$$\frac{\delta}{\delta\rho_0}\Big|_{m_{phys},\,g_0} (LSZ)\, G_R^{\rho_0}(\Phi_1,\cdots,\Phi_n) = 0 \tag{7.8}$$

under the condition that the arguments $\Phi_1,\cdots,\Phi_n$ correspond to physical components of classical fields (transverse part of a gauge boson, fermion field, or physical part of a Higgs field). The symbol (LSZ) means the application of the LSZ reduction formula projector to the connected Green function $G_R^{\rho_0}$.

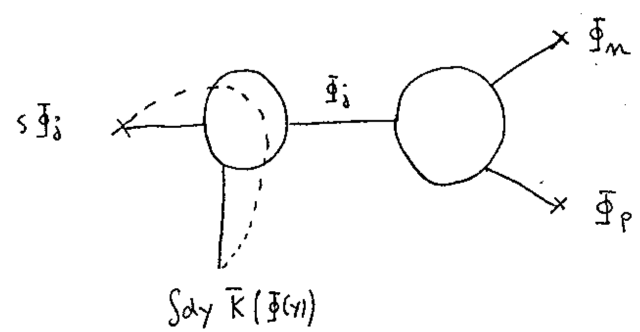
For the sake of notational conveniency we shall use a diagrammatic representation. Eq. (7.6) can be rewritten under the following form



(7.9)

where all blobs stand for the regularized "bare" connected Green functions defined in eq. (7.5c). Each external line symbolizes an argument Φ_p of $G_R^{\rho_0}$. The symbol "x" at the end of an external line symbolizes the amputation of external fields by LSZ reduction operators. The dotted line stand for the continuous ghost line connecting the BRS operators $s\Phi_j$ to the composite operator $\int d^d y\, \bar{K}(\Phi(y))$.

The LSZ operator acting on line j can only pick up the residue of a pole located at the mass of physical components of the field Φ_j. It follows that if one applies the LSZ operators on both sides of eq. (7.9), among all Feynman diagrams which contribute to the last Green function represented in the r.h.s. of eq. (7.9), only those with the following structure are not cancelled

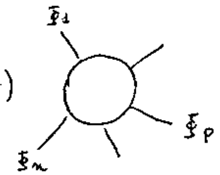


(7.10)

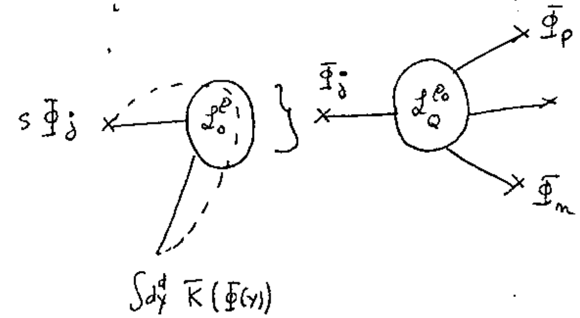
Indeed, any diagram with a different structure is cancelled either because there is no pole in the j- channel, i.e. when the diagram is 1PI in the j-channel, or because the j-channel corresponds to a ghost propagator whose pole is located at a mass different from a physical one. The latter situation occurs when j corresponds either to a Higgs field H with sH=c<H>+.... or to a gauge field with $sA_\mu = \partial_\mu c + \dots$.

Thus, eq. (7.9) can be rewritten under the following form

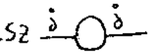
$$\frac{\delta}{\delta\rho_0}\Big|_{m_{phys},g_0} (LSZ)\, G_R^{\rho_0}(\Phi_1,\dots,\Phi_n) \equiv \frac{\delta}{\delta\rho_0}\Big|_{m_{phys},g_0} (LSZ)$$



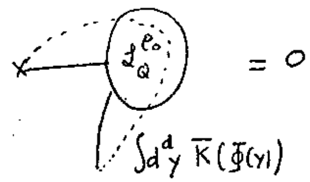
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{Z_i^{\rho_0}}} \sum_j \Bigg\{ \frac{1}{2} \frac{\frac{\delta Z_j^{\rho_0}}{\delta\rho_0}\Big|_{m_{phys},g_0}}{Z_j^{\rho_0}} + s\Phi_j$$



(7.11)

and it remains to prove that the terms which factor out in the r.h.s. of eq. (7.11) vanish identically. But the latter property is a direct consequence from of the constraint that the physical masses are the same in both theories generated from $\mathcal{L}_Q^{\rho_0}$ and $\mathcal{L}_Q^{\rho_0+\delta\rho_0}$. Indeed, consider eq. (7.11) for n=2. Then, the l.h.s. vanishes since LSZ  must be equal to 1 independently of ρ_0, by definition of LSZ formula, and one gets therefore the following equation which is true for all classical fields Φ_j

$$\frac{1}{2}\,\frac{\left.\frac{\delta Z_j^{\rho_0}}{\delta\rho_0}\right|_{m_{phys},g_0}}{Z_j^{\rho_0}} + s\Phi_j \quad = 0 \tag{7.12}$$



The latter equation implies in turn that the whole r.h.s. of eq. (7.11) vanishes identically.

Going back to the definition of ρ_0 $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ in eq. (7.3), we have therefore demonstrated the following theorem for the renormalized S-matrix elements

$$\left.\frac{\delta}{\delta\alpha_0}\right|_{m_{phys.},g_0} \langle phys'|S|phys\rangle_R = 0$$

$$\left.\frac{\delta}{\delta\beta_0}\right|_{m_{phys.},g_0} \langle phys'|S|phys\rangle_R = 0$$

$$\left.\frac{\delta}{\delta\gamma_0}\right|_{m_{phys.},g_0} \langle phys'|S|phys\rangle_R = 0$$

$$\left.\frac{\delta}{\delta\delta_0}\right|_{m_{phys.},g_0} \langle phys'|S|phys\rangle_R = 0$$

(7.13)

in which $|phys\rangle$ and $|phys\rangle'$ stand for any physical states. Now, since the coupling constant g is defined for each given set of parameters α_0, β_0, γ_0, δ_0 as the value of an on-shell vertex with external physical fields, eq. (7.13) imply as a particular case the following property of g

$$\left.\frac{\delta g}{\delta\rho_0}\right|_{m_{phys.},g_0} = 0 \quad , \text{ for } \quad \rho_0 = \alpha_0, \beta_0, \gamma_0, \delta_0 \tag{7.14}$$

We have therefore demonstrated the nontrivial result that the value of the renormalized coupling constant must be the same for all theories with arbitrary rescaled gauge parameters α_0, β_0, γ_0, δ_0 but the same physical masses and rescaled coupling constant g. This result proves the following theorem for the renormalization constant of the gauge coupling constant $Z_g=g_0/g$

$$\left.\frac{\delta Z_g}{\delta\rho}\right|_{m_{phys.},g_0} = 0 \quad , \text{ for } \quad \rho = \alpha, \beta, \gamma, \delta \tag{7.15}$$

Indeed one can verify that eq. (7.15) follows from eq. (7.14) by substitution of $Z_g^{-1} g_0$ in place of g and by use of the one-to-one correspondence which exists between (α_0, β_0, γ_0, δ_0) and (α, β, γ, δ) at fixed values of g_0 and physical masses.

Finally, using eqs. (7.15), we can rewrite eq. (7.13) under the following form

$$\left.\frac{\delta}{\delta\rho}\right|_{m_{phys.},g} \langle phys' | S | phys \rangle_R = 0$$

for $\rho = \alpha, \beta, \gamma$ and δ.

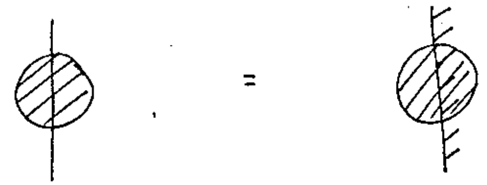
(7.13b)

We have therefore proven the independence of the physical part of the renormalized S-matrix with respect to variations of all renormalized gauge parameters, at fixed values of physical masses and coupling constant g. This is the desired gauge independence theorem for physics in perturbation theory.

One should note that we have implicitely assumed in this demonstration the existence of the S-matrix, i.e. the applicability of LSZ reduction formula. It means in particular that the proof is only valid when all gauge bosons ar massive. Whenever massless gauge bosons are present in the theory, all our arguments become formal. It is however believed that one can generalize them and prove the Gauge independence of physical observables, at least when there is only one massless gauge boson corresponding to a factor of the gauge group [0, 4, 5].

VII.3 - UNITARITY

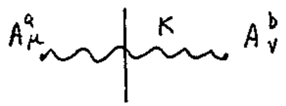
To prove the unitarity of the physical sub-part of the S-matrix, it is sufficient to prove the t'Hooft and Veltman unitarity equations order by order in perturbative theory[4]. These equations mean that a summation over all the possible intermediary states (classical and unphysical) within any given physical S-matrix element is equivalent to a summation restricted solely to the possible physical intermediary states. The unitary equation can be written graphically as follows[4]



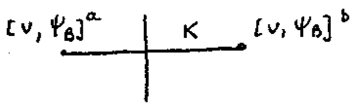
(7.16)

Here the blobs stand for a generic physical S-matrix element, defined by applying the on-shell reduction formula to a renormalized connected Green function whose arguments are physical fields (transverse gauge bosons, fermions or physical Higgs particle). The vertical straight line denotes all possible Cutkoski cuts on all intermediary states, classical and ghosts, and the shaded line denotes the Cutkoski cuts which apply only to physical intermediate states.

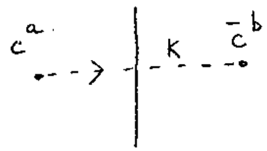
For demonstrating the unitarity equation we shall use the so-called Feynman-t'Hooft gauge. It is not a lost of generality since we have proven in the last Section the gauge independence of physical renormalized S-matrix elements. This gauge is defined by setting $\lambda_c = 0$ and $\lambda_\beta = \lambda_b = 1$ in Lagrangian (3.10). After the elimination of the b field through its algebraic equation of motion, this Lagrangian takes the form given in formula (B.1), with $\alpha=\beta=1$. Formula (B20) gives the corresponding renormalized Lagrangian which includes all the counterterms which are necessary to render the theory finite. Note that the gauge fixing term does not get renormalized. The advantage of this gauge is that there are no quartic ghost interactions and no free transitions between Higgs field unphysical components and gauge boson longitudinal components. Moreover, in this particular gauge, the masses of non-physical components of longitudinal gauge bosons, of Higgs fields and of Faddeev-Popov ghosts are equal to those of corresponding gauge boson physical components. These properties can be checked easily from the Feynman rules of Lagrangian (B.1). It implies that theCutkoski rules for cutting internal lines of gauge bosons, of unphysical Higgs particle and of ghost fields, have respectively the following form



$$\delta^{ab}\, g_{\mu\nu}\, \delta(K^2 - M^2_{ab})\, \Theta(K_0)$$



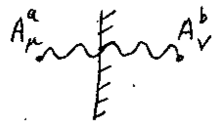
$$\delta^{ab}\, M^2_{ab}\, \delta(K^2 - M^2_{ab})\, \Theta(K_0) \tag{7.17}$$



$$\delta^{ab}\, \delta(K^2 - M^2_{ab})\, \Theta(K_0) \tag{7.18}$$

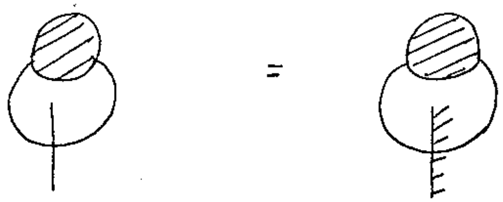
Note that we have also choosen a basis in the gauge group Lie algebra such that the mass matrix elements of unphysical field components are diagonal $M^2_{ab} \equiv ([\ ,v][v,\ ])_{ab} \equiv T^K_{ai} T^\delta_{bK} V_i V_\delta = M^2_a \delta_{ab}$. Furthermore, in the choosen gauge, M_a is equal to the physical mass of the transverse component A^a of gauge boson A.

On the other hand, the physical Cutkoski cut operators act in the diagrammatic expression of Green functions only upon lines of matter fields and physical transverse parts of gauge bosons but not on lines of ghost, unphysical Higgs fields or longitudinal unphysical components of gauge bosons. The Cutkoski rule for the physical transverse part of gauge bosons is defined as follows



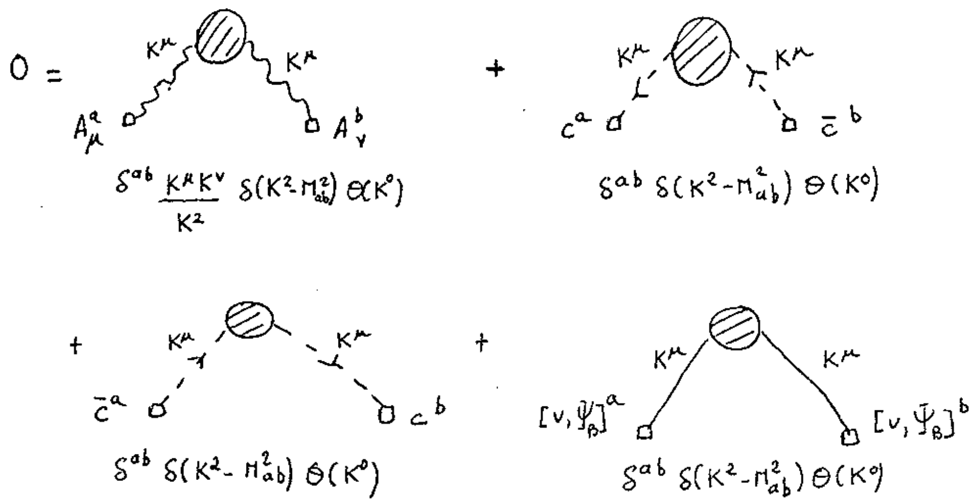
$$\delta^{ab}\left(g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}\right) \delta(K^2 - M^2_{ab})\, \Theta(K_0) \tag{7.19}$$

A straightforward induction proof allows one to reduce the demonstration of eq. (7.16)into that of the simpler equation[4])



(7.20)

In both sides of this equation, the Cutkoski cuts are applied only once on any internal line which can be possibly cut in the given S-matrix element. Then, by combining eq. (7.20) with the definitions (7.17-7.19) of Cutkoski rules, one finds that the proof of the unitarity equation reduces to that of the following equation



(7.21)

where the $\square$ symbol means that the blobs are amputated in the K^μ channel, while an overall integration over d^dK is done with a weight specified under each diagram. Observe that the terms which correspond to the $g_{\mu\nu}$ terms in the Cutkoski rules (7.17) and (7.19) have cancelled each others on both side of eq. (7.20).

The point is that eq. (7.21) holds true as a consequence of Ward identities. In order to demonstrate this, consider both

following renormalized connected Green functions which are set on shell for all arguments, except for those which are written explicitly, i.e. $\bar{c}$, $\partial^\mu A_\mu$, Ψ_B .

$$\langle \bar{c}^a(K), \partial A^b(-K), \ldots \rangle_R^{\text{on-shell}}$$

$$\langle [v, \Psi_B(K)]^a, \bar{c}^b(-K), \ldots \rangle_R^{\text{on-shell}} \tag{7.22}$$

Here the dots stand for all external on-shell physical fields which are unspecified around each blob in (7.16) or (7.20). It must be understood that these fields are set on shell by LSZ reduction operators, symbolized by the expression "on-shell" in eq. (7.22). The BRS transforms of the Green functions defined in eq. (7.22) vanish as a consequence of the BRS invariance of the theory, and furthermore the s_R transforms of physical fields are cancelled by application of reduction formula. Thus when one computes the BRS variation of these quantities, only remain the BRS variations of fields which have been made explicit in eq. (7.22). On the other hand, one has from eq. (B.21) that $s_R\,\bar{c} = -\partial A - [v, \Psi_B]$. Consequently, one gets the following "on-shell" Ward identities

$$\langle \partial A^a(K), \partial A^b(-K), \ldots \rangle_R^{\text{on shell}} + \langle [v, \Psi_B(K)]^a, \partial A(-K), \ldots \rangle_R^{\text{on-shell}}$$
$$+ \langle \bar{c}^a(K), K^2 c^b(-K) + K^\mu [A_\mu, c]^b(-K), \ldots \rangle_R^{\text{on-shell}} = 0$$

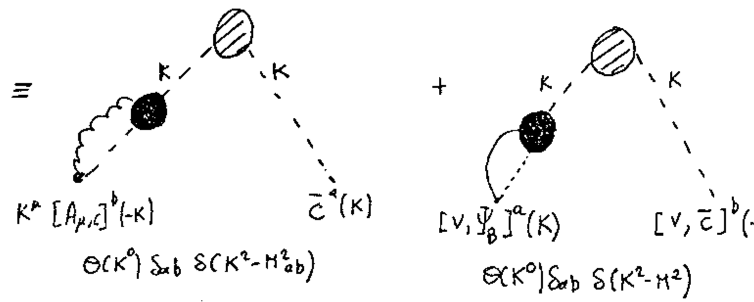
(7.23a)

$$\langle [v,\Psi_B(K)]^a, [v,\Psi_B(-K)]^b, \dots \rangle_R^{\text{on-shell}} + \langle [v,\Psi_B(K)]^a, \partial A^b(-K), \dots \rangle_R^{\text{on-shell}}$$
$$+ \langle [v,[v,c(K)]]^a + [v,[\Psi_B,c]]^a(-K), \bar{c}^b(-K), \dots \rangle_R^{\text{on-shell}} = 0 \qquad (7.23b)$$

Observe that, were any of on-shell fields in Green functions (7.22) either a ghost, an unphysical Higgs field or an unphyical gauge field component, its BRS transform would contribute on-shell to the BRS variations of eq. (7.22), and eqs. (7.23) would be wrong.

Now, it is easy to show that the Ward identities (2.23) allows one to prove eq. (7.20). Indeed, let us integrate eqs. (7.23) with the weight $\int d^dK\, \delta_{ab}\, \delta(K^2-M_{ab}^2)\, \Theta(K^0)/M_{ab}^2$. Then, by adding together both eqs. (7.23a) and (7.23b), the second terms of each equation cancel each others. Furthermore by using the on-shell relation $\delta(K^2-M_{ab}^2)\, M_{ab}^2 = \delta(K^2-M_{ab})\, K^2\, \delta_{ab}$ and the identity $\langle [v,[v,c]]^a, \bar{c}^b, \dots \rangle = \langle [v,c]^a, [v,\bar{c}]^b, \dots \rangle = \delta_{ab} M_{ab}^2 \langle c_a, \bar{c}_b, \dots \rangle$ the desired equation (7.20) is obtained up to the following term

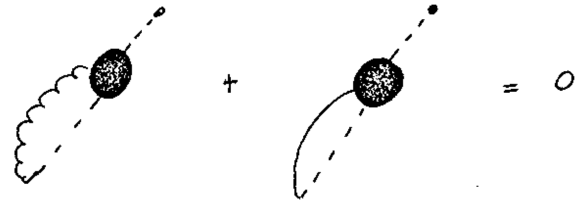
$$\Delta = \int d^dK\, \Theta(K^0)\, \delta_{ab}\, \delta(K^2-M_{ab}^2)\, \frac{1}{M_{ab}^2} \Big\{ \langle \bar{c}^a(K), K^\mu [A_\mu,c]^b(-K), \dots \rangle^{\text{on-shell}} + \langle [v,[\Psi_B,c]]^a(K), \bar{c}^b(-K), \dots \rangle^{\text{on-shell}} \Big\}$$

$\equiv$ 

(7.24)

where the black bubbles stands for the 1PI 2 point functions $\langle [A_\mu,c](K), \bar{c}(-K) \rangle^{1PI}$ and $\langle [v,\Psi_B](K), [v,\bar{c}(-K)] \rangle^{1PI}$

Consequently, to complete the proof of the unitarity equations, it remains only to demonstrate $\Delta = 0$. But this final equation is true owing to the equality



(7.25)

which is a direct consequence of the Ward identity eq. (B.14) demonstrated in Appendix B to all orders of perturbation theory in the choosen gauge.

$$\frac{\delta\Gamma}{\delta\bar{c}^a} = \partial_\mu \frac{\delta\Gamma}{\delta\bar{v}_{A_\mu^a}} + \left[v, \frac{\delta\Gamma}{\delta\bar{v}_{\Psi_B}} \right]^a \qquad (7.26)$$

The link between eqs. (7.25) and (7.26) is indeed trivial: by recalling the definition of $\bar{v}_A$ and $\bar{v}_{\Psi_B}$ as the sources of the BRS operators $sA_\mu = \partial_\mu c + [A_\mu, c]$ and $s\Psi_B = -c\,\Psi_B$, one can trivially verify that eq. (7.26) implies (7.25) by functional differentiation with respect to c.

Therefore, we have proven the unitarity equations in Feynman t'Hooft gauge. This demonstrates the unitarity of the physical part of the renormalized S-matrix for all gauges owing to the gauge invariance theorem demonstrated in the previous sub-section. Observe that the use of a physical renormalization scheme has been essential in order to give a sense to the cut diagrams.

VIII - CONCLUSION

In this review article, we have been motivated by the idea that the construction of a quantum theory associated with a given gauge invariance should be as simple as the construction of the classical theory. This idea is supported by the existence of an intrinsic BRS symmetry associated with any general gauge symmetry. We believe that our presentation reveals new features of the structure of gauge theories and also simplifies some of the aspects of the previous studies. Nevertheless a great deal of work remains clearly to be done in order to acquire a deeper understanding of the formalism in which one treats as one all components of an extended gauge field, classical and ghost. In a quite encouraging way, recent work has shown the relevance of such a unified formalism not only in the case of flat space Yang Mills theories, which have been quite extensively discussed in this article, but also for more general gauge theories, which can be coupled to gravity, with or without local supersymmetry.[28,29] The ultimate goal should be the construction of a truly unifying picture in which a single entity, containing as a whole the classical and ghost fields, would be the sole element of quantum field theory. We believe that such a formalism must exist, and could help to by-pass the technical difficulties which arise in those gauge theories which are not renormalizable by power counting.

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APPENDIX A

DETERMINATION OF THE MOST GENERAL YANG-MILLS BRS AND ANTI-BRS INVARIANT LAGRANGI

A.I - PURE GAUGE THEORY

Let us build the most general Lorentz invariant local polynomial K in the fields A , c, $\bar{c}$, b which has canonical dimension 4, ghost number 0 and is s and $\bar{s}$ invariant. The result, up to some exact divergences, is the Lagrangian written in eq. (3.9).

In full generality we can write this polynomial as

$$K = K_4 + s\bar{K}_3 + \bar{s}K_3 + s\bar{s}K_2 \tag{A.1}$$

where K_4, $\bar{K}_3$, K_3 and K_2 are themselves polynomials in the fields with dimension 4, 3, 3 and 2 and ghost number 0, -1, +1 and 0, respectively. s and $\bar{s}$ are given in eq. (2.11) and satisfy the nilpotency relations

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0 \tag{A.2}$$

Eq. (A.2) doesn't imply any constraint on K_2, but implies that $\bar{K}_3$ and K_3 are defined modulo the choice on K_2, and that K_4 is defined modulo the choice of K_2, K_3, $\bar{K}_3$. Furthermore K_2, K_3, $\bar{K}_3$ and K_4 are only defined up to a divergence.

By imposing that K has dimension 4 and ghost number 0, one immediately finds that

$$K_2 = \alpha_3 A_\mu^2 + \alpha_4 \bar{c}c \tag{A.3}$$

Now K_3 can only be a combination of the following three independent monomials :

$$b\bar{c} \qquad \text{(A.4a)}$$

$$b\bar{c} + \frac{1}{2} c\,[\bar{c},c] = -\,\bar{s}(c\bar{c}) \qquad \text{(A.4b)}$$

$$A^\mu \partial_\mu \bar{c} = \frac{1}{2}\,\bar{s}(A^2) \qquad \text{(A.4c)}$$

One can eliminate (A.4b) and (A.4c) because they are of the type $\bar{s}K_2$, and we have therefore

$$\bar{K}_3 = \alpha_5\, b\bar{c} \qquad \text{(A.5)}$$

with

$$s\bar{K}_3 = \alpha_5\, b^2 \qquad \text{(A.6)}$$

It is trivial to check that $\bar{s}s\bar{K}_3 = \alpha_5 \bar{s} b^2 = 0$.

In most Lie groups, but not in SU(2), there exists an invariant symmetric tensor of rank 3, d_{abc} . In these cases, one must add to formula (A.4) the following candidate :

$$d_{abc}\, A^a_\mu A^{\mu b} \bar{c}^c \equiv \bar{c}\,\{A_\mu, A^\mu\} \qquad \text{(A.7)}$$

This type of gauge has been studied in ref. (59) . It is easy to see that it leads to gauge conditions non-linear in the gauge fields. However, we discard them because, they break the $\bar{s}$ invariance of the Lagrangian. Indeed,

$$\bar{s}\,(s\bar{c}\{A^\mu, A_\mu\}) = \bar{s}\,(b\{A_\mu, A^\mu\} - 2\bar{c}\,\{D_\mu c, A^\mu\}) \neq 0 \qquad \text{(A.8)}$$

In order to determine K_3 one proceeds in the same way as for $\bar{K}_3$. The result is that

$$K_3 = \alpha'_5\, b c \qquad \text{(A.9)}$$

with

$$\bar{s} K_3 = -\,\alpha'_5\, b^2 \qquad \text{(A.10)}$$

Therefore, without loss of generality, one can write

$$s\bar{K}_3 = \bar{s} K_3 = \alpha_5 b^2 = \alpha_5\, s(b\bar{c}) = -\alpha_5\, \bar{s}(bc) \qquad \text{(A.11)}$$

Finally, we have to find K_4 . The terms depending on b, c and $\bar{c}$ can only be combinations of the following monomials :

$$b^2 \;,\; b\,\partial A \;,\; b\,[\bar{c},c] \;,\; \partial_\mu \bar{c}\, \partial^\mu c \;,\; \partial_\mu c\,[A_\mu, \bar{c}]$$

$$[A_\mu, \bar{c}]\,\partial^\mu c \;,\; [A_\mu, \bar{c}][A^\mu, c] \;,\; [\bar{c}, c]^2 \qquad \text{(A.12)}$$

By redefinition of K_2 and K_3 , we may absorb the terms $[\bar{c},c]^2$, b^2 and $\partial_\mu \bar{c}\,[A^\mu, c]$ since

$$s(\bar{c} b) = b^2$$

$$s\bar{s}\,[\bar{c},c] = b^2 + b\,[\bar{c},c] + \frac{1}{2}\,[\bar{c},c]^2$$

$$\frac{1}{2}\, s\bar{s}\, A_\mu^2 = \partial_\mu \bar{c}\,[A_\mu, c] + \partial_\mu \bar{c}\, \partial^\mu c \qquad \text{(A.13)}$$

Further, the term $b[\bar{c},c]$ is the only term which leads to a term of the type $[b,\bar{c}] \cdot [c,\bar{c}]$ upon an s transformation, and therefore must be discarded. Requiring that the s transform of the other terms doesn't depend on b and $\partial_\mu b$, one can then eliminate any linear combination of these terms. One can also eliminate in the same way all terms constructed from the d_{abc} tensor defined in eq. (A.7).

Thus, K_4 can only be a function of the gauge field A^{μ}. BRS invariance is then equivalent to the classical gauge invariance and it is well known that the only possible terms are

$$K_4 = \alpha_1 F_{\mu\nu}^2 + \alpha_2 \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{A.14}$$

We have therefore shown that the most general form satisfying our constraint is

$$K = \alpha_1 F_{\mu\nu}^2 + \alpha_2 \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + s\bar{s}\left(\alpha_3 A_\mu^2 + \alpha_4 \bar{c}\, c\right) + \alpha_5 b^2 \tag{A.15}$$

A.II - INCLUSION OF MATTER FIELDS

We show how to include matter fields in our formalism. The s and $\bar{s}$ transformations for a scalar field Ψ_B and a fermion field Ψ_F are the following

$$s\left(\Psi_B + v\right) = s\Psi_B := -c\cdot(\Psi_B + v) \qquad \bar{s}(\Psi_B + v) = \bar{s}\Psi_B = -\bar{c}\cdot(\Psi_B + v)$$
$$s\Psi_F = -c\,\Psi_F \qquad \bar{s}\,\Psi_F = -\bar{c}\,\Psi_F \tag{A.16}$$

Ψ_B and Ψ_F belong to arbitrary representations of the gauge group. When the shift v is different from zero, the symmetry is spontaneously broken.

In order to construct the most general s and $\bar{s}$ invariant, Lorentz invariant, 4-dimensional Lagrangian with ghost number zero, one proceeds similarly as in sub-section A.I.

Let us first consider the terms which are not obtained by action of s, $\bar{s}$ or $s\bar{s}$ on a polynomial of the fields. By inspection it can only be the classical locally gauge-invariant Ψ_B and Ψ_F dependent Lagrangian.

The remaining pieces can only be linear combinations of the following elements

$$\begin{array}{llll} s\bar{s}\,\Psi_B^2 & (a) & s\bar{s}\,(v\,\Psi_B) & (d) \\ s\,(\bar{c}\,\{\Psi_B,\Psi_B\}) & (b) & s\,(\bar{c}\,\{\Psi_B, v\}) & (e) \\ \bar{s}\,(c\,\{\Psi_B,\Psi_B\}) & (c) & \bar{s}\,(c\,\{\Psi_B, v\}) & (f) \end{array} \tag{A.17}$$

(fermions fields cannot be used because they have canonical dimension $\frac{3}{2}$). Now (a) is zero, (b) and (e) are s invariant but not $\bar{s}$ invariant, (c) and (f) are $\bar{s}$ invariant but not s invariant. If we had introduced several scalar fields which can be combined by pairs into group scalars, say $\Psi_{1B}\cdot\Psi_{2B}$, then one could have tried expressions such as (a), (b) and (c) whith the change $\Psi_B^2 \to \Psi_{1B}\cdot\Psi_{2B}$. Again (a) is zero, (b) is not $\bar{s}$ invariant and (c) is not s invariant.

Therefore, the only piece which survives to our analysis is $s\bar{s}v\Psi_B$

The most general s and $\bar{s}$ invariant Lagrangian which is a function of Ψ_B and Ψ_F is therefore

$$\mathcal{L}_{matter}(\Psi_B,\Psi_F) = \mathcal{L}_{cl.}(\Psi_B,\Psi_F,A_\mu) + \lambda\, s\bar{s}\,(v\cdot\Psi_b) \tag{A.18}$$

where λ is a new gauge parameter. By expanding $s\bar{s}(v\cdot\Psi_B)$, one finds

$$s\bar{s}(v\cdot\Psi_B) = b\cdot[v,\Psi_B] - [\Psi_B,c][v,\bar{c}] - [v,c][v,\bar{c}] \tag{A.19}$$

The second and third terms of eq. (A.4) give, respectively, the ghost-scalar interaction and ghost mass terms.

A.III - NON-COVARIANT GAUGES

All the non-covariant gauges which are renormalizable by power counting, can oe obtained by adding to the classical Lagrangian a linear combination of the following three expressions

$$-\frac{1}{2}\, s\bar{s}(\eta A)^2 = b(\eta\cdot\partial)(\eta\cdot A) - (\eta\,\partial\bar{c})(\eta\, Dc) \tag{A.20}$$

$$s(\bar{c}\,\eta\cdot A) = b\,\eta\cdot A - \bar{c}(\eta\cdot Dc) \tag{A.21}$$

$$\bar{s}(c\,\eta A) = b\,\eta A - \bar{c}(\eta\cdot Dc) + \eta\cdot A[\bar{c},c] \tag{A.22}$$

η^μ is a fixed Lorentz vector. The canonical dimension of η^μ is 0 in eq. (A.20) and 1 in eqs. (A.21) and (A.22).

The form (A.20) is s and $\bar{s}$ invariant by construction. The form (A.21) is $\bar{s}$ invariant, and the form (A.22) is $\bar{s}$ invariant but not s invariant.

One can, in fact, consider independent forms with different vectors η. The form (A.20) leads to the Coulomb gauge, and the forms (A.21) and (A.22) lead to the axial gauge. This can be easily seen by adding these forms to the classical Lagrangian, and then by integrating out the b field, with help of the identity

$$\int db\, \exp i \int b\, B\, dx = \delta(B) \tag{A.23}$$

Indeed, by adding to the classical Lagrangian the form $-\frac{1}{2}\sum_{i=1,2,3} s\bar{s}(\eta_i^\mu A_\mu)^2$, where the vectors η_i^μ are such that $\sum_{i=1,2,3}(\eta_i^\mu\partial_\mu)(\eta_i^\nu A_\nu) = \mathrm{div}\,\vec{A}$, and by integrating over the b field, one obtains the following functional :

$$\int [dA][db][d\bar{c}]\,\delta(\mathrm{div}\,\vec{A})\,\exp i\int dx\left(\mathcal{L}_{cl} + \lambda(\eta_i^\mu\partial_\mu\bar{c})(\eta_i^\nu D_\nu \bar{c})\right) \tag{A.24}$$

By adding to the classical Lagrangian either the term (A.21) or the form (A.22), and integrating over the b-field, one gets

$$\int [dA][dc][d\bar{c}]\;\delta(\eta^\mu A_\mu)\,\exp i\int dx\left(\mathcal{L}_{cl} - \lambda\bar{c}\,\eta^\mu\partial_\mu c\right) \tag{A.25}$$

In the Coulomb gauge the ghosts are coupled with the gauge fields, whereas they decouple in the axial gauge.

It is interesting to note that by starting from the different forms (A.21) and (A.22), one finally finds the same generating functional after using the $\delta(\eta\cdot A)$ function. This means that the s and $\bar{s}$ invariances of (A.24) cannot be determined, and this is another example of the difficulty of including a δ -function in the functional integral measure.

A.IV YANG MILLS CLASSICAL LAGRANGIAN IN HIGHER DIMENSIONS.

A classical Yang Mills Lagrangian in d-dimensional space-time is a Lorentz invariant function of A_μ and solution of the equation

$$\int d^d x \;\; s \; \mathcal{L}_{cl}(A) = 0 \tag{A.26a}$$

If we assume that the fields vanish at infinity, eq.(A.26a) is equivalent to the local equation

$$\left(s \, \mathcal{L}_{cl}(A)\right) d^d x = d \; \mathcal{L}^1_{d-1}(A) \tag{A.26b}$$

where $\mathcal{L}^1_{d-1}$ is a Lorentz (d-1)-form with ghost number 1. $\mathcal{L}_{cl}(A)$ is defined up to a pure gradient $\partial^\mu \Lambda_\mu$.

It is conjectured that the most general solution to eq. A.26 is of the form

$$\mathcal{L}_{cl}(A) d^d x = G\left((F_{\mu\nu})^2\right) d^d x + K_d(A,F) \tag{A.27}$$

where the function G is arbitrary, and the exterior d-form K_d only depends on exterior products of the 1-form $A = A_\mu dx^\mu$ and the 2-form $F = 1/2 \, F_{\mu\nu} \, dx^\mu \wedge dx^\nu$. Furthermore, K_d can only exist in odd dimension, when $d = 2p-1$, and is a general Chern-Simon form of rank 2p-1.

Therefore, there is a one to one correspondence between all possible forms K_d and the invariant polynomials $\mathcal{P}^{inv}_{\cdot p}$ made from exterior product of p field strengths F, which follows from the Chern formula

$$K_d(A,F) = \int_0^1 dt \;\; \mathcal{P}^{inv}_{p}(A, \underbrace{F_t, \dots, F_t}_{p-1})$$

$$\Leftrightarrow \;\; \mathcal{P}^{inv.}_{2p}(\underbrace{F, \dots, F}_{p}) = d \; K_d(A,F) \tag{A.28}$$

where $F_t = tdA + t^2/2 \, [A,A]$. One can easily verify that

$$s \, K_d(A,F) = -d\left(c \left.\frac{\delta}{\delta A}\right|_F K_d(A,F)\right) \tag{A.29}$$

and thus eq. (A.26) is satisfied.

Terms like $K_d(A,F)$ are usually known as topological mass terms[56].

APPENDIX B

LINEAR GAUGES AND CONSISTENCY OF THE FADDEEV AND POPOV METHOD

In the Faddeev Popov method, when one chooses a linear gauge function $\mathcal{G}(A,\Psi_B) = -\frac{1}{\alpha}(\partial A + \beta[v,\Psi_B])$ with $v = \langle \Psi_B \rangle$, the corresponding Faddeev Popov Lagrangian can be written as follows

$$\mathcal{L}_{FP} = \mathcal{L}_{cl}(A,\Psi_B,\Psi_F) + \partial\bar{c}\, Dc + \beta\,[\bar{c},v][c,\Psi_B] - \frac{1}{2\alpha}\left(\partial A + \beta[v,\Psi_B]\right)^2 \tag{B.1}$$

This Lagrangian is equivalent to Lagrangian (3.10) with $\lambda_c = 0$

$$\mathcal{L}_Q = \mathcal{L}_{cl} + s\bar{s}\left(\frac{1}{2}A^2 - \beta v \Psi_B\right) + \frac{\alpha}{2} b^2$$

$$\equiv \mathcal{L}_{cl} + s\left(-\bar{c}\,\partial A - \beta[v,\Psi_B]\bar{c} + \alpha\bar{c} b\right) \tag{B.2}$$

$\mathcal{L}_{FP}$ is characterized by the absence of 4-ghost interactions, and we shall demonstrate that this property is compatible with renormalization, thanks to a Ward identity which can be enforced to all orders in perturbative theory.

Consider the effective Lagrangian

$$\mathcal{L}_{FP\,eff} = \mathcal{L}_{FP} + \bar{V}_{A_\mu} sA_\mu + \bar{V}_{\Psi_F} s\Psi_F + \bar{V}_{\Psi_B} s\Psi_B - \bar{V}_c\, sc \tag{B.3}$$

and the effective tree action $I_o = \int dx\, \mathcal{L}_{FP\,eff}$. $\mathcal{L}_{FP}$ is invariant under the following transformation :

$$\begin{aligned} sA_\mu &= D_\mu c \\ s\Psi_B &= -[c,\Psi_B] \equiv -c\Psi_B \\ s\Psi_F &= -[c,\Psi_F] \equiv -c\Psi_F \\ sc &= -\tfrac{1}{2}[c,c] \\ s\bar{c} &= -\tfrac{1}{\alpha}\left(\partial A + \beta[v,\Psi_B]\right) \end{aligned} \tag{B.4}$$

This implies that I_o satisfies the Ward identity

$$\int dx \left\{ \sum_{\Phi = A,c,\Psi_F,\Psi_B} \frac{\delta I_o}{\delta \Phi(x)} \frac{\delta I_o}{\delta \bar{V}_\Phi(x)} - \frac{1}{2\alpha}\left(\partial A + \beta[v,\Psi_B]\right)\frac{\delta I_o}{\delta \bar{c}(x)} \right\} = 0 \tag{B.5}$$

On the other hand, the equation of motion of $\bar{c}$ is

$$\frac{\delta \mathcal{L}_{FP}}{\delta \bar{c}} = -\partial Dc + \beta[v,[c,\Psi_B]] = -\partial(sA) - \beta[v, s\Psi_B] \tag{B.6}$$

which implies the following functional identity for I_o

$$\frac{\delta I_o}{\delta \bar{c}(x)} + \partial^\mu \frac{\delta I_o}{\delta \bar{V}_{A^\mu}(x)} + \beta\left[v, \frac{\delta I_o}{\delta \bar{V}_{\Psi_B}(x)}\right] = 0 \tag{B.7}$$

If now, following Zinn-Justin, we define

$$\hat{I}_o = I_o + \frac{1}{2\alpha}\int \left(\partial A + \beta[v,\Psi_B]\right)^2 dx \tag{B.8}$$

we obtain an effective action $\hat{I}_o$ which is explicitely independent of the symmetry breaking direction v, and moreover $\hat{I}_o$ satisfies the following functional identities :

$$\int dx \left(\sum_{\Phi = A, c, \psi_B, \psi_F} \frac{\delta \hat{I}_0}{\delta \Phi(x)} \frac{\delta \hat{I}_0}{\delta \bar{v}_\Phi(x)} \right) = 0 \tag{B.9}$$

and

$$\frac{\delta \hat{I}_0}{\delta \bar{c}(x)} + \partial^\mu \frac{\delta \hat{I}_0}{\delta \bar{v}_{A_\mu}(x)} + \beta \left[v, \frac{\delta \hat{I}_0}{\delta \bar{v}_{\psi_B}(x)} \right] = 0 \tag{B.10}$$

We now proceed to radiative corrections, starting from I_0. An inductive proof analogous to that displayed in Section VI allows one to prove that provided there is no anomaly, a renormalized functional of 1PI Green functions, $\Gamma_R(A, \psi_B, \psi_F, c, \bar{c}, \bar{v}_A, \bar{v}_{\psi_B}, \bar{v}_{\psi_F}, \bar{v}_c)$, can be constructed order by order in perturbation theory, such that both Ward identities (B.9) and (B.10) can be imposed to the functional $\hat{\Gamma}_R$, defined as follows

$$\hat{\Gamma}_R = R_R + \frac{1}{2\alpha} \left(\partial A + \beta [v, \psi_B] \right)^2 \tag{B.11}$$

Moreover, if a BRS symmetry preserving regulator is used, one can demonstrate that Γ_R is generated by a renormalized local action I_R, such that the following quantity $\hat{I}_R$

$$\hat{I}_R = I_R + \frac{1}{2\alpha} \left(\partial A + \beta [v, \psi_B] \right)^2 \tag{B.12}$$

satisfies also the Ward identities (B.9) and (B.10). This determines the most general possible form of $\hat{I}_R$, and thus of I_R. We shall demonstrate the latter assertion, following the general method detailed in Section V. Let us expand $\hat{I}_R$ as follows :

$$\hat{I}_R = \int dx \left(\hat{\mathcal{L}}_R(A, \psi_B, \psi_F, c, \bar{c}) + \sum_{\Phi = A, \psi_B, \psi_F, c} \bar{v}_\Phi \, t_\Phi(\Phi) \right) \tag{B.13}$$

where $t_\Phi(\Phi)$ are unknown polynomials of fields that we shall determine from identities satisfied by I_R

$$\int dx \sum_{\Phi = A, \psi_B, \psi_F, c} \frac{\delta \hat{I}_R}{\delta \Phi(x)} \frac{\delta \hat{I}_R}{\delta \bar{v}_\Phi(x)} = 0 \tag{B.14a}$$

$$\frac{\delta \hat{I}_R}{\delta \bar{c}(x)} + \partial_\mu \frac{\delta \hat{I}_R}{\delta \bar{v}_{A_\mu}(x)} + \beta \left[v, \frac{\delta \hat{I}_R}{\delta \bar{v}_{\psi_B}} \right] \tag{B.14b}$$

We introduce the differential operator S_R built from unknown functions $t_\Phi(\Phi)$

$$S_R = \int dx \left(t_{A_\mu} \frac{\delta}{\delta A_\mu(x)} + t_c \frac{\delta}{\delta c(x)} + t_{\psi_B} \frac{\delta}{\delta \psi_B(x)} + t_{\psi_F} \frac{\delta}{\delta \psi_F(x)} \right) \tag{B.15}$$

Then, we can rewrite eq. (B.14a) as

$$\int dx \left(s_R \hat{\mathcal{L}}_R + \sum_{\Phi = A, \psi_B, \psi_F, c} \bar{v}_\Phi \, s_R^2 \Phi \right) = 0 \tag{B.16}$$

and eq. (B.14b) as

$$\frac{\delta \hat{\mathcal{L}}_R}{\delta \bar{c}(x)} + s_R \left(\partial A + \beta [v, \psi_B] \right) = 0 \tag{B.17}$$

On the one hand eq. (B.16) implies $s^2_R = 0$ on the fields A_μ, c, ψ_B, ψ_F. But, as already deponstrated in Section V.2, this determines s_R as follows

$$s_R A_\mu = \tilde{Z}_3(\partial_\mu c + Z_g\sqrt{Z_3}[A_\mu, c]) \equiv \tilde{Z}_3 D_{\mu R} c$$
$$s_R c = -\tfrac{1}{2}\tilde{Z}_3 Z_g\sqrt{Z_3}[c,c] \equiv -\tfrac{1}{2}\tilde{Z}_3 [c,c]_R$$
$$s_R \psi_F = -\tilde{Z}_3 Z_g\sqrt{Z_3}[c,\psi_F] \equiv -\tilde{Z}_3[c,\psi_F]$$
$$s_R \psi_B = -\tilde{Z}_3 Z_g\sqrt{Z_3}[c,\psi_B] \equiv -\tilde{Z}_3[c,\psi_B]$$

(B.18)

where the Z factors will be shortly interpreted as renormalizable constants.

Further, the equation $s_R \hat{\mathcal{L}}_R = 0$ implies that the ghost independent part of $\hat{\mathcal{L}}_R$ is gauge independent, the renormalized classical gauge transformations being determined by the definitions of $s_R A$, $s_R\psi_F$ and $s_R\psi_B$ in eq. (B.18). On the other hand eq. (B.17) shows that the ghost dependence in $\mathcal{L}_R$ is quadratic in c and $\bar{c}$. Moreover, eq. (B.17) can be explicitely integrated on $\bar{c}$ since we have determined the action of s_R on all fields. Therefore, we obtain finally the most general possible form of $\hat{\mathcal{L}}_R$ as satisfying both eqs.(B.16) and (B.17)

$$\hat{\mathcal{L}}_R = -\tfrac{1}{4}Z_3\left(F_{\mu\nu R}\right)^2 - \tfrac{1}{2}Z_{\psi_B}\left(D_{\mu R}\psi_B\right)^2 - Z_{\psi_F}\bar{\psi}_F \not{D}_R \psi_F + \tilde{Z}_3\bar{c}\left(\partial D_R c + \beta[v,[\psi_B,c]_R]_R\right)$$

(B.19)

while the complete renormalized Lagrangian $\mathcal{L}_{FP_R}$ is

$$\mathcal{L}_{FP_R} = \hat{\mathcal{L}}_R - \frac{1}{2\alpha}\left(\partial A + \beta[v,\psi_B]\right)^2 \qquad \text{(B.20)}$$

Finally, by defining the renormalized BRS trunsform of $\bar{c}$ as

$$s_R\bar{c} = -\frac{1}{\alpha}\left(\partial A + \beta[v,\psi_B]\right) \qquad \text{(B.21)}$$

it follows that

$$s_R \mathcal{L}_{FP_R} = 0 \qquad \text{(B.22)}$$

and

$$s^2_R \bar{c} = \frac{1}{\alpha}\frac{\delta\mathcal{L}_{FP_R}}{\delta c} \qquad \text{(B.23)}$$

The only constraints on Z factors which have been introduced in eqs. (B.18) and (B.19) in that $\mathcal{L}_{FP_R}$ leads by construction to a finite generating functional. Thereby, they are determined in perturbation theory as a formal sery in g and ϵ, and the six independent constants Z_3, $\tilde{Z}_3$, Z_{ψ_B}, Z_{ψ_F}, Z_g and $\langle\psi_B\rangle_R/V$ can be interpreted as multiplicative renormalization factors. It is interesting to note that the gauge fixing term do not need to be renormalized at any order of perturbation theory in these gauges.

We have therefore shown that the Faddeev-Popov Lagrangian is multiplicatively renormalizable for any linear Lorentz invariant gauge condition, apart from the gauge fixing term which is left unrenormalized.

In other words, in the most general BRS and anti-BRS invariant Lagrangians, the value 0 of the gauge parameter λ_c which modulates the 4-ghost interactions is a fixed point under renormalization effects. One has the same conclusion for the value $\lambda_c = 1$, since the change $\lambda_c \leftrightarrow 1 - \lambda_c$ corresponds to a mere redefinition $c \leftrightarrow - \bar{c}$ in all the formula. It means that the Faddeev-Povpov procedure survives the radiative corrections whenever one chooses a linear gauge condition. This result is also confirmed by the work of Babelon and Viallet [10]. The gauges $\lambda = \frac{1}{2}$ are also stable because one has in their cases the invariance of the Lagrangian under the discrete symmetry $c \leftrightarrow - \bar{c}$. For other non linear gauges, there is no Ward identity like (B.7), and 4-ghost interactions generally occur in a way which is compatible with the BRS and/or anti-BRS symmetry, thereby jeopardizing the Faddeev and Popov prescription.