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A GEOMETRIC INTERPRETATION OF BRST SYMMETRY

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Abstract

In a geometric and unified approach to gauge, (super)gravity and (super)string theories, all the symmetries are seen as diffeomorphisms along the directions of an appropriate group manifold G. We show how to incorporate BRST symmetry in this framework, by enlarging Lie(G) to contain a central fermionic generator Q, which satisfies $Q^2 = 0$ and plays the rôle of the BRST charge.

In this Letter we provide a geometric interpretation of BRST symmetry [1], along the lines of the so called group manifold approach (for a review, see refs. in [2]).

The basic idea is to enlarge with an extra grassmann coordinate θ the group manifold G of the original theory, be it a gauge, a (super)gravity or a (super)string theory. Adding θ to the spacetime coordinates was already considered in refs. [3-4], and indeed in this way one achieves a *superspace* formulation of BRST invariant theories. Here we want to take a step further, and consider the theory as living on the enlarged *group manifold* G + Q, obtained by adding a fermionic central charge Q to the original group generators T_A, satisfying $Q^2 = 0$. This is, in our opinion, the natural geometric arena of BRST-invariant theories. Also, our formulation is easily extended to describe the BRST structure of theories containing antisymmetric tensors. In this case, the relevant geometry is that of an enlarged free differential algebra *, whose "potentials" can be forms of arbitrary degree.

Let us briefly recall the logical steps of the group manifold construction: the starting point is the choice of a Lie algebra G. Let $g(y)$ be a group element labelled by the group manifold coordinate y. The vielbein one-form on the group manifold G=exp$G$:

$$\mu(y) = g^{-1}(y)dg(y) \tag{1}$$

is taken to be the fundamental field of the theory. The one-form μ is Lie algebra valued:

$$\mu(y) = \mu^A(y)T_A \tag{2}$$

where T_A are the generators of G, whose commutators are:

$$[T_A, T_B] = C^C{}_{AB}T_C \tag{3}$$

For example, if G = Poincaré algebra, we have:

$$\mu = V^a P_a + \omega^{ab} M_{ab} \tag{4}$$

where P_a and M_{ab} are the generators of translations and Lorentz rotations, respectively. The one-forms V^a and ω^{ab} are identified with the ordinary vielbein and spin connection.

From its definition, μ obviously satisfies

$$d\mu + \mu \wedge \mu = 0 \quad \text{(Cartan – Maurer equation)} \tag{5}$$

or, using eq.(3):

$$d\mu^A + \frac{1}{2}C^A{}_{BC}\mu^B \wedge \mu^C = 0 \tag{6}$$

Eq. (5) can be considered the dual formulation of the Lie algebra (3). The closure of the exterior derivative $d^2 = 0$ is equivalent to the Jacobi identities on $C^A{}_{BC}$. Eq. (5)

* for a review on free differential algebras (FDA's) see ref. [2]

deals with forms, and is therefore better suited to the purpose of constructing integrands (lagrangians). Also, it naturally extends to p-forms ($p > 1$).

A dynamical theory for the "potential" $\mu^A(y)$ requires the notion of "soft" group manifold $\tilde{G}$, i.e. a deformation of G whose vielbein μ does not satisfy the Cartan-Maurer equation $d\mu + \mu\wedge\mu = 0$. The extent of deformation is measured by the curvature two-form

$$R^A \equiv d\mu^A + \frac{1}{2} C^A{}_{BC}\mu^B \wedge \mu^C \tag{7}$$

Its definition implies the Bianchi identities:

$$(\nabla R)^A \equiv dR^A - C^A{}_{BC} R^B \mu^C = 0 \tag{8}$$

(using $d^2 = 0$, and the Jacobi identities on $C^A{}_{BC}$). In the case of $G =$ Poincaré, R^A takes the form *

$$\begin{aligned} R^a &= dV^a - \omega^{ab}V^b \\ R^{ab} &= d\omega^{ab} - \omega^{ac}\omega^{cb} \end{aligned} \tag{9}$$

defining respectively the torsion and the Riemann curvature. These satisfy the Bianchi identities:

$$\begin{aligned} dR^a + R^{ab}V^b - \omega^{ab}R^b &\equiv \mathcal{D}R^a + R^{ab}V^b = 0 \\ dR^{ab} + R^{ac}\omega^{cb} - \omega^{ac}R^{cb} &\equiv \mathcal{D}R^{ab} = 0 \end{aligned} \tag{10}$$

where $\mathcal{D}$ is the Lorentz covariant derivative.

Bianchi identities are a powerful tool in determining the invariance structure of the theory. When the symmetry algebra closes only on shell, as in the case of supergravity without auxiliary fields, the Bianchi identities actually determine the full dynamical structure of the theory, i.e. the field equations.

By using only diffeomorphic invariant operations (exterior derivative and wedge product), we ensure from the start the invariance under diffeomorphisms on the soft group manifold $\tilde{G}$. These have the form:

$$\begin{aligned} \delta\mu^A(y) &= \mu^A(y+\delta y) - \mu^A(y) = \\ &= l_{\delta\vec{y}}\mu^A(y) \equiv (i_{\delta\vec{y}}d + d\, i_{\delta\vec{y}})\mu^A(y) = d\delta y^A + i_{\delta\vec{y}}d\mu^A = \\ &= d\delta y^A + i_{\delta\vec{y}}R^A - C^A{}_{BC}\delta y^B\mu^C \equiv (\nabla\delta y)^A + i_{\delta\vec{y}}R^A \end{aligned} \tag{11}$$

where we have used the definition (7); $l_{\delta\vec{y}}$ is the Lie derivative along the tangent vector $\delta\vec{y} = \delta y^A \frac{\partial}{\partial y^A}$, and the contraction $i_{\vec{t}}$ along a tangent vector $\vec{t}$ is defined on p-forms

$$\omega_{(p)} = \omega_{B_1\ldots B_p}\mu^{B_1}\wedge\ldots\wedge\mu^{B_p} \tag{12}$$

as

* wedge products are omitted in the following

$$i_{\vec{t}}\,\omega_{(p)} = pt^A\omega_{AB_2\ldots B_p}\mu^{B_2}\wedge\ldots\wedge\mu^{B_p} \tag{13}$$

All the invariances of the theory are contained in eq. (11). In particular, suppose that the two-form $R^A = R^A{}_{BC}\mu^B\wedge\mu^C$ has vanishing components along the directions of a subgroup H of G:

$$R^A{}_{BH} = 0 \qquad \begin{array}{l}\text{A runs on G}\\ \text{H runs on H}\end{array} \tag{14}$$

Then we say that R^A is *horizontal* on H, and the diffeomorphisms along the H-directions reduce to *gauge transformations*:

$$\delta\mu^A(y) = (\nabla\delta y)^A \tag{15}$$

Moreover, the dependence on the y^H coordinates becomes inessential, in the sense that it factorizes after a finite gauge transformation. Then $\mu^A(y)$ really lives on the coset space $\tilde{G}/H$. The theory "remembers" the invariance under y^H-diffeomorphisms by retaining the gauge invariance under H (eq.(15)), with δy^H interpreted now as a gauge parameter.

In Poincaré gravity, we assume horizontality of the curvatures along the Lorentz directions: then the fields V^a and ω^{ab} live on the coset space

$$\frac{\text{G}}{\text{H}} = \frac{\text{Poincare'}}{\text{Lorentz}} \tag{16}$$

i.e. on ordinary spacetime. The resulting theory is invariant under spacetime diffeomorphisms, and under local Lorentz rotations.

BRST symmetry is a global fermionic symmetry. Global symmetries are described in our formalism by rigid translations along some group manifold coordinates y^B. Also, the would be gauge potentials associated to the corresponding generators T_B become pure gauge, and thus effectively disappear from the theory, if one imposes $R^B = 0$.

In the case of BRST symmetry, we assume therefore that the Q-curvature R[Q] vanishes. Moreover, in order to remove the θ-dependence in the fields of the theory, we impose the horizontality constraints:

$$R^A{}_{\theta C} = R^A{}_{C\theta} = R^A{}_{\theta\theta} = 0 \tag{17}$$

The enlarged Lie algebra we start from is given by the (anti)commutations:

$$\begin{aligned} [T_A, T_B] &= C^C{}_{AB}T_C \\ [T_A, Q] &= 0 \\ \{Q, Q\} &= 0 \end{aligned} \tag{18}$$

Using the structure constants of (18) in eq. (7), we find the curvature definitions:

$$R^A \equiv d\mu^A + \frac{1}{2} C^A{}_{BC} \mu^B \wedge \mu^C$$
$$R[Q] \equiv d\mu[Q] \tag{19}$$

where $\mu[Q]$ is the potential corresponding to Q. The Bianchi identities are:

$$dR^A - C^A{}_{BC}\, d\mu^B \wedge \mu^C = 0$$
$$dR[Q] = 0 \tag{20}$$

Note that the horizontality constraints in eq. (17) and the "rigidity" constraint

$$R[Q] = 0 \tag{21}$$

are consistent with the Bianchi identities (20).

Our claim is that the gauging of the extended Lie algebra (18), supplemented with the constraints (17) and (21), yields a BRST-invariant theory. The proof is simple. First we expand the vielbein one-form μ on the basis of differentials $(dy^\alpha, d\theta)$:

$$\mu^A(y,\theta) = \mu^A_\alpha\, dy^\alpha + \mu^A_\theta\, d\theta \equiv \mu^A_\alpha\, dy^\alpha + g^A\, d\theta \tag{22}$$
$$\mu[Q](y,\theta) = \mu[Q]_\alpha\, dy^\alpha + \mu[Q]_\theta\, d\theta \tag{23}$$

$\mu[Q]$ being a pure gauge because of eq. (21), we will concentrate on the transformation laws for μ^A. Note that in eq. (22) we have renamed g^A the $d\theta$ component of μ^A. The reason is that the fermionic zero-form g^A will play the role of the ghost field associated to the gauge potential μ^A_α. Thus, gauge fields and ghost fields are parts of the same fundamental field μ^A.

Consider now the general formula (11) for all the symmetry transformations of the theory. The coordinate variation δy^A has a flat (adjoint) index A, and can be expressed in terms of coordinate variations with curved indices as:

$$\delta y^A = \delta y^\alpha \mu^A_\alpha + \delta\theta\ \mu^A_\theta \tag{24}$$

Let us specialize the variation $(\delta y^\alpha, \delta\theta)$ to describe a rigid translation in the θ direction. Then $\delta y^\alpha = 0$, $\delta\theta =$ constant and eq. (11) takes the form:

$$\delta\mu^A = \nabla(\delta\theta\ g^A) = d(\delta\theta\ g^A) + C^A{}_{BC}\mu^B(\delta\theta\ g^C)$$
$$= (-dg^A - C^A{}_{BC}\mu^B g^C)\delta\theta \tag{25}$$

where the curvature term drops because of horizontality. Projecting on the differentials dy^α, $d\theta$ yields the BRST transformation laws of the gauge fields μ^A_α and ghost fields g^A:

$$\delta\mu^A_\alpha = -(\nabla_\alpha g^A)\delta\theta \tag{26}$$

$$\delta g^A = (-\partial_\theta g^A - C^A{}_{BC} g^B g^C)\delta\theta = -\frac{1}{2} C^A{}_{BC} g^B g^C \delta\theta \tag{27}$$

In the last equation we have used the curvature definition (7) to express $\partial_\theta g^A$ as:

$$\partial_\theta g^A = -\frac{1}{2} C^A{}_{BC} g^B g^C \tag{28}$$

This concludes the proof. The theory is BRST invariant, this invariance being on the same conceptual footing as the other invariances of the theory. All of them have the same geometric origin, i.e. are relics of diffeomorphism invariance on the enlarged group manifold $\tilde{G}$+Q.

A few final comments are in order.

The whole discussion can be straightforwardly extended to the case of free differential algebras. It suffices to enlarge the FDA to FDA+Q.

Anti-BRST transformations are easily included in the game, just by considering another nilpotent central charge $\bar{Q}$, and gauging the augmented algebra $G + Q + \bar{Q}$. This of course introduces another grassmann coordinate $\bar{\theta}$, and the antighosts $\bar{g}^A$.

Our discussion provides a geometric rationale to the algorithm (the "Russian formula"*) due to Stora [5], a systematic way to introduce ghosts and BRST transformations.

* for its curious etymology , ask Raymond Stora.

References

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