IFUM–827–FT
LBNL–57265
UCB–PTH–05/05

Euler angles for G_2

Sergio L. Cacciatori[1,3*], Bianca L. Cerchiai[4,5†], Alberto Della Vedova[2‡]
Giovanni Ortenzi[2§], Antonio Scotti[1¶]

[1] Dipartimento di Matematica dell'Università di Milano,
Via Saldini 50, I-20133 Milano, Italy.

[2] Dipartimento di Matematica ed Applicazioni, Università Milano Bicocca,
via R. Cozzi, 53- I-20126 Milano, Italy.

[3] INFN, Sezione di Milano,
Via Celoria 16, I-20133 Milano, Italy.

[4] Lawrence Berkeley National Laboratory
Theory Group, Bldg 50A5104
1 Cyclotron Rd, Berkeley, CA 94720-8162, USA.

[5] Department of Physics, University of California,
Berkeley, CA 94720, USA.

Abstract

We provide a simple coordinatization for the group G_2, which is analogous to the Euler coordinatization for $SU(2)$. We show how to obtain the general element of the group in a form emphasizing the structure of the fibration of G_2 with fiber $SO(4)$ and base $\mathcal{H}$, the variety of quaternionic subalgebras of octonions. In particular this allows us to obtain a simple expression for the Haar measure on G_2. Moreover, as a by-product it yields a concrete realization and an Einstein metric for $\mathcal{H}$.

---

[*] cacciatori@mi.infn.it
[†] BLCerchiai@lbl.gov
[‡] alberto.dellavedova@unimib.it
[§] giovanni.ortenzi@unimib.it
[¶] Graduate visitor (visitatore laureato), ascotti@mindspring.com

1 Introduction

The relevance of Lie groups in physics is a well established fact: They appear both in classical and in quantum problems. In this context an important role is played by the Haar measure, needed e.g. for the construction of a consistent path integral in lattice gauge theories [1]. The canonical 1-form θ of the compact Lie group G is the fundamental structure customarily used to find this measure. In effect the components of θ w.r.t. a basis of $Lie(G)$ are everywhere linear independent, smooth, left-invariant 1-form on G [2]. So their wedge product gives us a left-invariant volume form. In order to perform explicit calculations we have to choose a suitable local chart and the related local expression for the Haar measure. The logarithmic coordinates [3] are the most obvious choice for a coordinatization of G. In this case the canonical 1-form becomes

$$\theta(X) = \int_0^1 e^{s\,\mathrm{ad}X}\ ds.$$

The related volume form is

$$\omega = \prod_{\lambda\in\sigma(\mathrm{ad}X)} \frac{1-e^{-\lambda}}{\lambda} d\alpha_1 \wedge \cdots \wedge d\alpha_n,$$

where $\sigma(\mathrm{ad}X)$ denotes the spectrum of $\mathrm{ad}X$ and α_i are a basis of $Lie(G)^*$. However these coordinates do not display generally the subgroup structure of G which usually are relevant in the physical applications. The difficulties of such a kind of coordinatization arise when one needs to explicitly determine the global range of the coordinates. A coordinatization which yields a simple form for the Haar measure and at the same time allows a simple determination of the range for the angles can become crucial for numerical computations, e.g. in lattice gauge theories or in random matrix models. For unitary groups such a coordinatization has been constructed in [4], generalizing the "Euler angle parameterization" for $SU(2)$.

In this paper we provide an analogous simple coordinatization for the exceptional Lie group G_2. We start by showing how a simple matrix realization of the algebra can be obtained starting from the octonions. Then a proposal for a representation of the group elements, based on the previous construction and emphasizing the $SO(4)$ subgroup embedded in G_2, is given and it is proven to cover the whole group.

After computing the left-invariant currents in a way that respects the structure of the fibration, the infinitesimal invariant measure is determined with a suitable normalization. We then use topological tools and symmetry arguments to determine the correct range of the coordinates. As a by-product we obtain a coordinatization and an Einstein metric for the eight-dimensional variety of quaternionic subalgebras of octonions.

Motivations for considering models with G_2 symmetries are provided by different physical systems, for example they arise in the study of deconfinement phase transitions [5], in random matrix models [6] or in the new matrix models related to $D-$brane physics [7].

2 The G_2 algebra

The octonions $\mathbb{O}$ are an eight-dimensional real algebra whose generic element a is a pair of quaternions (α_1, α_2) with the following multiplication rules

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) \quad = \quad (\alpha_1\beta_1 - \bar{\beta}_2\alpha_2, \beta_2\alpha_1 + \alpha_2\bar{\beta}_1). \tag{2.1}$$

Here $(\alpha_1, \alpha_2), (\beta_1, \beta_2)$ are generic octonions. This algebra comes naturally equipped with an involution called conjugation

$$\overline{(\alpha_1, \alpha_2)} = (\bar{\alpha}_1, -\alpha_2).$$

Denoting by $1, i, j, k$ the usual basis of the quaternions yields the following canonical basis for the algebra $\mathbb{O}$

$$e_0 = (1,0),\ \ e_1 = (i,0),\ \ e_2 = (j,0),\ \ e_3 = (k,0),\ \ e_4 = (0,1),\ \ e_5 = (0,i),\ \ e_6 = (0,j),\ \ e_7 = (0,k).$$

Using this basis it follows easily that the subspace $\mathbb{H}$ spanned by $\{1, e_1, e_2, e_3\}$ is in fact a quaternionic subalgebra that we call the canonical quaternionic subalgebra. Moreover we can consider $\mathbb{O}$ as a two-dimensional module over $\mathbb{H}$, i.e. every octonion z can be decomposed as $z = x + ye_4$, where x, y are suitable quaternions.
The octonions, together with $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, are the only normed division algebras. The norm is induced by the standard Euclidean structure of the underlying real vector space. They are neither commutative nor associative, but they are alternative, i.e. any subalgebra generated by two octonions is associative. This weak form of associativity implies the Moufang identities [8], which are multiplication laws among octonions and will prove very useful in the following

$$\begin{aligned} (ax)(ya) &= a(xy)a, \\ a(x(ay)) &= (axa)y, \\ y(a(xa)) &= y(axa). \end{aligned} \tag{2.2}$$

The relevance of the octonions in mathematics is due to their deep connection with the exceptional Lie groups ([9] and references therein). We are interested in the group G_2. In this case the link is easy to understand: G_2 is the automorphism group of the octonions. For every octonion a we denote by l_a the left multiplication $l_a(x) = ax$. Under suitable hypothesis the composition of left multiplications generates elements of G_2. In fact it holds

Proposition 1 *Let $g = l_{a_1} \dots l_{a_n}$, where $a_1, \dots, a_n$ are unitary purely imaginary octonions. If $g(1) = 1$, then $g \in G_2$.*

Proof We have to show that $g(x)g(y) = g(xy)$ for all $x, y \in \mathbb{O}$. To this end we prove by induction on n that

$$g(x)g(y) = g((xb)y), \tag{2.3}$$

where $b = (\dots(a_1a_2)\dots)a_n$.

For the moment we avoid the hypothesis $g(1) = 1$.

If $n = 1$ we have

$$(ax)(ay) = -a^2((ax)(ay)) = -a((a(ax)a)y) = a((xa)y),$$

where the first equality holds because a is purely imaginary and the others by the Moufang identities.

Now we suppose the statement is true for n. So we have

$$g(ax)g(ay) = g(((ax)b)(ay)) = -g(a^2(((ax)b)(ay))) = g(a((x(ba))y)),$$

which is the equation (2.3), if we replace g by $g\,l_a$ and b by ba.

Finally, to complete the proof we have to show that $b = (\ldots(a_1a_2)\ldots)a_n = 1$. So, applying the operator $l_{a_n}\ldots l_{a_1}$ to both members of the equation $1 = g(1) = a_1(\ldots(a_{n-1}a_n)\ldots)$ gives us $a_n(a_{n-1}(\ldots(a_2a_1)\ldots)) = (-1)^n$, and then by conjugation we obtain $(-1)^n(\ldots(a_1a_2)\ldots)a_n = (-1)^n$ which implies $b = 1$.

□

We can get some interesting subgroups of G_2 by imposing additional conditions on g. If we add the hypothesis $g(e_1) = e_1$ we obtain a $SU(3)$ subgroup denoted in the following by P. Moreover, imposing also $g(e_2) = e_2$, the resulting subgroup is a copy of $SU(2)$, which we call S.

In order to write down generalized Eulerian coordinates we need to describe the embedding of $SO(4)$ in G_2 [10].

To this end we identify as usual $SU(2)$ with the 3–sphere of unitary quaternions and we consider the following homomorphism

$$\begin{array}{rcl} \gamma : SU(2)\times SU(2) & \to & G_2 \\ (a,b) & \mapsto & \gamma_{ab}, \end{array} \tag{2.4}$$

where $\gamma_{ab}(x + ye_4) = ax\bar{a} + (by\bar{a})e_4$. Using the Moufang identities it is not hard to check that γ_{ab} is truly an octonion automorphism. Fixing $a = 1$ provides us with the embedding γ_{1b} of $SU(2)$ in G_2, whose image is the subgroup S. On the other hand the image of the embedding γ_{a1} is a $SU(2)$, which we denote by Σ and which is not conjugate to S.

The map γ is not injective and its kernel is the subgroup $\mathbb{Z}_2 = \{(1,1),(-1,-1)\}$. By the homomorphism theorem the image of the map γ in G_2 is isomorphic to $(SU(2)\times SU(2))/\mathbb{Z}_2$, which is $SO(4)$ as well known. In the following sections we will refer to the image of γ simply as $SO(4)$.

The homogeneous space $\mathcal{H} = G_2/SO(4)$ is the eight-dimensional variety of the quaternionic subalgebras of $\mathbb{O}$. So $\mathbb{H}$ can be thought as a point of $\mathcal{H}$. G_2 acts transitively on $\mathcal{H}$ and the stabilizer of $\mathbb{H}$ is the image of γ [10]. Then we have the fibration

$$\begin{tikzcd} SO(4) \arrow[r, hook] & G_2 \arrow[d] \\ & \mathcal{H} \end{tikzcd}$$

The G_2 generators provided in proposition 1 are useful in particular to find a basis of the Lie algebra $Lie(G_2)$. Actually, let us take an element of G_2 of the form $g_{abc} = -l_{(cb)a}l_al_bl_c$, where $(cb)a, a, b, c$ are unitary and purely imaginary (i.e. $a + \bar{a} = 0$). Notice that the choice of $-(cb)a$ guarantees $g_{abc}1 = 1$. The condition that $(cb)a$ be purely imaginary amounts to $(cb)\perp a$.

Consider now a path $g_{a_t bc} = -l_{(cb)a_t} l_{a_t} l_b l_c$ where $a_t = c\cos(t) + a\sin(t)$ and with the additional requirements $b \perp c$, $b \perp a$ and $a \perp c$.

By definition $C_{abc} = \frac{d}{dt}\Big|_{t=0} g_{abc}(t)$ is an element of $Lie(G_2)$. With suitable choices of the elements a, b and c among the elements of the canonical basis of $\mathbb{O}$ we can find a basis of this algebra. The representative matrices, written below with respect to the canonical basis, are normalized with the condition $Tr(C_I C_J) = -4\delta_{IJ}$. We remark that they are seven-dimensional because of the trivial action on the real unity.

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad C_{10} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_{11} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_{12} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_{13} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_{14} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This basis satisfy the commutation rules summarized in the antisymmetric matrix $B_{IJ} = [C_I, C_J]$ given in the appendix A.

Among that matrices we can recognize the Lie algebras corresponding to the subgroups of G_2 mentioned above. The first eight matrices generate $Lie(P)$ and they are reminiscent of the Gell-Mann matrices. Moreover, the matrices $\{C_1\ , C_2\ , C_3\}$ generate $Lie(S)$ and finally $\{C_8\ , C_9\ , C_{10}\}$ generate $Lie(\Sigma)$.

Since the elements C_5 and C_{11} commute, they generate a Cartan subalgebra of $Lie(G_2)$ which is the Lie algebra of the maximal torus T of G_2. Notice that the commutators among the basis of $Lie(S)$, $Lie\Sigma$ and $Lie(T)$ generate the whole basis of $Lie(G_2)$.
Our previous observations lead us to the conjecture that a good coordinatization for the generic element $g \in G_2$ can be defined by

$$g = \sigma(a_1, a_2, a_3) s(a_4, a_5, a_6) e^{\sqrt{3}a_7 C_{11}} e^{a_8 C_5} u(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}), \tag{2.5}$$

where

$$s(x, y, z) = e^{xC_3} e^{yC_2} e^{zC_3}\ , \tag{2.6}$$

$$\sigma(x', y', z') = e^{\sqrt{3}x' C_8} e^{\sqrt{3}y' C_9} e^{\sqrt{3}z' C_8}\ , \tag{2.7}$$

$$u(x, y, z; x', y', z') = s(x, y, z)\sigma(x', y', z') \tag{2.8}$$

are elements respectively of S, Σ and $SO(4)$.
In this paper we prove that this is in fact a good coordinatization for G_2, which could be used to determine a simple form for the Haar measure on the group. In order to achieve this we will first determine the corresponding invariant metric and then compute the range of the coordinates $a_1, a_2, \ldots, a_{14}$.

3 The invariant metric for G_2

In order to compute the invariant metric over the Lie group G_2 we will first show how our coordinatization (2.5) is related to the fibration described in the previous section.

It is well known that for a simple group, and therefore in particular for G_2, the invariant metric is uniquely defined (up to a normalization constant) by the Killing form over the algebra. More precisely, the Killing form defines a metric (and a Lebesgue measure) over the tangent space to the identity, which can be pulled back via the left (or right) multiplication. If ${f_{IJ}}^K$ are the structure constants of the algebra, then the Killing metric has components

$$K_{IJ} = (C_I, C_J) = -k {f_{IL}}^M {f_{JM}}^L \ , \tag{3.1}$$

where k is a normalization constant. In our case we find $K_{IJ} = 16k\delta_{IJ}$, which suggests to choose $k = \frac{1}{16}$ conveniently in such a way that the generators $\{C_I\}_{I=1}^{14}$ are orthonormal: $(C_I, C_J) = \delta_{IJ}$. By right multiplication we can associate to every matrix C_I a vector field in the tangent bundle. Its dual (defined using the pullback of the Killing form) is a left-invariant 1-form and the collection of those 1-forms provides a trivialization of the cotangent bundle. In a coordinate patch $\{w^I\}$ the canonical 1-form J becomes

$$J = J^I C_I = g^{-1} \frac{\partial g}{\partial w^J} dw^J \ , \tag{3.2}$$

so that the invariant metric is

$$ds^2 = g_{IJ} dw^I \otimes dw^J = J^L \otimes J^M (C_L, C_M) \ . \tag{3.3}$$

Therefore, if we define the matrix $\underline{J} = \{{J^I}_K\}$ with components $J^I = {J^I}_K dw^K$ and remember our normalization, we find for the components of the metric

$$g_{IJ} = \delta_{LM} {J^L}_I {J^M}_J \ , \tag{3.4}$$

This means that the right currents define a 14–bein over the Lie group G_2. In particular the invariant volume form is

$$\omega = J^1 \wedge \ldots \wedge J^{14} = \det(\underline{J}) dx^1 \wedge \ldots \wedge dx^{14} \tag{3.5}$$

and the associated Haar measure is

$$d\mu = \det(\underline{J}) \prod_{I=1}^{14} dw^I \ . \tag{3.6}$$

Now we use our coordinatization (2.5), which we rewrite in the form

$$g = h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) u(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}) \ . \tag{3.7}$$

Note that u is the generic element of the subgroup $SO(4)$, while h is an element of $S\Sigma T$ which is not a subgroup of G_2. We can express the currents associated to the elements u and h respectively as

$$J_u = duu^{-1} = \sum_{i \in A} J_u^i C_i \ , \qquad J_h = h^{-1} dh = \sum_{i=1}^{14} J_h^i C_i \ , \tag{3.8}$$

where $A = \{1, 2, 3, 8, 9, 10\}$. Note that for J_U we have chosen the left currents. This is because from the orthonormality condition it follows that

$$ds^2 = \sum_{i \in A} \left(J_u^i + J_h^i\right)^2 + \sum_{i \notin A} \left(J_h^i\right)^2 \ . \tag{3.9}$$

This particular form of the metric stress the ralation with the fibration of G_2 over the variety $\mathcal{H}$. In effect it shows explicitly the separation between the base and the fiber and if we fix the coordinates $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$, it reduces to the metric on the fiber $SO(4)$. On the other hand the term $\sum_{i \notin A} \left(J_h^i\right)^2$ corresponds to the acht-bein $\tilde{J}_h$ obtained after projecting the currents J_h orthogonally to the fiber, and therefore it has to coincide with the metric on the base.
This decomposition greatly simplifies the explicit computation of the metric, the main task being the computation of J_h.
In order to determine J_u remember that $u(x, y, z; x', y', z') = s(x, y, z)\sigma(x', y', z')$ and that s and σ commute, so that in terms of $J_s = dss^{-1}$ and $J_\sigma = d\sigma\sigma^{-1}$, we get $J_u = J_s + J_\sigma$ with

$$\begin{aligned} J_S(x, y, z) = &[-\sin(2x)dy + \cos(2x)\sin(2y)dz]C_1 + [\cos(2x)dy + \sin(2x)\sin(2y)dz]C_2 \\ &+[dx + \cos(2y)dz]C_3 \ , \\ J_\Sigma(x, y, z) = &\sqrt{3}[dx + \cos(2y)dz]C_8 + \sqrt{3}[\cos(2x)dy + \sin(2x)\sin(2y)dz]C_9 \\ &+\sqrt{3}[\sin(2x)dy - \cos(2x)\sin(2y)dz]C_{10} \ . \end{aligned} \tag{3.10}$$

Using Mathematica we have found for the currents J_h the expressions written in the Appendix B.

We are now able to compute the Haar measure in our coordinates. In fact, if $\underline{\tilde{J}_h}$ is the 8×8 matrix given by the acht-bein $\tilde{J}_h$ and $\underline{J_s}$, $\underline{J_\sigma}$ are the 3×3 matrices associated to the drei-bein J_s and J_σ respectively, then $\det(\underline{J}) = \det(\underline{\tilde{J}_h}) \det(\underline{J_s}) \det(\underline{J_\sigma})$ so that

$$d\mu = 27\sqrt{3} f(2a_7 \ , 2a_8) \sin(2a_2) \sin(2a_5) \sin(2a_{10}) \sin(2a_{13}) \prod_{i=1}^{14} da_i \ , \tag{3.11}$$

where

$$\begin{aligned} f(\alpha, \beta) &= \sin(\frac{\beta - \alpha}{2}) \sin(\frac{\beta + \alpha}{2}) \sin(\frac{\beta - 3\alpha}{2}) \sin(\frac{\beta + 3\alpha}{2}) \sin(\alpha) \sin(\beta) \\ &= \frac{1}{4}(\cos(\alpha) + \cos(\beta))(\cos(3\alpha) + \cos(\beta)) \sin(\alpha) \sin(\beta) \end{aligned} \tag{3.12}$$

This, however, is not the end of the story. We need in fact to determine the range of the coordiantes which covers the whole group G_2, apart from a subset of zero measure. To this end we will use a topological argument.

4 The range of the coordinates

Before entering into more details, let us explain our strategy.
Looking at the measure (3.11) one immediately sees that for some values of the coordiantes it

vanishes. This happens for certain values of the angles $a_2, a_5, a_7, a_8, a_{10}, a_{13}$. Let us suppose we choose the range for these coordinates in such a way that it delimits a region where $d\mu$ is vanishing only on the boundary. For the other coordinates the range is fixed in such a way that each of them goes around a closed orbit exactly once.
To this end we describe a 14-dimensional closed cycle V which represents an element of the homology group $[V] \in H_{14}(G_2, \mathbb{Z})$ [1]. Using the pairing $B : H^k \times H_k \longrightarrow \mathbb{R}$ given by

$$B([\xi], [W]) = \int_W \xi \ , \tag{4.1}$$

we define the normalized form

$$\tau = B([\omega], [G_2])^{-1}\omega. \tag{4.2}$$

It is clear that the function $B([\tau], \cdot)$ takes only integer values. In particular $B([\tau], [W])$ counts the number of times the cycle W wraps G_2. Once this is known all we need to do is to find how to restrict the range of coordinates until we obtain a cycle V which wraps G_2 once.
We are now going to enter into more details and compute τ in three steps.

4.1 The evaluation of $B([\omega], [G_2])$

There is a simple way to compute the total volume of a connected simple Lie group, described by Macdonald in [11]. It works as follows.
If G is the group, $t \subset Lie(G)$ a Cartan subalgebra, and $t_{\mathbb{Z}}$ the integer lattice generated in t by the simple roots, then $T = t/t_{\mathbb{Z}}$ is a torus with the same dimension as t. Let $\alpha > 0$ denote the positive roots and $|\alpha|$ their length.
Then from Hopf theorem, the rational homology of G is equal to the rational homology of a product of odd-dimensional spheres: $H_*(G, \mathbb{Q}) \sim H_*\left(\prod_{i=1}^k (S^{2i+1})^{n_i}, \mathbb{Q}\right)$, where n_i is the number of times the given sphere appears. Let $Vol(S^{2i+1}) = 2\pi^{i+1}/i!$ be the volume of the $(2i+1)$-dimensional unit sphere and $Vol(T)$ be the volume of the torus computed using the measure induced by the Lebesgue measure on the algebra.
Then the whole volume determined via the pullback of the Lebesgue measure on the algebra is

$$Vol(G) = Vol(T) \cdot \prod_{i=1}^k Vol(S^{2i+1})^{n_i} \cdot \prod_{\alpha>0} \frac{4}{|\alpha|^2} \ . \tag{4.3}$$

The roots of G_2 computed with our choice for the algebra and the normalization, are shown in the Appendix C. From the figure we see that there are three positive roots of length 2 and three of length $\frac{2}{\sqrt{3}}$. The torus associated to the simple roots α_1 and α_2 is generated by the coroots H_1 and H_2. Remembering the relations $|H| = 2/|\alpha|$, we find $Vol(T) = \frac{\sqrt{3}}{2}$. Since S^3 and S^{11} are the odd spheres which generate the rational homology of G_2, we obtain the desired result

$$B(\mu, G_2) = Vol(G_2) = 9\sqrt{3}\frac{\pi^8}{20} \ . \tag{4.4}$$

We are now ready for the next step.

[1] However for our purposes it would be enough to consider the homology with rational or real coefficients.

4.2 The construction of the cycle V

Let us look at (2.5): to determine a closed cycle we first observe that for each one of the two $SU(2)$ subgroups S and Σ it is possible to choose the range of the coordinates in such a way as to cover the whole 3−sphere. The method to do this is well-known (Euler angles) and here we give only the final result:

$$\begin{aligned}
&0 \le a_1 \le 2\pi \ , \qquad 0 \le a_2 \le \frac{\pi}{2} \ , \qquad 0 \le a_3 \le \pi \ , \\
&0 \le a_4 \le 2\pi \ , \qquad 0 \le a_5 \le \frac{\pi}{2} \ , \qquad 0 \le a_6 \le \pi \ , \\
&0 \le a_9 \le 2\pi \ , \qquad 0 \le a_{10} \le \frac{\pi}{2} \ , \qquad 0 \le a_{11} \le \pi \ , \\
&0 \le a_{12} \le 2\pi \ , \qquad 0 \le a_{13} \le \frac{\pi}{2} \ , \qquad 0 \le a_{14} \le \pi \ .
\end{aligned} \tag{4.5}$$

To complete the cycle we need to determine the range for a_7 and a_8 in such a way that $d\mu$ does not vanish. To this end we solve the inequality $f(x, y) > 0$ and obtain a tiling of the fundamental region

$$(2a_7 \ , 2a_8) \in [0, 2\pi] \times [0, 2\pi] \ , \tag{4.6}$$

as we show in the Appendix D. There, we also prove that every region of the tiling gives the same (absolute value) contribution to the measure.

The cycle V is then obtained by choosing any one of these regions, for example the one denoted by B in the figure.

4.3 The evaluation of τ and the range of the Euler angles for G_2

We can now evaluate the degree of the map $V \to G_2$. Using (4.5) for the range of the coordinates, (3.11) for $d\mu$ and (4.4) for $Vol(G_2)$, we easily find $B(\tau, V) = 16$. Therefore, our next task is to understand the origin of this factor.

A factor 4 can be easily accounted for in the following way. We have built the cycle V starting from the closed submanifolds S and Σ corresponding to the two $SU(2)$ embeddings. Thus, naively, we would expect to find a $SU(2) \times SU(2)$ submanifold embedded in G_2. But a direct inspection shows that this is not exactly true. In fact, varying for example $a_1 \in [0, 2\pi]$ provides a double covering of the six-dimensional submanifold obtained by taking $a_1 \in [0, \pi]$ and $a_2, \ldots, a_6$ as in (4.5). This is because the image of $SU(2) \times SU(2)$ in G_2 is $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$, as previously remarked in Section 2.

Similarly, we must reduce the range of a_{12} to $a_{12} \in [0, \pi]$. The new cycle we obtain in this way wraps G_2 four times.

Now, let us consider the torus $T(a_7, a_8) := e^{\sqrt{3}a_7 C_{11}} e^{a_8 C_5}$. We need to determine the subgroup of 7×7 orthogonal matrices A of $SO(4)$, which leaves each element $T(a_7, a_8)$ invariant under the adjoint action

$$AT(a_7, a_8)A^t = T(a_7, a_8) \ . \tag{4.7}$$

It turns out that it is a finite group generated by the idempotent matrices σ ($\sigma = \sigma^{-1}$) and η ($\eta = \eta^{-1}$)

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{4.8}$$

Considering the action of σ one finds

$$\begin{aligned} g &= U(a_4, a_5, a_6; a_1, a_2, a_3)T(a_7, a_8)U(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}) \\ &= U(a_4, a_5, a_6; a_1, a_2, a_3)\sigma T(a_7, a_8)\sigma U(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}) \\ &= U(a_4, a_5, a_6 + \frac{\pi}{2}; a_1, a_2, a_3 + \frac{\pi}{2})T(a_7, a_8)U(a_9 + \frac{\pi}{2}, a_{10}, a_{11}; a_{12} + \frac{\pi}{2}, a_{13}, a_{14}). \end{aligned} \tag{4.9}$$

Therefore, we can restrict $0 \leq a_3 < \frac{\pi}{2}$.

Analogously, let us now look at the symmetry generated by η:

$$\begin{aligned} g &= U(a_4, a_5, a_6; a_1, a_2, a_3)T(a_7, a_8)U(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}) \\ &= U(a_4, a_5, a_6; a_1, a_2, a_3)\eta T(a_7, a_8)\eta U(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14}), \end{aligned} \tag{4.10}$$

and restrict our attention to the factor ηU on the right. (Similar relations are true for the factor on the left.) A direct computation using the explicit expression of the matrices shows that the left action of η on U is equivalent to the shift

$$\begin{aligned} &a_9 \mapsto \frac{\pi}{4} - a_9 \ , \qquad a_{10} \mapsto a_{10} + \frac{\pi}{2} \ , \qquad a_{11} \mapsto a_{11} \ , \\ &a_{12} \mapsto -\frac{\pi}{4} - a_{12} \ , \qquad a_{13} \mapsto a_{13} + \frac{\pi}{2} \ , \qquad a_{14} \mapsto a_{14} \ . \end{aligned} \tag{4.11}$$

The analysis is more complicated than for the action of σ, because now some of the angles are mapped to values which are outside of the range (4.5) we have fixed. For example, $a_{10} + \frac{\pi}{2} \in [\frac{\pi}{2}, \pi]$, when $a_{10} \in [0, \frac{\pi}{2}]$. Therefore, we need to use other equivalence relations to map the angles back to the original region (4.5).

In fact, the following symmetries hold for S and Σ:

$$\begin{aligned} &S[a_9, a_{10}, a_{11}] \sim S[a_9 + \frac{\pi}{2}, \pi - a_{10}, a_{11} + \frac{\pi}{2}] \\ &\Sigma[a_{12}, a_{13}, a_{14}] \sim \Sigma[a_{12} + \frac{\pi}{2}, \pi - a_{13}, a_{14} + \frac{\pi}{2}] \ . \end{aligned} \tag{4.12}$$

These are the known symmetries which have been used to determine (4.5) in the first place. Thus the action (4.11) of η is equivalent to

$$\begin{aligned} &a_9 \mapsto \frac{3}{4}\pi - a_9 \ , \qquad a_{10} \mapsto \frac{\pi}{2} - a_{10} \ , \qquad a_{11} \mapsto a_{11} + \frac{\pi}{2} \ , \\ &a_{12} \mapsto \frac{\pi}{4} - a_{12} \ , \qquad a_{13} \mapsto \frac{\pi}{2} - a_{13} \ , \qquad a_{14} \mapsto a_{14} + \frac{\pi}{2} \ . \end{aligned} \tag{4.13}$$

where now a_{10} and a_{13} stay inside the allowed intervals. The next point to notice is that for the remaining coordinates it is possible to use similarity relations which does not involve a_{10} and a_{13}, e.g.

$$S[a_9, a_{10}, a_{11}]\Sigma[a_{12}, a_{13}, a_{14}] \sim S[a_9 + \pi, a_{10}, a_{11}]\Sigma[a_{12} + \pi, a_{13}, a_{14}] \tag{4.14}$$

and similar symmetries. Moreover, η is a linear transformation, so that it is enough to restrict the range of one of the angles. Therefore, luckily, we actually do not need to know the action of η on the whole set of coordinates. The solution of our problem simply consists in restricting the range of either a_{10} or a_{13} in the $SO(4)$ factor $U(a_9, a_{10}, a_{11}; a_{12}, a_{13}, a_{14})$ on the right. Alternatively, the same result can be achieved by restricting the range of either a_2 or a_5 in the factor $U(a_4, a_5, a_6; a_1, a_2, a_3)$ on the left. Since we prefer a coordinatization which respects the fibration described in Section 2 and we want the angles $a_9, \dots, a_{14}$ to span the whole fiber $U(a_9, \dots, a_{14})$, we choose the second option and restrict the range of a_5.

Finally, we can summarize our results for the range of the angles describing G_2:

$$\begin{aligned}
&0 \le a_1 \le \pi \ , && 0 \le a_2 \le \frac{\pi}{2} \ , && 0 \le a_3 \le \frac{\pi}{2} \ , \\
&0 \le a_4 \le 2\pi \ , && 0 \le a_5 \le \frac{\pi}{4} \ , && 0 \le a_6 \le \pi \ , \\
&0 \le a_9 \le 2\pi \ , && 0 \le a_{10} \le \frac{\pi}{2} \ , && 0 \le a_{11} \le \pi \ , \\
&0 \le a_{12} \le \pi \ , && 0 \le a_{13} \le \frac{\pi}{2} \ , && 0 \le a_{14} \le \pi \ , \\
&0 \le a_7 \le \frac{\pi}{6} \ , && 3a_7 \le a_8 \le \frac{\pi}{2} \ .
\end{aligned} \tag{4.15}$$

5 Conclusions

We have found a coordinatization of the G_2 group with a one to one correspondence between the range of coordinates and a full measure subset of the group. In particular, this has allowed us to obtain a quite simple expression for the Haar measure, which should make numerical computations involving the geometry of G_2 much easier, for example in lattice gauge theories or in random matrix models.

However, note that to find an Haar measure on the group we would not have actually needed the last step of the work, i.e. the determination of the correct range of the coordinates which yields an injective map.

In fact, if $d\mu$ indicates the measure over the cycle V, it is possible to simply get an Haar measure $d\tilde{\mu}$ over the Lie group G_2 by just taking

$$d\tilde{\mu} := \frac{1}{16} d\mu \ , \tag{5.1}$$

16 being the degree of the map $V \longrightarrow G_2$.

On the other hand the determination of the correct range of coordinates, which covers the whole G_2 wrapping it exactly once except for a subset of vanishing measure, provides a new result, since it determines also a coordinatization of the homogeneous space $\mathcal{H} = G_2/SO(4)$ of quaternionic subalgebras of octonions.

Moreover, we have also computed the induced metric of $\mathcal{H}$

$$ds^2_{\mathcal{H}} = \delta_{ab} \tilde{J}^a_h \otimes \tilde{J}^b_h \ , \tag{5.2}$$

and shown that it is an Einstein metric. This result is proven in the Appendix E.

Acknowledgments

SC and GO would like to thank G. Berrino, M. Bertini, S. Bertini and B. Van Geemen for helpful conversations. BLC would like to thank O. Ganor, B. Zumino, M. Papucci for very useful discussions. ADV would like to thank P. Cotta Ramusino for interesting discussions on the octonions and exceptional Lie groups. BLC is supported by the INFN under grant 8930/01. This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the US Department of Energy, Grant No.DE-FG02-03ER41262. The work of SC is supported by INFN, COFIN prot. 2003023852_008 and the European Commission RTN program MRTN–CT–2004–005104, in which SC is associated to the University of Milano–Bicocca.

A Commutators

$$
B = \begin{pmatrix}
0 & 2C_3 & -2C_2 & C_7 & C_6 & -C_5 & -C_4 & 0 & 0 & 0 & C_{14} & C_{13} & -C_{12} & -C_{11} \\
* & 0 & 2C_1 & -C_6 & C_7 & C_4 & -C_5 & 0 & 0 & 0 & -C_{13} & C_{14} & C_{11} & -C_{12} \\
* & * & 0 & C_5 & -C_4 & C_7 & -C_6 & 0 & 0 & 0 & C_{12} & -C_{11} & C_{14} & -C_{13} \\
* & * & * & 0 & C_3+\sqrt{3}C_8 & -C_2 & C_1 & -\sqrt{3}C_5 & -C_{14} & -C_{13} & 0 & 0 & C_{10} & C_9 \\
* & * & * & * & 0 & C_1 & C_2 & \sqrt{3}C_4 & C_{13} & -C_{14} & 0 & 0 & -C_9 & C_{10} \\
* & * & * & * & * & 0 & C_3-\sqrt{3}C_8 & \sqrt{3}C_7 & -C_{12} & C_{11} & -C_{10} & C_9 & 0 & 0 \\
* & * & * & * & * & * & 0 & -\sqrt{3}C_6 & C_{11} & C_{12} & -C_9 & -C_{10} & 0 & 0 \\
* & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_{10} & -\frac{2}{\sqrt{3}}C_9 & -\frac{1}{\sqrt{3}}C_{12} & \frac{1}{\sqrt{3}}C_{11} & \frac{1}{\sqrt{3}}C_{14} & -\frac{1}{\sqrt{3}}C_{13} \\
* & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_8 & C_7-\frac{2}{\sqrt{3}}C_{14} & \frac{2}{\sqrt{3}}C_{13}-C_6 & C_5-\frac{2}{\sqrt{3}}C_{12} & \frac{2}{\sqrt{3}}C_{11}-C_4 \\
* & * & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_{13}+C_6 & \frac{2}{\sqrt{3}}C_{14}+C_7 & -\frac{2}{\sqrt{3}}C_{11}-C_4 & -\frac{2}{\sqrt{3}}C_{12}+C_5 \\
* & * & * & * & * & * & * & * & * & * & 0 & -\frac{1}{\sqrt{3}}C_8+C_3 & \frac{2}{\sqrt{3}}C_{10}-C_2 & -\frac{2}{\sqrt{3}}C_9+C_1 \\
* & * & * & * & * & * & * & * & * & * & * & 0 & \frac{2}{\sqrt{3}}C_9+C_1 & \frac{2}{\sqrt{3}}C_{10}+C_2 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & \frac{1}{\sqrt{3}}C_8+C_3 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0
\end{pmatrix}
$$

B The left-invariant 1-forms J_h

$$
\begin{aligned}
J_h^1 = & \left[\cos^3(a_7)\sin(2a_6)\cos a_8 - \sin^3(a_7)\cos(2a_6)\sin a_8\right] da_5 \\
& -\sin(2a_5)\left[\cos^3(a_7)\cos(2a_6)\cos a_8 + \sin^3(a_7)\sin(2a_6)\sin a_8\right] da_4 \\
& +\frac{3}{2}\sin(2a_7)\left[\cos(2a_3)\sin(a_7)\cos a_8 + \sin(2a_3)\cos(a_7)\sin a_8\right] da_2 \\
& +\frac{3}{2}\sin(2a_7)\sin(2a_2)\left[\sin(2a_3)\sin(a_7)\cos a_8\right. \\
& \left. -\cos(2a_3)\cos(a_7)\sin a_8\right] da_1 \qquad \text{(B.1)} \\
J_h^2 = & \left[\cos^3(a_7)\cos(2a_6)\cos a_8 - \sin^3(a_7)\sin(2a_6)\sin a_8\right] da_5 \\
& +\sin(2a_5)\left[\cos^3(a_7)\sin(2a_6)\cos a_8 + \sin^3(a_7)\cos(2a_6)\sin a_8\right] da_4 \\
& -\frac{3}{2}\sin(2a_7)\left[\sin(2a_3)\sin(a_7)\cos a_8 + \cos(2a_3)\cos(a_7)\sin a_8\right] da_2
\end{aligned}
$$

$$+\frac{3}{2}\sin(2a_7)\sin(2a_2)\left[\cos(2a_3)\sin(a_7)\cos a_8 - \sin(2a_3)\cos(a_7)\sin a_8\right] da_1 \tag{B.2}$$

$$J_h^3 = \frac{1}{4}\left(3\cos(2a_7)+\cos(2a_8)\right)\left(da_6 + cos(2a_5)da_4\right) - \frac{\sqrt{3}}{4}\left(\cos(2a_7)-\cos(2a_8)\right)\left(da_3 + cos\,(2a_2)\,da_1\right) \tag{B.3}$$

$$J_h^4 = -\frac{1}{2}\sin(2a_8)da_6 - \frac{1}{2}\cos(2a_5)\sin(2a_8)da_4 - \frac{3}{2}\sin(2a_8)da_3 - \frac{3}{2}\sin(2a_8)\cos(2a_2)\,da_1 \tag{B.4}$$

$$J_h^5 = da_8 \tag{B.5}$$

$$\begin{aligned} J_h^6 = &\left[\sin^3(a_7)\cos(2a_6)\cos a_8 + \cos^3(a_7)\sin(2a_6)\sin a_8\right] da_5 \\ &+\sin(2a_5)\left[\sin^3(a_7)\sin(2a_6)\cos a_8 - \cos^3(a_7)\cos(2a_6)\sin a_8\right] da_4 \\ &+\frac{3}{2}\sin(2a_7)\left[-\sin(2a_3)\cos(a_7)\cos a_8 + \cos(2a_3)\sin(a_7)\sin a_8\right] da_2 \\ &+\frac{3}{2}\sin(2a_7)\sin(2a_2)\left[\cos(2a_3)\cos(a_7)\cos a_8 \right.\\ &\left.+\sin(2a_3)\sin(a_7)\sin a_8\right] da_1 \end{aligned} \tag{B.6}$$

$$\begin{aligned} J_h^7 = &\left[\sin^3(a_7)\sin(2a_6)\cos a_8 + \cos^3(a_7)\cos(2a_6)\sin a_8\right] da_5 \\ &-\sin(2a_5)\left[\sin^3(a_7)\cos(2a_6)\cos a_8 - \cos^3(a_7)\sin(2a_6)\sin a_8\right] da_4 \\ &+\frac{3}{2}\sin(2a_7)\left[\cos(2a_3)\cos(a_7)\cos a_8 - \sin(2a_3)\sin(a_7)\sin a_8\right] da_2 \\ &+\frac{3}{2}\sin(2a_7)\sin(2a_2)\left[\sin(2a_3)\cos(a_7)\cos a_8 \right.\\ &\left.+\cos(2a_3)\sin(a_7)\sin a_8\right] da_1 \end{aligned} \tag{B.7}$$

$$J_h^{11} = da_7 \tag{B.8}$$

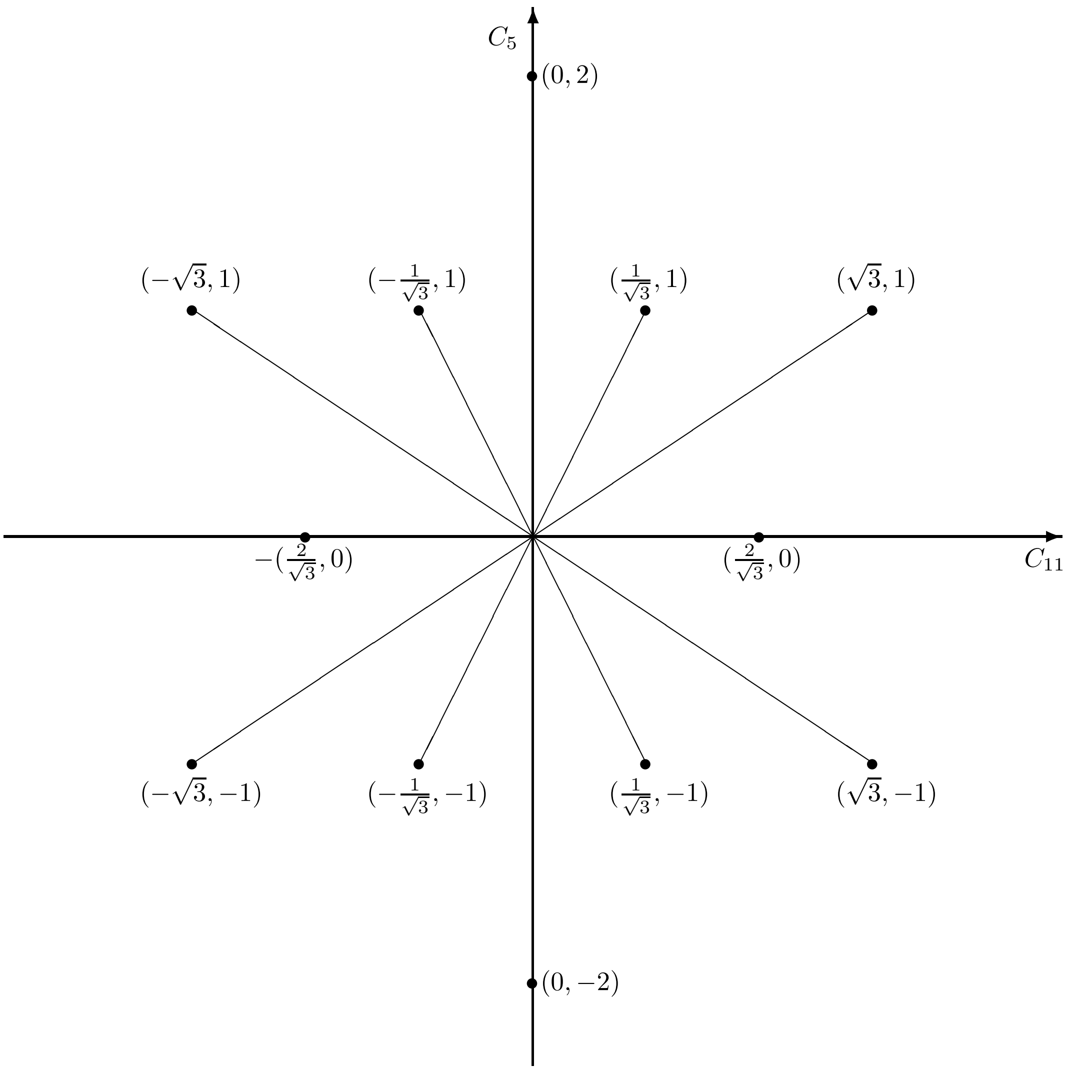
$$J_h^{12} = \frac{\sqrt{3}}{2}\sin(2a_7)\,da_6 + \frac{\sqrt{3}}{2}\cos(2a_5)\sin(2a_7)\,da_4 - \frac{\sqrt{3}}{2}\sin(2a_7)\,da_3 - \frac{\sqrt{3}}{2}\sin(2a_7)\cos(2a_2)\,da_1 \tag{B.9}$$

$$\begin{aligned} J_h^{13} = &-\frac{\sqrt{3}}{2}\sin(2a_7)\left[\cos(a_7)\cos(2a_6)\cos a_8 + \sin(a_7)\sin(2a_6)\sin a_8\right] da_5 \\ &+\frac{\sqrt{3}}{2}\sin(2a_7)\sin(2a_5)\left[-\cos(a_7)\sin(2a_6)\cos a_8 + \sin(a_7)\cos(2a_6)\sin a_8\right] da_4 \\ &+\sqrt{3}\left[\sin(2a_3)\sin(a_7)\left(3\sin^2(a_7)-2\right)\cos a_8 \right.\\ &\left.-\cos(2a_3)\cos(a_7)\left(3\cos^2(a_7)-2\right)\sin a_8\right] da_2 \\ &-\sqrt{3}\sin(2a_2)\left[\cos(2a_3)\sin(a_7)\left(3\sin^2(a_7)-2\right)\cos a_8 \right.\\ &\left.+\sin(2a_3)\cos(a_7)\left(3\cos^2(a_7)-2\right)\sin a_8\right] da_1 \end{aligned} \tag{B.10}$$

$$\begin{aligned} J_h^{14} = &\frac{\sqrt{3}}{2}\sin(2a_7)\left[\cos(a_7)\sin(2a_6)\cos a_8 + \sin(a_7)\cos(2a_6)\sin a_8\right] da_5 \\ &-\frac{\sqrt{3}}{2}\sin(2a_7)\sin(2a_5)\left[\cos(a_7)\cos(2a_6)\cos a_8 - \sin(a_7)\sin(2a_6)\sin a_8\right] da_4 \\ &+\sqrt{3}\left[\cos(2a_3)\sin(a_7)\left(3\sin^2(a_7)-2\right)\cos a_8 \right.\\ &\left.-\sin(2a_3)\cos(a_7)\left(3\cos^2(a_7)-2\right)\sin a_8\right] da_2 \\ &+\sqrt{3}\sin(2a_2)\left[\sin(2a_3)\sin(a_7)\left(3\sin^2(a_7)-2\right)\cos a_8 \right.\\ &\left.+\cos(2a_3)\cos(a_7)\left(3\cos^2(a_7)-2\right)\sin a_8\right] da_1 \end{aligned} \tag{B.11}$$

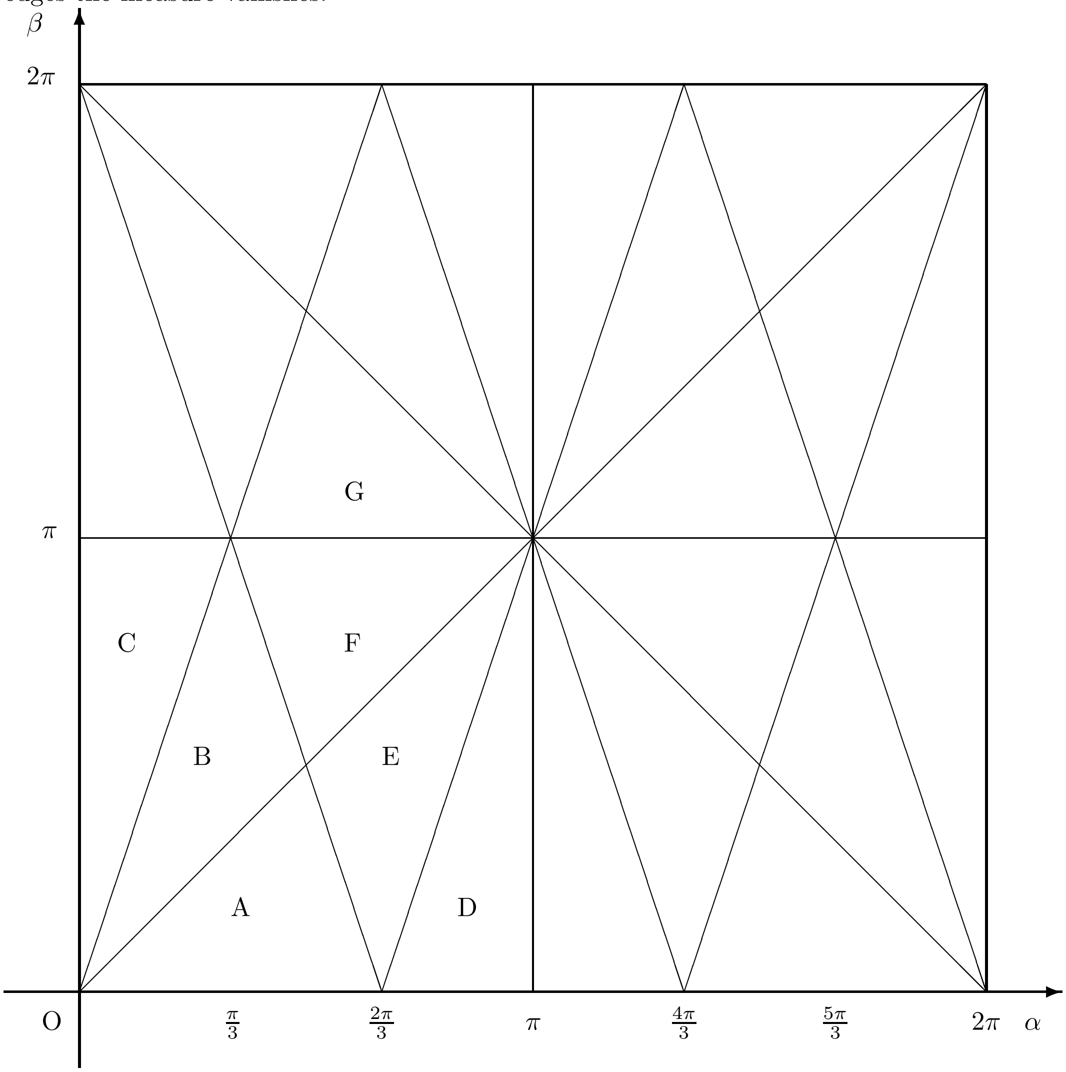
C The root system

Here we show the roots computed using C_5 and C_{11} normalized to 1. The long roots have length 2 and the short ones have length $2/\sqrt{3}$.



D The range for a_7 and a_8

Here we show a plot of the fundamental region for the variables a_7 and a_8, which is determined by the condition $f(\alpha, \beta) > 0$ where f is given in (3.12). We obtain a tiling of the torus in 24 triangles, over which the sign of the measure alternates, starting with a positive sign in the region B. In the edges the measure vanishes.



We now show that any sector of this tiling gives exactly the same contribution (obviously up to a sign) to the volume of G_2. To this end we describe some symmetry property of the function $f(\alpha, \beta)$. The translation symmetries $f(\alpha, \beta) = f(\alpha+\pi, \beta+\pi) = f(\alpha-\pi, \beta+\pi)$ and the reflection symmetries $-f(\alpha, \beta) = f(-\alpha, \beta) = f(\alpha, -\beta)$ allows to restrict us to the square $[0, \pi] \times [0, \pi]$. Moreover, the translations gives the equivalence of the triangles A, B and D with F, E and C

respectively.
At this point we are left with only three different kinds of triangles: A, B and D. The symmetry $f(\alpha, \beta) = f(\frac{\alpha+\beta}{2}, \frac{3\alpha-\beta}{2})$ maps A to B, B to A and D to G. This proves that we can choose whatever triangle, for example A, as the fundamental region.

E A metric for $\mathcal{H}$

Here we give the expression for the metric on $\mathcal{H}$ induced by the metric on G_2, and show that it is an Einstein metric. Let us introduce the 1–forms

$$\begin{aligned}I_1(x,y,z) &:= \sin(2y)\cos(2z)dx - \sin(2z)dy \ , \\ I_2(x,y,z) &:= \sin(2y)\sin(2z)dx + \cos(2z)dy \ , \\ I_3(x,y,z) &:= dz + \cos(2y)dx \ .\end{aligned} \tag{E.1}$$

Thus we can write

$$\begin{aligned}ds^2_{\mathcal{H}} = da_8^2 + da_7^2 &+ \left[\sin^2 a_8 \cos^2 a_7 + \cos^2 a_8 \sin^2 a_7\right]\left(da_5^2 + \sin^2(2a_5)da_4^2\right. \\ &\left.+3da_2^2 + 3\sin^2 2a_2 da_1^2\right) \\ &\frac{1}{2}\cos(2a_8)\cos 2a_7 \sin^2 2a_7 \left\{[I_1(a_4,a_5,a_6) + 3I_2(a_1,a_2,a_3)]^2\right. \\ &\left.+[I_2(a_4,a_5,a_6) - 3I_1(a_1,a_2,a_3)]^2\right\} \\ &+\frac{3}{4}\sin^2 2a_7 \,[I_3(a_4,a_5,a_6) - I_3(a_1,a_2,a_3)]^2 \\ &+\frac{1}{4}\sin^2(2a_8)\,[I_3(a_4,a_5,a_6) + 3I_3(a_1,a_2,a_3)]^2 \ .\end{aligned} \tag{E.2}$$

We will now compute the curvature of such a metric. Let us use capital indices for the full algebra, $I = 1, \dots, 14$, Latin indices for the $so(4)$ subalgebra, $i \in \{1,2,3,8,9,10\}$, and Greek indices for the complementary elements, $\alpha \in \{4,5,6,7,11,12,13,14\}$. Let f_{IJK} be the structure constants with one index lowered through the identity matrix. It is then clear that the non vanishing structure constants are those with either none or two Greek indices: f_{ijk}, $f_{\alpha\beta k}$ and permutations. Using this fact and the Maurer-Cartan equations[2], the components of the Riemann tensor with respect to the acht-bein are found using only the algebra.
In fact by construction we have that (E.2) takes the form $ds^2_{\mathcal{H}} = g_{\alpha\beta}da^\alpha \otimes da^\beta$, with

$$g_{\alpha\beta} = \tilde{J}^\gamma_\alpha \tilde{J}^\delta_\beta \delta_{\gamma\delta} \ . \tag{E.3}$$

Let us note that, using notations as in section 3, we have

$$J_h = \tilde{J}_h + J^i_h C_i \ . \tag{E.4}$$

On the other hand, using $J_h = h^{-1}dh$ one finds

$$dJ_h = -J_h \wedge J_h = -\frac{1}{2}J^I \wedge J^J f_{IJ}{}^K C_K \ . \tag{E.5}$$

The spin connection can be determined computing $d\tilde{J}_h$ from (E.4). Being $\tilde{J}^\alpha_h = J^\alpha_h$ we can omit the tilde in what follows and write

$$dJ^\alpha_h = -\frac{1}{2}J^I J^J f_{IJ}{}^\alpha = -\frac{1}{2}J^\beta J^\gamma f_{\beta\gamma}{}^\alpha - J^i J^\beta f_{i\beta}{}^\alpha \ , \tag{E.6}$$

[2]We use the following notations: The spin connection is uniquely defined by $d\tilde{J}^\alpha_h = -\omega^\alpha{}_\beta \tilde{J}^\beta_h$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$; the Riemann tensor field is then $\mathrm{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$ with components $R_{\alpha\beta}{}^\gamma{}_\delta$ such that $\mathrm{R}^\gamma{}_\delta = \frac{1}{2}R_{\alpha\beta}{}^\gamma{}_\delta J^\alpha_h \wedge J^\beta_h$

where we used the properties of the structure constants. From this we can read the expression for the spin connection one-form, which can be written in the form

$$\omega^{\alpha}{}_{\beta} = J^{I}_{h} f_{I\beta}{}^{\alpha} - \frac{1}{2} J^{\gamma}_{h} f_{\gamma\beta}{}^{\alpha} \ . \tag{E.7}$$

The curvature tensor can be then computed directly

$$\mathrm{R}^{\alpha}{}_{\beta} = J^{\lambda}_{h} J^{\mu}_{h} \left[-f_{\lambda i}{}^{\alpha} f_{\mu\beta}{}^{i} + \frac{1}{4} f_{\lambda\mu}{}^{\nu} f_{\nu\beta}{}^{\alpha} - \frac{1}{4} f_{\mu\nu}{}^{\alpha} f_{\lambda\beta}{}^{\nu} - f_{\lambda\nu}{}^{\alpha} f_{\mu\beta}{}^{\nu} \right] \ , \tag{E.8}$$

or in components

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} = -f_{\alpha I}{}^{\gamma} f_{\beta\delta}{}^{I} + f_{\beta I}{}^{\gamma} f_{\alpha\delta}{}^{I} + \frac{1}{2} f_{\alpha\beta}{}^{I} f_{I\delta}{}^{\gamma} - \frac{1}{4} f_{\beta\delta}{}^{I} f_{I\alpha}{}^{\gamma} + \frac{1}{4} f_{\alpha\delta}{}^{I} f_{I\beta}{}^{\gamma} \ . \tag{E.9}$$

The Ricci tensor $\rho_{\alpha\beta} := R_{\gamma\alpha}{}^{\gamma}{}_{\beta}$ is then

$$\rho_{\alpha\beta} = \frac{1}{4} f_{\alpha}{}^{IJ} f_{\beta IJ} + \frac{1}{2} f_{\alpha}{}^{\gamma i} f_{\beta\gamma i} + \frac{1}{2} f_{\beta}{}^{\gamma i} f_{\alpha\gamma i} \ . \tag{E.10}$$

The explicit form of the structure constants in our base then yields

$$\rho_{\alpha\beta} = 8\delta_{\alpha\beta} \ , \tag{E.11}$$

or in curvilinear coordinates $\rho_{\mu\nu} dx^{\mu} \otimes dx^{\nu} = 8 ds^2_{\mathcal{H}}$.

References

[1] H. J .Rothe, "Lattice Gauge Theories: An Introduction", World Scientific Lecture Notes in Physics-Vol.43.

[2] B. Simon, "Representations of finite and compact groups", Graduate studies in Mathematics 10, AMS.

[3] J.J. Duistermaat, J.A.C. Kolk, "Lie Groups", Universitext, Springer.

[4] T. Tilma and E. C. G. Sudarshan, "Generalized Euler angle parameterization for SU(N)", J. Phys. A: Math. Gen. **35** (2002) 10467-10501; "Generalized Euler Angle Parameterization for $U(N)$ with Applications to $SU(N)$ Coset Volume Measures", J. Geom. Phys. **52**, 3 (2004) 263-283.

[5] K. Holland, P. Minkowski, M. Pepe and U. J. Wiese, "Exceptional confinement in G(2) gauge theory", Nucl. Phys. B **668** (2003) 207; K. Holland, M. Pepe and U. J. Wiese, "The deconfinement phase transition in Yang-Mills theory with general Lie group G", Nucl. Phys. Proc. Suppl. **129** (2004) 712 ; M. Pepe, "Deconfinement in Yang-Mills: A conjecture for a general gauge Lie group G", arXiv:hep-lat/0407019.

[6] J.P. Keating, N.Linden and Z. Rudnick Random Matrix Theory, "The exceptional Lie groups, and L-functions", J. Phys. A 36 no. 12 (special RMT volume) (2003), 2933-2944.

[7] W. Krauth and M. Staudacher, "Yang-Mills integrals for orthogonal, symplectic and exceptional groups", Nucl. Phys. B **584** (2000) 641 [arXiv:hep-th/0004076].

[8] R. Schafer, "An Introduction to Nonassociative Algebra", Dover Ed.

[9] J. C. Baez, "The Octonions", Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 2, 145–205

[10] J. H. Conway and D. Smith, "On Quaternions and Octonions", A K Peters

[11] I. G. Macdonald, "The volume of a compact Lie group", Invent. math. **56**, 93-95 (1980).