A DIFFERENTIAL GEOMETRIC SETTING FOR BRS TRANSFORMATIONS AND ANOMALIES I

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Introduction.

This paper provides a detailed introduction to the differential geometric and cohomological framework underlying BRS transformations and anomalies of gauge fields. These items both appeared in the study of perturbative renormalization of gauge fields — renormalization being required to yield in particular physical answers independent of the choice of gauge [3]. The vanishing of anomalies thus appears as a criterion for the relevance of fundamental field theories.[0] However anomalies play also a positive role in a different context, that of phenomenological theories for the search of which they provide a means of writing "effective lagrangians". It is in this context that anomalies were first found [11]. Though they arose in a quantum (field theory) context, BRS transformations and anomalies ultimately appear as purely classical (differential geometric) objects, which can be isolated as such from the original quantum context — this is what we do in the present paper. In fact the anomalies themselves -- and the algorithms which are useful for their description -- are elements of certain vector valued Lie algebra cohomologies related to the ambient Yang-Mills principal bundle. This fact was realized following the discovery of the Wess-Zumino compatibility condition, which in fact characterizes 1-cocycle of the cohomology of the Lie algebra of the gauge group with values "local" functionals of the potentials (connection one-forms).

Our paper comprises seven sections, amongst which sections 1,2,3 and 5 describe prerequisites to the actual subject matter in sections 4,6 and 7. We included these prerequisites in order to be complete, and also because of the necessity of fixing notation. Section 1 describes the De Rham complex Λ^* of a principal bundle P with values in the symmetric tensors on the Lie algebra L of the structure group G, and defines the commuting actions, on this De Rham complex, of G and of the gauge group $\mathcal{G}$. Section 2 describes the subcomplex A^* of fixpoints of the action of G (i.e. Ad-equivariant element of Λ^*). Section 3 describes the cohomology algebra of the Lie algebra $\mathcal{L}$ of the gauge group. After these prerequisites, Section 4 describes the cohomology of $\mathcal{L}$ with values in A^*, with the ensuing double complex and differential algebra structures. This furnishes the framework of the BRS relations, to which section 5 is devoted, as well as a framework for the construction of anomalies, described in section 7. The cohomology algebra of $\mathcal{L}$ with values in local functionals of connection one-forms -- the receptable for anomalies -- is defined in section 6.

0) See, however, [10] where it is suggested that gauge theories with anomalies may have a consistent interpretation at the non-perturbative level.

This expository paper leaves aside important aspects to which we shall return later, e.g.,

(i) The additional analytical apparatus arising from the fact, realized in physics, that the structure group G is a linear group (a group of matrices -- its Lie algebra L consisting then also of matrices). We here look for G a general Lie group, in the spirit of the general theory of smooth principal bundle.

(ii) The Chevalley cohomology of the gauge group Lie algebra $\mathcal{X}$ with values in the local functionals of the potentials (resp., in a S(L)-valued De Rham complex) has a version utilizing equivariant differential forms on the gauge group $\mathfrak{G}$ himself, with the operators stemming from the exterior derivative of $\mathfrak{G}$.

(iii) The homotopy formula should be generalized in two respects: one can avoid the assumption of triviality of the principal bundle at hand by introducing a background field. On the other hand, it is useful to consider multidimensional generalizations of "transgression" involving more than one potential. This necessitates the replacement of S(L) by an algebra of "graded symmetric" forms.

§1. The Gauge group $\mathfrak{G}$ of a smooth principal bundle $\mathbb{P} = (P\xrightarrow{\pi}M,G)$. Actions of G and $\mathfrak{G}$ on the real and vector-valued De Rham algebras $(A^*(P,\mathbb{R}),d,\wedge)$, $(A^*(P,S(L)),d,\times)$, and $(A^*(P,L),d,[\wedge])$.

Our basic object in this paper is a smooth principal bundle $\mathbb{P}$: $P\xrightarrow{\pi}M$, with basis M and (compact) structural Lie group G. We shall denote L the Lie algebra of G, and write [u,v] for the Lie bracket of u,v $\in$ L. We denote by R the right action of G on P:

(1,1) $$R_s z = zs, \qquad s \in G, \qquad z \in P.$$

[1.1]. The gauge group $\mathfrak{G}$ and its Lie algebra $\mathcal{X}$.

The gauge group $\mathfrak{G}$ is the group of automorphisms of $\mathbb{P}$ inducing the identity on M. Specifically $\mathfrak{G}$ consists of the diffeomorphisms: P⟶P commuting with all R_s, s $\in$ G, and mapping each fiber into itself. Since Ψ acts on the fiber, we have

(1,2) $$\Psi(z) = zg(z), \qquad z \in P,$$

where g is a smooth map: P⟶G, <u>ad-equivariant</u> in the sense

(1,3) $$g(zs) = \mathrm{Ad}s^{-1}(g(z)) = s^{-1}g(z)s, \quad z \in P, \quad s \in G.$$

this expressing commutativity of Ψ and R_s. Relation (1,2) in fact establishes a bijection between the elements Ψ of $\mathfrak{G}$ and the smooth ad-equivariant maps g: P⟶G, whereby products and inverses in $\mathfrak{G}$ are turned into pointwise products, resp. inverses:

(1,4) $$\left\{\begin{array}{l}\Psi \leftrightarrow g\\ \Psi' \leftrightarrow g'\end{array}\right. \Rightarrow \left\{\begin{array}{ll}\Psi^{-1}\leftrightarrow g^{-1}, & g^{-1}(z) = g(z)^{-1}\\ \Psi\Psi'\leftrightarrow gg', & (gg')(z) = g(z)g'(z)\end{array}\right., \quad z \in P.$$

$\mathfrak{G}$ is an "infinite dimensional Lie group" (a <u>diffeological</u> <u>group</u> in the sense of Souriau [13]). As such it possesses a Lie algebra which we denote $\mathcal{X}$. We can view $\mathcal{X}$ as the set of smooth maps Ω: P⟶L, <u>Ad-equivariant</u> in the sense:

(1,5) $$\Omega(zs) = \mathrm{Ad}\, s^{-1}(\Omega(z)) = \text{"}s^{-1}\Omega(z)s,\text{"} \; s \in G$$

(here Ad s is the tangent map of ad s = $s \cdot s^{-1}$ at the unit of G). Lie bracket and exponential are then obtained pointwise:

(1,6) $$[\Omega,\Omega'](z) = [\Omega(z),\Omega(z')], \quad \Omega,\Omega' \in \mathcal{X}, \quad z \in P$$

(left hand side bracket in $\mathcal{X}$, right hand side bracket in L).

(1,7) $$e^{\Omega}(z) = e^{\Omega(z)}, \quad \Omega \in \mathcal{X}, \quad z \in P.$$

Setting, for $\psi \in \mathcal{C}^{\infty}(M)$, $(\psi\Omega)(z) = \psi(\pi(z))\Omega(z)$, $\Omega \in \mathcal{X}$, $z \in P$, we thus obtain an action of $\mathcal{C}^{\infty}(M)$ on $\mathcal{X}$ commuting with Lie-brackets; thus $\mathcal{X}$ <u>is a Lie algebra over $\mathcal{C}^{\infty}(M)$</u>.

[1.2]. The (real valued) De Rham complex $(\Lambda^*(P,\mathbb{R}),d,\wedge)$ as a GCDA. Action of the Lie algebra $\mathfrak{X}(P)$ on $\Lambda^*(P,\mathbb{R})$. Representation of G and $\mathfrak{V}$ on $\Lambda^*(P,\mathbb{R})$.

We write $\Lambda^*(P,\mathbb{R}) = \oplus\Lambda^p(P,\mathbb{R})$, with $\Lambda^*(P,\mathbb{R})$ the set of smooth real-valued differential p-forms on P. Denoting by $\mathfrak{X}(P)$ the Lie algebra of smooth vector fields on P, we can view $\Lambda^*(P,\mathbb{R})$ as the set of $\mathcal{C}^{\infty}(P)$-valued, alternate $\mathcal{C}^{\infty}(P)$-linear p-forms on $\mathfrak{X}(P)$ ($\mathcal{C}^{\infty}(P) = \Lambda^{0}(P,\mathbb{R})$). The *wedge product* $\wedge$, *exterior derivative* d *Lie derivative* $L(\xi)$ *along* $\xi \in \mathfrak{X}(P)$ *and inner product* $i(\xi)$ *by* $\xi \in \mathfrak{X}(P)$, are then defined as follows: for $\xi_0, \xi_1,\dots,\xi_{p+q} \in \mathfrak{X}(P)$, $\alpha \in \Lambda^*(P,\mathbb{R})$, $\beta \in \Lambda^*(P,\mathbb{R})$ we have

(1,8) $$(\alpha\wedge\beta)(\xi_1,\dots,\xi_{p+q}) = \frac{1}{p!}\frac{1}{q!}\sum_{\sigma\in\Sigma_{p+q}} \chi(\sigma)\alpha(\xi_{\sigma 1},\dots,\xi_{\sigma p})\beta(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})$$

(1,9) $$(d\alpha)(\xi_0,\dots,\xi_p) = \sum_{i=0}^{p}(-1)^i\xi^i(\alpha(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_p)) + \sum_{0\leq i<j\leq p}(-1)^{i+j}\alpha([\xi_i,\xi_j],\xi_0,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_p)$$

(1,10) $$(L(\xi)\alpha)(\xi_1,\dots,\xi_p) = \xi(\alpha(\xi_1,\dots,\xi_p)) - \sum_{i=1}^{p}\alpha(\xi_1,\dots,\xi_{i-1},[\xi,\xi_i],\xi_{i+1},\dots,\xi_p)$$

(1,11) $$\begin{cases} (i(\xi)\alpha)(\xi_1,\dots,\xi_{p-1}) = \alpha(\xi,\xi_1,\dots,\xi_{p-1}) \\ i(\xi) = 0 \text{ on } \Lambda^{0}(P,\mathbb{R}). \end{cases}$$

Through definitions (1,8), (1,9) <u>$(\Lambda^*(P,\mathbb{R}),d,\wedge)$ now becomes a GCDA</u>. <u>And definitions (1,10) and (1,11) determine an action (L,i) of the Lie algebra $\mathfrak{X}(P)$ on this GCDA</u>. For a proof of these well known facts we refer e.g. to [1], Corollary [10].

In addition to the previous structure <u>the De Rham complex $\Lambda^*(P,\mathbb{R})$ is both a G-space and a $\mathfrak{V}$-space</u>. We obtain the action of $s \in G$ on $\Lambda^*(P,\mathbb{R})$ as $r(s) = R_s^*$, where $R_s^*\alpha$ denotes the pull back of the differential form α by R_s, specifically[1]:

(1,12) $$(r(s)\alpha)(z,Z_i) = \alpha(zs,R_{s*g}Z_i), \quad z \in P, \; Z_i \in T_z^P$$

<u>In this way G is represented in the zero grade automorphisms of the GCDA $(\Lambda^*(P,\mathbb{R}),d,\wedge)$</u> (indeed r(s), $s \in G$, commutes with d and with the wedge product).The corresponding representation of L:

(1,13) $$\theta(u) = \frac{d}{dt}\Big|_{t=0} R^*_{e^{tu}}, \quad u \in L,$$

then arises as the composition

(1,14) $$\theta(u) = L(\xi^u), \quad u \in L,$$

1) Since both the right action and the pull back are product-inverting, we obtain indeed a representation $s \longrightarrow R_s^*$ of G on $\Lambda^*(P,\mathbb{R})$. For a definition in terms of sections see [1.5] below.

with ξ^u the principal field

$$\xi^u_z = L_{z*e}u, \quad u \in L, \tag{1,15}$$

(here L_z, $z \in P$, is the map: $G \longrightarrow P$ determined by

$$L_z s = zs \; (= R_s z), \quad z \in P, \; s \in G \tag{1,16}$$

From this and the convention

$$i(u) = i(\xi^u), \quad u \in L, \tag{1,17}$$

we get an action (θ, i) of the Lie algebra L on $\Lambda^*(P,\mathbb{R})$.

We now describe the representation ρ of $\mathcal{D}$ on $\Lambda^*(P,\mathbb{R})$: for $\Psi \in \mathcal{D}$, $\rho(\Psi)$ is obtained by pulling back the (inverse) action of $\mathcal{D}$[2]

$$\rho(\Psi)\alpha = \Psi^{-1*}\alpha, \quad \Psi \in \mathcal{D}, \; \alpha \in \Lambda^*(P,\mathbb{R}), \tag{1,18}$$

specifically we have[3]

$$(\rho(\Psi)\alpha)(z, Z_i) = \alpha(\Psi^{-1}(z), (\Psi^{-1})_{*z} Z_i), \quad z \in P, \; Z_i \in T^P_z. \tag{1,18a}$$

2) Since $\mathcal{D}$ acts on P on the left, we now have to pull back the inverse of Ψ.

3) For a definition in terms of sections, see [1,5] below.

We shall also denote ρ the accompanying representation of $\mathcal{X}$:

$$\rho(\Omega) = \frac{d}{dt}\Big|_{t=0} \rho(e^{t\Omega}), \quad \Omega \in \mathcal{X}. \tag{1,19}$$

Specifically, one has, for $\Psi = e^{t\Omega}$

$$(\Psi^{-1})_{*z} = (R_{e^{-t\Omega(z)}})_{*z} - t(L_{ze^{-t\Omega(z)}})_{*e} d\Omega(z, \cdot) \tag{1,20}$$

$\mathcal{D}$ (resp. $\mathcal{X}$) are thereby represented in the zero grade automorphisms (resp. derivatives commuting with d) of the GCDA $(\Lambda^*(P,\mathbb{R}), d, \wedge)$ (immediate consequence of the commutativity of the pull back (1,18) with d and with the wedge product). Furthermore, since $\Psi \in \mathcal{D}$ commutes with R_s, $s \in G$, the representations of G and $\mathcal{D}$ on $\Lambda^*(P,\mathbb{R})$ commute.

So much about real-valued differential forms on P. We now describe the differential forms on P with values in L, or more generally in symmetric tensors over L.

[1.3]. The S(L)-valued De Rham complex $(\Lambda^*(P,S(L)), d, \vee)$ as a GCDA. Representations of G and $\mathcal{D}$ on $\Lambda^*(P,S(L))$

We denote by S(L) the symmetric algebra over L:

$$\begin{cases} S(L) = \bigoplus_{k \in \mathbb{N}} S_k(L) \\ S_k(L) = L^{\vee k} = S_k L^{\otimes k}, \quad S_0(L) = \mathbb{R} \end{cases} \tag{1,21}$$

equipped with the symmetric product

$$f \vee f' = S_{k+\ell}(\varphi \otimes \varphi'), \quad f \in S_k(L), \; f' \in S_\ell(L) \tag{1,21a}$$

Here S_k, $k > 0$ the idempotent projecting $L^{\otimes k}$ onto the symmetric tensors -- vanishing on the $f \otimes f' - (-1)^{i+j} f' \otimes f$, $i,j \in \mathbb{N}$, $i+j = k$. And $S_0 = id_L$. Note that the dual $S_k^*(L)$ can be identified with the set of symmetric k-linear forms P on L[4] by writing

(1,22) $$P(u_1,\dots,u_k) = P(u_1 \vee \dots \vee u_k),\ u_s,\dots,u_k \in L$$

A subset of $S_k^*(L)$ of particular interest is the subset $I_k(L)$ of Ad-invariant symmetric k-forms characterized by

(1,23) $$P(Ads(u_1),\dots,Ads(u_k)) = P(u_1,\dots,u_k),\ s \in G,\ u_1,\dots,u_k \in L$$

or equivalently

(1,23a) $$\sum_{i=1}^{n} P(u_1,\dots,u_{i-1},[u,u_i],u_{i+1},\dots,u_k) = 0,\ u,u_1,\dots,u_k \in L.$$

Now we consider the S(L)-valued De Rham complex of P:

(1,24) $$\Lambda^*(P,S(L)) = \bigoplus_p \Lambda^p(P,S(L))$$

4) The latter are in turn one-to-one with the polynomials of order k on L, the passage from k-forms to polynomials arising by restriction to the diagonal, and inversely by polarization.

with $\Lambda^p(P,S(L))$ the set of S(L)-valued smooth differential forms on P, alternatively

(1,25) $$\Lambda^p(P,S(L)) = S(L) \otimes \Lambda^p(P,\mathbb{R})$$

with the identification

(1,25a) $$\begin{cases} \tau = f \otimes \alpha \iff \langle \tau(\xi_1,\dots,\xi_p) = \alpha(\xi_1,\dots,\xi_p) f, \text{ for all } \xi_1,\dots,\xi_p \\ \tau \in \Lambda^p(P,S(L)),\ \alpha \in \Lambda^p(P,\mathbb{R}),\ f \in S(L) \end{cases}$$

Of course $\Lambda^p(P,S(L))$ decomposes into "homogeneous components":[5]

(1,26) $$\begin{cases} \Lambda^p(P,S(L)) = \bigoplus_{k \in \mathbb{N}} \Lambda_k^p(P,S(L)) \\ \Lambda_k^p(P,S(L)) = \Lambda^p(P,S_k(L)) = \Lambda^p(P,\mathbb{R}) \otimes S_k(L) \end{cases}$$

the term $k = 0$ arising from our convention $S_0(L) = \mathbb{R}$, which implies

(1,27) $$\Lambda_0^p(P,S(L)) = \Lambda^p(P,\mathbb{R})$$

(incorporating in this way the real valued De Rham complex has a convenient unifying virtue)

5) We accordingly write $\Lambda_k^*(P,S(L)) = \bigoplus_p \Lambda_k^p(P,S(L))$.

On $\Lambda^*(P,S(L))$ we now define[6]

— a bilinear product $\times$ by requiring

(1,28) $$(f\otimes\alpha)\times(f'\otimes\alpha') = (f\vee f')\otimes(\alpha\wedge\alpha'), \quad \begin{cases} \alpha,\alpha'\in\Lambda(P,\mathbb{R}) \\ f,f'\in S(L) \end{cases}$$

— operators, d, i(u) θ(u), u $\in$ L as follows:

(1,29) $$d = id_{S(L)}\otimes d$$

(1,30) $$i(u) = id_{S(L)}\otimes i(u)$$

(1,31) $$\theta(u) = id_{S(L)}\otimes\theta(u) + \mathrm{Ad}\,u\otimes id_{\Lambda^p(P,\mathbb{R})}$$

(with the following definition of Adu on S(L)

(1,32) $$\begin{cases} \mathrm{Ad}u(u_1\vee\ldots\vee u_k) = \sum_{i=1}^{k} u_1\vee\ldots\vee u_{i-1}\vee[u,u_i]\vee u_{i+1}\vee\ldots\vee u_k) \\ u_1,\ldots,u_k \in L \end{cases},$$

— actions r and ρ of G, resp. $\mathfrak{F}$, as follows:

(1,33) $$r(s) = \mathrm{Ad}s\otimes R_s^*, \qquad s \in G$$

6) Note that these definitions extend d, i(u) and $\rho(\Psi)$ by requiring them to be trivial on $S(L) = S(L)\otimes 1$, 1 the unit function on P. In contrast θ(u) and r(s) are obtained from tensorizing with the adjoint representation. This will result in a departure from a "Lie action situation", see (ii) below.

(with the following definition of Ads on S(L):

(1,34) $$\mathrm{Ads}(u_1\vee\ldots\vee u_k) = \begin{cases} \mathrm{Ads}(u_1)\vee\ldots\vee\mathrm{Ads}(u_k), \ u_1,\ldots,u_k \in L) \\ \mathrm{Ads} = id \text{ on } S_0(L) \end{cases}$$

(1,35) $$\rho(\Psi) = id_{S(L)}\otimes\rho(\Psi)$$

the latter yielding as usual

(1,36) $$\rho(\Omega) = \frac{d}{dt}\Big|_{t=0}\rho(e^{t\Omega}), \qquad \Omega \in \mathfrak{X}.$$

We note that these definitions reduce on $\Lambda_0^*(P,S(L))$ to our former definitions (1,8), (1,9), (1,13), (1,17), (1,12) and (1,18), in accordance with embedding (1,27).

The foregoing definitions now imply the following:

(i) $\Lambda^*(P,S(L)),d,\times)$ is a GCDA whose sub-GCDA $(\Lambda_0^*(P,S(L),d,\times)$ is isomorphic with $(\Lambda^*(P,\mathbb{R}),d,\wedge)$.

(ii) The pair (θ,i) behaves as an action of $\mathfrak{X}(P)$ on this GCDA, but for the fact that one has[7]

(1,37) $$i(u)d + di(u) = id_{S(L)}\otimes\theta(u), \ (= \theta(u) - \mathrm{Ad}u\otimes id_{\Lambda^p(P,\mathbb{R})}), \quad u \in L.$$

(iii) r as defined in (1,33) is a representation of G on $\Lambda^*(P,S(L))$ by zero-grade automorphisms of the latter as a GCDA, moreover such that the accompanying representation of L coincides with θ[8]:

7) In other terms, with the replacement $\mathfrak{A}\longrightarrow\Lambda^*(P,S(L))$, the product $\cdot \longrightarrow$ the product $\times$, one has properties (A,1) through (A,16) in Appendix A except property (A,14) to be replaced by (1,37).

8) Coherent with the fact that the θ(u) are derivations commuting with d (cf. A,13) and (A,15).

(1,38) $$\frac{d}{dt}\Big|_{t=0} r(e^{tu}) = \theta(u), \qquad u \in L$$

(iv) <u>ρ as defined in (1,35) (resp. (1,36)) is a representation of $\mathfrak{D}$ (resp. $\mathfrak{X}$) on $\Lambda^*(P,S(L))$ by zero grade automorphisms of the latter as a GCDA (resp. by derivations of $(\Lambda^*(P,S(L)),\wedge)$ commuting with d)</u>

(v) <u>The representations r and ρ commute</u>[9]:

(1,39) $$\rho(\psi)r(s) = r(s)\rho(\psi), \qquad s \in G,\ \psi \in \mathfrak{D},$$

<u>accordingly</u>

(1,40) $$\rho(\Omega)\theta(u) = \theta(u)\rho(\Omega), \qquad u \in L,\ \Omega \in \mathfrak{X}.$$

These facts are classical. For a proof of (i) we refer to, e.g. [1, Theorem 1,8] with the replacement $L \longrightarrow \mathcal{X}(P)$, $A \longrightarrow \mathcal{C}^\infty(P)$, $V \longrightarrow S(L)\otimes\mathcal{C}^\infty(P)$, $\rho(\xi) \longrightarrow id_{S(L)}\otimes\xi$, $d_\rho \longrightarrow d$, $\cdot \longrightarrow \times$. The proof of (ii) is as follows: denote $\theta_1(u)$, resp. $\theta_2(u)$ the first, resp. second term r.h.s of (1,31): (θ_1,i) is a bona fide action of the Lie algebra L on the GCDA $\Lambda^*(P,S(P))$, obtained by tensoring by $id_{S(L)}$ the action (θ,i) of L on $\Lambda^*(P,\mathbb{R})$. We examine the changes in (A,9), (A,12) through (A,16) brought about by the change $\theta_1 \longrightarrow \theta_1+\theta_2$. Since $\theta_2(u)$ is a zero grade derivation, (A,9) stays unchanged. Since θ_1, and θ_2 are mutually commuting representation of the Lie algebra L, (A.12) is maintained.

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9) We could thus consider $G\times\mathfrak{D} \ni (s,\psi) \longmapsto r(s)\rho(\psi)$ as a representation of the direct product of the groups G and $\mathfrak{D}$.

We check (A,13), from which (A,16) follows: we have, from (1,30), (1,31)

(1,41) $$\begin{aligned}\theta(u)i(v) - i(v)\theta(u) &= id_{S(L)}\otimes(\theta_1(u)i(v) - i(v)\theta(u)) \\ &= id_{S(L)}\otimes i([u,v]) = i([u,v])\end{aligned}$$

We examine (A,14): we have, from (1,21),(1,30):

(1,42) $$\begin{aligned}i(u)d + di(u) &= id_{S(L)}\otimes(i(u)d + di(u)) \\ &= id_{S(L)}\otimes\theta(u)\end{aligned}$$

Finally we have from (1,29), (1,31)

(1,43) $$d\theta(u) - \theta(u)d = id_{S(L)}\otimes(d\theta(u) - \theta(u)d) = 0$$

hence (A,15) stays unchanged.

<u>Remark.</u> Defining $L(\xi)$ and $i(\xi)$, $\xi \in \mathcal{X}(P)$ on $\Lambda^*(P,S(L))$ as

(1,43) $$L(\xi) = id_{S(L)}\otimes L(\xi)$$

(1,44) $$i(\xi) = id_{S(L)}\otimes i(\xi)$$

we get an action of the Lie algebra $\mathcal{X}(M)$ on the GCDA $\Lambda^*(P,S(L))$ yielding the above action (θ_1,i) by composition with ξ in (1,15).

[1.4].

Our next concern is the "<u>descent</u>" from $S_k(L)$-valued to real differential forms by means of elements of $S_k^*(L)$. Consider a "polynomial" $P \in S_k^*(L)$ of order k: taking its covalue with a p-form $\tau \in \Lambda_k^p(P,S(L))$:

(1,43) $$(P(\tau))(\xi_1,\dots,\xi_p)(z) = P(\tau(\xi_1,\dots,\xi_p)(z)), \quad \xi_1,\dots,\xi_p \in \mathcal{X}(P), \qquad z \in P$$

(in other terms:

(1,43a) $$P(f\otimes\alpha) = P(f)\alpha,\quad \alpha \in \Lambda^p(P,\mathbb{R}),\ f \in S_k(L))$$

one gets an element of $\Lambda^p(P,\mathbb{R})$. We can thus view the dual $S_k^*(L)$ of $S_k(L)$ as providing linear maps

(1,44) $$P:\ \Lambda_k^p(P,S(L)) \ni \tau \longrightarrow P(\tau) \in \Lambda_k^p(P,\mathbb{R}),$$

with the properties

(1,45) $$P\circ d = d\circ P$$

(1,46) $$P\circ i(u) = i(u)\circ P,\qquad u \in L$$

(1,47) $$P\circ r(s) = r(s)\circ P\circ Ads,\qquad s \in G$$

Proof. (1,45), (1,46) follows from (1,43), (1,29), (1,30):

(1,48) $$\begin{aligned} P\circ d(f\otimes\alpha) &= P(f\otimes d\alpha) = P(f)d\alpha \\ &= d(P(f)\alpha) = d\circ P(f\circ\alpha) \end{aligned}$$

(1,49) $$\begin{aligned} P\circ i(u)(f\otimes\alpha) &= P(f\otimes i(u)\alpha) = P(f)i(u)\alpha \\ &= i(u)(P(f)\alpha) = i(u)\circ P(f\circ\alpha) \end{aligned}$$

On the other hand, (1,47) follows from (1,33), (1,43) (cf. (1,23))

(1,50) $$\begin{aligned} P\circ r(s)(f\otimes\alpha) &= P(Ads(f)\otimes R_s^*\alpha) \\ &= P(Ads(f))\circ R_s^*\alpha \\ &= R_s^* P(Ads(f))\cdot\alpha \\ &= r(s)(P\circ Ads)(f\otimes d) \end{aligned}$$

The 1-tensor part $\Lambda^*(P,L)$ of $\Lambda^*(P,S(L))$ deserves a special examination, since it inherits from the Lie bracket of L a graded Lie algebra structure (essential for expressing the BRS relations)

[1.5]. The L-valued De Rham complex $(\Lambda^*(P,L),d,[\wedge])$ as a DGL.

Recalling the identification

(1,47) $$\begin{cases} \Lambda_1^p(P,S(L)) = \Lambda^p(P,L) = L\otimes\Lambda^p(P,\mathbb{R}) \\ \lambda = u\otimes\alpha \iff \lambda(\xi_1,\ldots,\xi_p) = \alpha(\xi_1,\ldots,\xi_p)u \\ \lambda \in \Lambda^p(P,L),\ \alpha \in \Lambda^p(P,\mathbb{R}),\ \xi_1,\ldots,\xi_p \in \mathfrak{X}(P) \end{cases}$$

we define as follows the Schouten product $[\wedge]$ on $\Lambda^*(P,L)$: for $\lambda \in \Lambda^p(P,L)$, $\mu \in \Lambda^q(P,L)$, $p,q \in \mathbb{N}$, $\xi_1,\ldots,\xi_{p+q} \in L$ we set

(1,48) $$[\lambda\wedge\mu](\xi_1,\ldots,\xi_{p+q}) = \frac{1}{p!}\frac{1}{q!}\sum_{\sigma\in\Sigma_{p+q}} \chi(\sigma)[\lambda(\xi_{\sigma 1},\ldots,\xi_{\sigma p}),\mu(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)})]$$

where [,] s.h.s. denotes a Lie bracket in L. Alternative specification:

(1,48a) $$[u\otimes\alpha \wedge v\otimes\beta] = [u,v]\otimes(\alpha\wedge\beta),\quad \begin{cases} \alpha,\beta \in \Lambda^*(P,\mathbb{R}) \\ u,v, \in L \end{cases}$$

With this definition, we have that $(\Lambda^*(P,L),d,[\wedge])$ is a DGL. Moreover r, resp. ρ, restricted to $\Lambda^*(P,L)$ are commuting representations of the group G, resp. $\mathfrak{G}$, by zero-grade automorphisms of $(\Lambda^*(P,L),d,[\wedge])$. Correlatively, θ, resp., ρ are commuting representations of the Lie algebras L, resp. $\mathfrak{g}$ by zero grade derivations of $(\Lambda^*,[\wedge])$ commuting with d. These facts are classical. A proof of the DGL property of $(\Lambda^*(P,L),d,[\wedge])$ can be inferred from [1, Theorem 1,8] with the replacement $L\longrightarrow\mathfrak{X}(P)$, $A\longrightarrow\mathcal{C}^\infty(P)$, $V\longrightarrow\mathcal{C}^\infty(P)\otimes L$, $\rho(\xi) = \xi\otimes id_L$, $d_\rho\longrightarrow d$, $\cdot\longrightarrow[\wedge]$. We know from the preceding paragraph that r(s) and $\rho(\psi)$, $s \in G$, $\psi \in \mathfrak{G}$, commute with d. Moreover (1,48a) shows that they commute with the Schouten product: indeed we saw that R_s^* and $\rho(\psi)$ acting on $\Lambda^*(P,\mathbb{R})$ commute with the wedge product. And Ads commutes with the Lie bracket of L:

(1,49) $$[Ads(u),Ads(v)] = Ads([u,v]),\qquad s \in G,\ u,v \in L$$

[1.6]. The groups G and $\mathfrak{D}$ as acting on sections.

We mentioned that $\Lambda^*(P,\mathbb{R})$ can be considered as the set of $\mathcal{C}^\infty(P)$-valued, alternate, $\mathcal{C}^\infty(P)$-linear p-forms on $\mathfrak{X}(P)$ and formulated the definitions (1,8) through (1,11) and (1,14), (1,17) in this context. We here give for completeness the corresponding definitions of the representations r and ρ. Letting G and $\mathfrak{D}$ act on $\mathfrak{X}(P)$[10] as follows:

$$(1,50) \qquad (r(s)\xi)_z = (R_s)_{*zs^{-1}}\xi_{zs^{-1}}, \qquad s \in G$$

$$(1,51) \qquad (\rho(\Psi)\xi)_z = (\Psi^{-1})_{*\Psi(z)}\xi_{\Psi(z)}, \qquad \Psi \in \mathfrak{D}$$

the definitions (1,33), (1,35) are alternatively phrased as follows: one has, for $\tau \in \Lambda^p(P,S(L))$

$$(1,33a) \qquad (r(s)\tau)(\xi_1,\dots,\xi_p) = Ads(\tau(r(s)\xi_1,\dots,r(s)\xi_p)\circ R_s), \qquad s \in G$$

$$(1,35a) \qquad (\rho(\Psi)\tau)(\xi_1,\dots,\xi_p) = \tau(\rho(\Psi)\xi_1,\dots,\rho(\Psi)\xi_p)\circ\Psi^{-1}) \qquad \Psi \in \mathfrak{D}$$

It then follows from

$$(1,52) \qquad r(s)\xi_u = \xi_{Ads^{-1}(u)}, \qquad u \in L,\ s \in G$$

that one has

$$(1,53) \qquad r(s)i(u) = i(Ads(u))r(s), \qquad u \in L,\ s \in G$$

10) On the right.

§2. The differential subalgebras. $(A^*,d,\wedge)$, $(A_0^*,d,\times)$ and $(A_1^*,d,[\wedge])$ of real invariant, resp. S(L) and L-valued Ad-equivariant differential forms.

The fixed point set, for the action of G, of the differential algebras of the last sections, are differential subalgebras of direct relevance for gauge-field theory. We devote this section to their description.

[2.1]. The GCDA $(A^*,d,\times)$ of Ad-equivariant elements of $\Lambda^*(P,S(L))$.

We say that $\tau \in \Lambda^*(P,S(L))$ is Ad-equivariant whenever one has

$$(2.1) \qquad \begin{cases} R_s^*\tau = Ads^{-1}\tau \text{ for all } s \in G \\ \text{i.e. } \tau(zs, R_{s*z}Z_i) = Ads^{-1}\tau(z,Z_i),\ s \in G,\ z \in P,\ Z_i \in T_z^P \end{cases}$$

This is tantamount to requiring that

$$(2,1a) \qquad r(s)\tau = \tau \text{ for all } s \in G$$

(cf. 1,33), or else, if G is connected[11]

$$(2,1b) \qquad \theta(u)\tau = 0 \text{ for all } u \in L$$

Note that, since Ad acts trivially on the zero degree part $\Lambda_0^*(P,S(L)) = \Lambda^*(P,\mathbb{R})$, condition (2,1) restricted to the latter simply means invariance of the real valued form $\alpha \in \Lambda^*(P,\mathbb{R})$:

11) Generally (2,1b) amounts to requiring (2,1a) for all $s \in G$ within the connected component of the identity.

(2,1a) $$\begin{cases} R_s^*\alpha = \alpha \text{ for all } s \in G \\ \text{i.e. } \alpha(zs, R_{s*z}Z_i) = \alpha(z,Z_i),\ z \in P,\ Z_i \in T_z^P,\ s \in G. \end{cases}$$

We denote by A^* (resp. A^*, A_k^*, A_k^P, $p,k \in \mathbb{N}$) the respective subsets of Ad-equivariant forms in $\Lambda^*(P,S(L))$ (resp. in $\Lambda^p(P,S(L))$, $\Lambda^*(P,S_k(L))$, $\Lambda^p(P,S_k(L))$).

Since the $r(s)$, $s \in G$, are zero-grade automorphisms of the GCDA $(\Lambda^*(P,L),d,\times)$ commuting with all $\rho(\psi)$, $\psi \in \mathfrak{G}$, and leaving the degree k invariant, we have that A^* decomposes as

(2,1) $$A^* = \oplus_p A^p = \oplus_k A_k^* = \oplus_{p,k} A_k^p$$

and is a sub-GCDA of $(\Lambda^*(P,S(L)),d,\times)$ stable under the the action ρ of $\mathfrak{G}$ thus also of $\mathcal{X}$ (as well as every component of A_k^p) and containing A_0^* as a sub-GCDA isomorphic to the GCDA $(\Lambda^*(P,\mathbb{R}),d,\wedge)$.

We note that Ad-invariance allows to simplify the explicit expression of $\rho(\psi)$. First note that, due to (1,35), $\rho(\psi)\tau$ is given for $\tau \in \Lambda^p(P,S(L))$ by the same algorithm as for a real valued differential from (cf. 1,13a):

(2,3) $$(\rho(\psi)\lambda)(z,Z_i) = \lambda(\psi^{-1}(z),(\psi^{-1})_{*z}Z_i),\ z \in P,\ Z_i \in T_z^P$$

with $(\psi^{-1})_{*z}$ given by (1,20) for $\psi = e^{t\Omega}$, $\Omega \in \mathcal{X}$. Plugging (1,20) in (2,3) and using the Ad-equivariance property (2,1) now yields the following explicit form of $\rho(\Omega)$ on A^*:

(2,4) $$\begin{cases} (\rho(e^{t\Omega})\tau)(z,Z_i) \\ \quad = \mathrm{Ad}\, e^{t\Omega(z)}(\tau(z,Z_i - t(L_z)_{*e}\mathrm{Ad}\, e^{-t\Omega(z)}(d\Omega(z,Z)))) \\ \tau \in A^*,\ \Omega \in \mathcal{X},\ z \in P,\ Z_i \in T_z^P \end{cases}$$

leading to

(2,5) $$\begin{cases} (\rho(\Omega)\tau)(z,Z_1,\dots,Z_p) = [\Omega(z),\tau(z,Z_1,\dots,Z_p)] \\ \quad - \sum_{i=1}^{p} \tau(z,Z_1,\dots,Z_{i-1},(L_z)_{*e}d\Omega(z,Z_i),Z_{i+1},\dots,Z_p) \\ \tau \in A^*,\ \Omega \in \mathcal{X},\ z \in P,\ Z_i \in T_z^P \end{cases}$$

(for $\tau \in A_0$ one should set $\mathrm{Ad}\, e^{t\Omega} = \mathrm{id}$ in (2,4) and omit the first term r.h.s of (2,5)). Note that (2,4) reduces to

(2,4h) $$\begin{cases} (\rho(e^{t\Omega}\tau))(z,Z_i) = \mathrm{Ad}\, e^{t\Omega(z)}(p(z,Z_i))(\tau(z,Z_i)) \\ \tau \in hA^*,\ \Omega \in \mathcal{X},\ z \in P,\ Z_i \in T_z^P \end{cases}$$

on the subset $hA^* \subset A^*$ of horizontal Ad-equivariant differential forms, singled out as

(2,6) $$\begin{cases} hA^* = \oplus_{p,k} hA_k^p \\ hA_k^p = (\tau \in A_k^p;\ i(u)\tau = 0 \text{ for all } u \in L) \end{cases}$$

(equivalently, the Ad-equivariant $\tau \in A^*$ is horizontal whenever it vanishes as soon as one of its arguments is vertical i.e. tangent to the fiber). hA^* is a graded commutative subalgebra of A^* preserved by the action of $\mathfrak{G}$ as follows from (2,4h). Warning: A^* is not a subGCDA of A^*, since not stable under the exterior derivative d. However

[2.2]. The set hA_0^* of real valued horizontal G-invariant differential forms is a sub-GCDA of $(A_0^*,d,\wedge)$ stable under the action of $\mathfrak{G}$ and isomorphic as a GCDA to the real-valued De Rham complex $\Lambda^*(M,\mathbb{R}),d,\wedge)$ on the base.

We recall that the isomorphism is given as follows[12]

---

12) One has $d\alpha \leftrightarrow d\alpha'$ and $\alpha_1 \wedge \alpha_2 \leftrightarrow \alpha_1' \wedge_2'$ because the pull back π^* commutes with d and with the wedge product.

$$(2,7)\quad \begin{cases} h \in A_0^p \ni \alpha \leftrightarrow \alpha' \in \Lambda^p(M,\mathbb{R}) \\ \alpha' \longrightarrow \alpha = \pi^*\alpha' \\ \alpha \longrightarrow \alpha' :\ \alpha'(x,X_i) = \alpha(z,Z_i) \\ \qquad (\text{any } z \in P \text{ and } Z_i \in T_z^P \text{ with } \pi(z) = x,\ \pi_{*z}Z = X_i) \end{cases}$$

The fact that hA_0 is a sub-GCDA of A_0 is seen as follows: remembering that (θ, i) is an action of the Lie algebra L on $\Lambda^*(P,\mathbb{R})$, $A_0 = \bigcap_{u \in L} \ker u$ is closed for the wedge product due to the derivation property of $i(u)$, $u \in L$; and closed for d, because, for $\alpha \in A^0$ (cf. Appendix A, (A,14)) $i(u)d\alpha = -di(u)\alpha + \theta(u)\alpha = 0$.

[2.3]. The DGL $(A_1^*, d, [\wedge])$ of Ad-equivariant L-valued differential form.

Since the $r(s)$, $s \in G$, are zero-grade automorphism of the DGL$(\Lambda^*(P,L)), d, [\wedge])$ commuting with all $\rho(\psi)$, $\psi \in \mathcal{G}$, we have that A_1 is a sub-DGL of $(\Lambda^*(P,L), d, [\wedge])$ stable under the action ρ of $\mathcal{G}$ (thus also of $\mathcal{X}$)[13]. The importance of A_1 lies in the fact that it contains as subset both $\mathcal{X}$ and the set $\mathfrak{A}$ of connection one-forms on P. Our identification of $\mathcal{X}$ with the smooth Ad-equivariant functions on P (cf. (1,5)) is now expressed as

$$(2,8)\qquad \mathcal{X} = A_1^0$$

We recall, on the other hand, that the set $\mathfrak{A}$ of connection one-forms on P is the subset of $a \in A_1^1$ singled out by the following specification of the value of a on vertical vectors:

$$(2,9)\qquad a(z,Z) = L_{z*e}^{-1} Z, \quad z \in P,\ Z \in T_z^P \text{ with } \pi_{*z}Z = 0$$

alternatively

$$(2,9a)\qquad a(z,\theta(u)) = u, \quad u \in L.$$

We recall that $\mathfrak{A}$ is an affine subspace of A_1^1 modelled on hA_1^1[14]. The curvature of a is

$$(2,10)\qquad F^a = da + \tfrac{1}{2}[a \wedge a],$$

it is an element of A_1^2. The exterior covariant derivation determined by a is the map D^a: $A_1^* \longrightarrow A_1^*$ given by[15]

$$(2,11)\qquad D^a\lambda = d\lambda + [a,\lambda], \quad \lambda \in A_1^*.$$

We state for further reference the expression of (2,5) for $\tau = a \in \mathfrak{A}$, and $\tau = \Omega' \in \mathcal{X}$: we have

$$(2,12)\qquad \rho(\Omega)a = -[a \wedge \Omega] - d\Omega, \quad \Omega \in \mathcal{X},\ a \in \mathfrak{A}$$

(where we used (2,9)) and

$$(2,13)\qquad \rho(\Omega)\Omega' = [\Omega \wedge \Omega'], \quad \Omega, \Omega' \in \mathcal{X}$$

---

13) Note however that the replacement of (A,14) by (1,37) prevent $hA_k^* = \{\tau \in A_k^*;\ i(u)\tau = 0 \text{ for all } u \in L\}$ — hence in particular hA_1^* — to be stable under d: hA_1^* is merely closed for the product $[\wedge]$ (and hA^* for the product $\times$).

14) i.e. $a, a' \in \mathfrak{A}$, $\lambda, \lambda' \in \mathbb{R}$, $\lambda + \lambda' = 1$ imply $\lambda a + \lambda' a' = \mathfrak{A}$; and $a' - a \in \mathfrak{A}$.

15) unlike d, D^a, $a \in \mathfrak{A}$, leaves hA_1 stable.

We end up this section with a remark on the "descent" by means of the $P \in S^*(L)$ (cf. [1.4]): using (1,47) we see that, if P is Ad-equivariant, i.e. if $P \cdot \mathrm{Ad}s = P$, $s \in G$, P commutes with $r(s)$, $s \in G$: for $P \in I_k(S)$ we have

$$P \circ r(s) = r(s) \circ P \tag{2,14}$$

<u>hence P leaves A^* invariant</u>.

<u>§3.</u> The cohomology algebra $(\Phi^*, \delta, \wedge)$ of $\mathcal{X}$.

Constructing the cohomology algebra of a Lie algebra is a standard procedure which can be applied to any Lie algebra over an arbitrary abelian ring. We now describe this construction in the case where we need it: that of the Lie algebra $\mathcal{X}$, taken a Lie algebra over $\mathcal{C}^\infty(M)$. We noted in [1.1] that $\mathcal{X} = A_1^0$ is a Lie algebra over $\mathcal{C}^\infty(M)$ acting by pointwise multiplication -- abstractly the multiplication of elements of A_1^0 by those of A_0^0, with the following identification of $\mathcal{C}^\infty(M)$ with A_0^0:

$$\mathcal{C}^\infty(M) \ni a = a' \in A_0^0 \text{ iff } \begin{cases} a'(z) = a(\pi z), & z \in P \\ \text{i.e. } a'(x) = a(z), & z \in P,\ \pi z = x \end{cases} \tag{3,1}$$

We now consider the direct sum

$$\Phi^* = \bigoplus_{\alpha \in \mathbb{N}} \Phi^\alpha \tag{3,2}$$

where Φ^α is the set of alternate $\mathcal{C}^\infty(M)$-valued α-forms φ[16] on $\mathcal{X}$ ($\Phi^0 = \mathcal{C}^\infty(M)$) local in the following sense: the forms φ are of the type:

$$\varphi: (\Omega_1, \dots, \Omega_\alpha) \in \mathcal{X} \times \text{..}(\alpha \text{ times})\text{..} \times \mathcal{X} \longrightarrow \theta(D_1\Omega_1, \dots, D_\alpha\Omega_\alpha) \in \mathcal{C}^\infty(M) \tag{3,5}$$

with θ a $\mathcal{C}^\infty(M)$-valued $\mathcal{C}^\infty(M)$-linear α-forms on $\mathcal{X}$ and the D_i, $i=1,\dots,\alpha$, linear differential operators: $\mathcal{X} \longrightarrow \mathcal{X}$. Φ^* becomes a GCDA $(\Phi^*, \delta, \wedge)$ if equipped with the <u>wedge product</u>

---

16) The subset of those φ obtained with operators D_i of degree zero is a sub-GCDA $(\Phi_0^*, \delta, \wedge)$ of $(\Phi^*, \delta, \wedge)$.

$$
(3,6)\quad \begin{cases} (\varphi\wedge\psi)(\Omega_\alpha,\dots,\Omega_{\alpha+\beta}) \\ \quad = \frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{\alpha+\beta}} \chi(\sigma)\varphi(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha})\psi(\Omega_{\sigma(\alpha+1)},\dots,\Omega_{\sigma(\alpha+\beta)}) \\ \Omega_1,\dots,\Omega_{\alpha+\beta}\in\mathcal{L} \end{cases}
$$

and the <u>coboundary</u> <u>operator</u> (of Lie algebra cohomology):

$$
(3,7)\quad \begin{cases} \delta\varphi(\Omega_0,\dots,\Omega_\alpha) = \sum_{0\leq i<j\leq\alpha}(-1)^{i+j}\varphi([\Omega_i,\Omega_j],\Omega_0,\dots,\hat{\Omega}_i,\dots,\hat{\Omega}_j,\dots\Omega_\alpha) \\ \quad \Omega_0,\dots,\Omega_p\in\mathcal{L} \\ \delta = 0 \text{ on } \Phi^0 \end{cases}
$$

Explicitly, δ is a linear operator of grade 1 fulfilling

$$
(3,8)\qquad \delta^2 = 0
$$

$$
(3,9)\qquad \delta(\varphi\wedge\psi) = (\delta\varphi)\wedge\psi + (-1)^\alpha\varphi\wedge\delta\psi,\quad \varphi\in\Phi^\alpha,\ \psi\in\Phi^*
$$

For the proof of these classical facts we refer to, e.g. [1, Corollary 9] (case A = V = $\mathcal{C}^\infty(M)$).

<u>§4.</u> Cohomology of $\mathcal{L}$ relative to the representation spaces Λ_k^*. The double complexes $(\Lambda_k^{**},d,\delta)$, $(\Lambda_k^{**},D,\delta)$. The GCDAs $({}^*\Lambda,\Delta,\times)$, $({}^*\Lambda_0,\Delta,\times)$, $({}^*A,\Delta,\times)$, $({}^*A_0,\Delta,\times)$. The DGLs $({}^*\Lambda_1,\Delta,[\wedge])$, $({}^*A_1,\Delta,[\wedge])$.

Having at hand all prerequisites, we now come to our subject proper, a combination of the structures in sections 2 and 3. The matching is obtained by a canonical construction available whenever one has a representation ρ of a Lie algebra (over some abelian ring) on a module over this ring. In our case the Lie algebra is $\mathcal{L}$, the ring $\mathcal{C}^\infty(M)$, the $\mathcal{C}^\infty(M)$-module Λ^*; and the representation is ρ (direct sum of the restrictions of ρ to the component Λ_k^p). The additional structure of Λ^*, resp. Δ_0^* as GCDAs (and of Λ_1^* as a DGL) produces interesting extra features: products $\times$ and $[\wedge]$, a double complex and associated total complex, GCDA and GDL structures, the BRS transformations, etc. We now describe this "Chevalley cohomology" which is "local" by construction.

From here on we use the shorthands $\Lambda^* = \Lambda^*(P,S(L))$, $\Lambda^p = \Lambda^p(P,S(L))$, $\Lambda_k^p = \Lambda^p(P,S_k(L))$ (in particular $\Lambda_0^p = \Lambda^p(P,\mathbb{R})$, $\Lambda_1^p = \Lambda^p(P,L)$); and the corresponding shorthands for Λ replaced by A.

[4.1]. The double complex (Λ^{**},d,δ).

Consider the tensor product over $\mathcal{C}^\infty(M) = A_0^0$ 17)

---

17) All the subsequent tensor products are over $\mathcal{C}^\infty(M)$. It is tempting at this point to work with tensorproducts over $\mathcal{C}^\infty(M)$. However, one can, alternatively consider the purely algebraic theory where tensor products are over $\mathbb{C}$. The subsequent results are then maintained, with appropriate replacements of $\mathcal{C}^\infty(M)$ by $\mathbb{C}$. On the other hand the present construction can be made replacing Φ^* by Φ_0^* (see footnote 16)) with maintenance of all results.

(4,1) $$\Lambda^{**} = \Lambda^{*}\otimes\Phi^{*}$$

doubly graded as the direct sum of subspaces

(4,2) $$\Lambda^{p\alpha} = \Lambda^{p}\otimes\Phi^{\alpha}, \quad \alpha,p \in \mathbb{N}$$

themselves splitting into subspaces

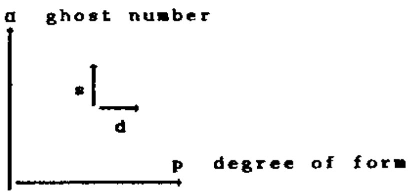
(4,3) $$\Lambda^{p\alpha}_k = \Lambda^{p}_k\otimes\Phi^{\alpha} \text{ (hence } \Lambda^{**} = \bigoplus_k \Lambda^{**}_k = \bigoplus_{p,\alpha} \Lambda^{p}_k\otimes\Phi^{*}\text{))}$$

(α is the "ghost number", p the degree of form, k the tensor type). The elements of Λ^{**} are interpreted as Λ^*-valued multilinear forms on $\mathcal{X}$[18], according to the identification

(4,4) $$U = \tau\otimes\varphi \Longleftrightarrow \begin{cases} U(\Omega_1,\dots,\Omega_\alpha) = \varphi(\Omega_1,\dots,\Omega_\alpha)\tau \\ \text{for all } \Omega_1,\dots,\Omega_\alpha \in \mathcal{X} \end{cases}, \quad \begin{cases} U \in \Lambda^{**} \\ \varphi \in \Phi^{\alpha} \\ \tau \in \Lambda^{*} \end{cases}$$

We now turn Λ^{**} into a double complex with horizontal differential d and vertical differential s:

(4,5)



This arises by setting

18) in other terms Λ^{**} is identified with the Λ^*-valued Grassmann space over the algebraic dual L^* of L.

$$(4,6)\qquad (dU)(\Omega_1,\dots,\Omega_\alpha) = d(U(\Omega_1,\dots,\Omega_\alpha)),\quad \begin{cases} U \in \Lambda^{*\alpha} \\ \Omega_1,\dots,\Omega_p \in \mathcal{X} \end{cases}$$

and on the other hand[19]

$$(4,7)\qquad \begin{cases} (sU)(\Omega_0,\dots,\Omega_\alpha) = -(-1)^p\Big(\sum_{i=0}^{p}(-1)^i\rho(\Omega_i)U(\Omega_0,\dots,\hat{\Omega}_i,\dots,\Omega_\alpha) \\ \qquad + \sum_{0<i<j\leq\alpha}(-1)^{i+j}U([\Omega_i,\Omega_j],\Omega_0,\dots,\hat{\Omega}_i,\dots,\hat{\Omega}_j,\dots,\Omega_\alpha)\Big) \\ \qquad U \in \Lambda^{*\alpha},\ \alpha \geq 1;\ \Omega_0,\dots,\Omega_\alpha \in \mathcal{X} \\ (s\lambda)(\Omega_0) = -\rho(\Omega_0),\ \lambda \in \Lambda^{*\circ} \end{cases}$$

(in fact $sU = -(-1)^p\delta_\rho U$, with δ_ρ <u>the coboundary operator of the cohomology of</u> $\mathcal{X}$ <u>relative to the representation space</u> Λ^*). With these definitions Λ^{**} (in fact each Λ_k^{**}, $k \in \mathbb{N}$, acquires the structure of a double complex i.e., we have

$$(4,8)\qquad \begin{cases} d^2 = 0 \\ s^2 = 0 \\ sd + ds = 0 \end{cases}$$

The corresponding <u>total complex</u> $({}^*\Lambda,\Delta)$ is defined as follows: the (single) <u>total grading</u> is

$$(4,9)\qquad \begin{cases} {}^*\Lambda\ (= \Lambda^{**}) = \oplus_n {}^n\Lambda \\ \text{with } {}^n\Lambda = \bigoplus_{\alpha+p=n} \Lambda^{p\alpha} \end{cases}$$

and the <u>total derivative</u> is

$$(4,10)\qquad \Delta = d + s$$

---

19) The minus sign r.h.s. of (4,1) is to ensure the traditional minus signs r.h.s. of the B.R.S. relations cf. Section 5 below.

fulfilling

$$(4,8a)\qquad \Delta^2 = 0$$

(note that d,s and Δ are of grade 1 for the total grading). Proof of these facts: (4,8): $d^2 = d^2\otimes id_{\Phi^*} = 0$. Last line (4,8): the coboundary operator δ_ρ commutes with d since acting "internally" (on the arguments of U) whilst d acts "externally" (on the value of U) — the factor $-(-1)^p$ then turns commutation into anticommutation. Second line (4,8): s is known to be a coboundary operator: for a proof we refer to e.g. [1, Corollary 9].

[4.2]. The GCDAs $(\Lambda^*,\Delta,\times)$ and $(\Lambda_0^*,\Delta,\times)$.

We now introduce a bilinear product $\times$ on Λ^{**}. We set

$$(4,11)\qquad (\tau\otimes\varphi)\times(\tau'\otimes\varphi') = (-1)^{\alpha p'}\cdot(\tau\times\tau')\otimes(\varphi\wedge\varphi'),\quad \begin{cases} \tau \in \Lambda^*,\ \tau' \in \Lambda^{p'} \\ \varphi \in \Lambda^\alpha,\ \varphi' \in \Phi^* \end{cases}$$

alternatively

$$(4,11a)\qquad \begin{cases} (U\times V)(\Omega_1,\dots,\Omega_{\alpha+\beta}) \\ \quad = (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{\alpha+\beta}}\chi(\sigma)U(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha})\times V(\Omega_{\sigma(\alpha+1)},\dots,\Omega_{\sigma(\alpha+\beta)}), \\ U \in \Lambda^{**},\ V \in \Lambda^{q\beta},\ \Omega_1,\dots,\Omega_{\alpha+\beta} \in \mathcal{X} \end{cases}$$

We now have that <u>$({}^*\Lambda,\Delta,\times)$ is a GCDA with $({}^*\Lambda_0,\Delta,\times)$ as a sub-GCDA</u>.[20] This is proven as follows: the fact that $({}^*\Lambda,\times)$ is a graded-commutative algebra is proven in Appendix A, cf (A,30), (A,31). We already proved that Δ is of grade 1 and square zero. Now <u>d and s, and hence Δ, are derivatives for the product</u> $\times$.

---

20) ${}^*\Lambda_0$ is Λ_0^{**} equipped with the total grading (4,9).

We check that for d: given $U \in \Lambda^{p\alpha}$, $V \in \Lambda^{q\beta}$, $\Omega_\alpha, \dots, \Omega_{\alpha+\beta} \in \mathcal{X}$ we have, from (4,6), since d is a graded derivasion for $\times$:

$$(4,12) \qquad d(U\times V)(\Omega_1,\dots,\Omega_{\alpha+\beta})$$

$$= (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{\alpha+\beta}} \chi(\sigma)\cdot$$

$$(dU(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha}) \times (V(\Omega_{\sigma(\alpha+1)},\dots,\Omega_{\sigma(\alpha+\beta)}) +$$

$$(-1)^p U(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha}) \times dV(\Omega_{\sigma(\alpha+1)},\dots,\Omega_{\sigma(\alpha+\beta)}))$$

$$= (-1)^{\alpha q}((-1)^{\alpha q}(dU\times V)(\Omega_1,\dots,\Omega_{\alpha+\beta}) + (-1)^{p+\alpha(q+1)}(U\times dV)(\Omega_1,\dots,\Omega_{\alpha+\beta}))$$

The proof for s is obtained as follows invoking Theorem [1,8](vi) of [1]: denoting by ∇ the product on Λ^{**} obtained by discarding the factor $(-1)^{\alpha q}$ r.h.s. of definition (4,12a), the fact that $\rho(\Omega)$, $\Omega \in \mathcal{X}$, is a derivation of $(\Lambda^*, \times)$ implies that δ_ρ is a derivation for the product ∇. We then have, for $U \in \Lambda^{p\alpha}$, $V \in \Lambda^{q\beta}$

$$(4,13) \quad s(U\times V) = -(-1)^{p+q+\alpha q}\delta_\rho(U\nabla V)$$

$$= -(-1)^{p+q+\alpha q}(\delta_\rho U)\nabla V + (-1)^\alpha U\nabla\delta_\rho V)$$

$$= -(-1)^{(\alpha+1)q}(sU)\nabla V + (-1)^{p+\alpha+\alpha q}U\nabla sV$$

$$= sU\times V + (-1)^{p+\alpha}U\times sV$$

Note that, since d and s preserve the tensor type, each $\{\Lambda_k^{**}, d, \delta\}$ is a double complex with total complex $(^*\Lambda_k, \Delta)$. Moreover $(^*\Lambda_0, \Delta, \times)$ is a sub-GCDA of $(^*\Lambda, \Delta, \times)$. The fact that $^*\Lambda_0$ is closed for the product $\times$ stems from the fact that Λ_0^* is closed for $\times$. And $^*\Lambda_0$ is stable under Δ (in fact under d and s) because the latter preserve the tensor type.

[4.3]. The DGL $(^*\Lambda_1, \Delta, [\wedge])$.

The case $k = 1$ deserves special attention. We define a bilinear product on Λ_1^{**} by setting

$$(4,14) \qquad [\lambda\otimes\varphi\wedge\mu\otimes\psi] = (-1)^{\alpha q}(\lambda\wedge\mu)\otimes(\varphi\wedge\psi) \left\{ \begin{array}{l} \lambda \in \Lambda^*, \ \mu \in \Lambda^q \\ \varphi \in \Phi^\alpha, \ \psi \in \Phi^* \end{array} \right.$$

equivalently

$$(4,14a) \quad \left\{ \begin{array}{l} [A\wedge B](\Omega_1,\dots,\Omega_{\alpha+\beta}) = (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{\alpha+\beta}} \chi(\sigma) \\ \qquad [A(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha})\wedge B(\Omega_{\sigma(\alpha+1)},\dots,\Omega_{\sigma(\alpha+\beta)}))] \\ A \in \Lambda_1^{*\alpha}, \ B \in \Lambda^{q\beta}, \ \Omega_1,\dots,\Omega_{\alpha+\beta} \in \mathcal{X} \end{array} \right.$$

Then $(^*\Lambda_1, \Delta, [\wedge])$ becomes a DGL[21].

Proof. The fact that $(^*\Lambda_1, [\wedge])$ is a graded Lie algebra is shown in Appendix A(iv) cf. (A,34). Δ preserves Λ_1^* and is of square zero and total grade 1. On the other hand d and s, and thus Δ, acts as derivations of $^*\Lambda_1$ for the product $[\wedge]$. For d this follows from [A,2](ii) in Appendix A. The proof for s is again obtained by invoking Theorem [1,8](vi) of [1]: denoting by $\square$ the product on Λ obtained by discarding the factor $(-1)^{\alpha q}$ r.h.s. of definition (4,14a), the fact that $\rho(\Omega)$, $\Omega \in \mathcal{X}$ is a derivation of $(\Lambda_1^*, [\wedge])$ implies that δ_ρ is a derivation of Λ_1^*. We then have, for $A \in \Lambda_1^{p\alpha}$, $B \in \Lambda^{q\beta}$,

$$(4,15) \quad s[A\wedge B] = -(-1)^{p+q+\alpha q}\delta_\rho(A\square B)$$

$$= -(-1)^{p+q+\alpha q}(\delta_\rho A\square B + (-1)^\alpha A\square\delta_\rho V)$$

$$= -(-1)^{(\alpha+1)q}(sA)\square B + (-1)^{p+\alpha+\alpha q}A\square sB$$

$$= [sA\wedge B] + (-1)^{p+\alpha}[A\wedge sB]$$

---

21) $^*\Lambda_1$ is Λ_1^{**} equipped with the total grading (4,9).

[4.4]. Action r of G on $\Lambda^{**} = {}^*\Lambda$. The operators $\theta(u)$ and $i(u)$, $u \in L$, on Λ^{**}. Commutation and derivation properties.

On $\Lambda^{**} = \Lambda^*\otimes\Phi^*(= {}^*\Lambda)$ we define the action r of the group G, and the operators $\theta(u)$, $r(u)$, $u \in L$, by tensoring with id_{Φ^*}:

(4,16) $$r(g) = r(g)\otimes id_{\Phi^*}, \qquad g \in G$$

(4,17) $$\theta(u) = \theta(u)\otimes id_{\Phi^*}, \qquad u \in L$$

(4,18) $$i(u) = i(u)\otimes id_{\Phi^*}, \qquad u \in L$$

alternatively, for $U \in \Lambda^{**}$, $\Omega_1,\dots,\Omega_\alpha \in \mathfrak{X}$

(4,16a) $$(r(g)U)(\Omega_1,\dots,\Omega_\alpha) = r(g)(U(\Omega_1,\dots,\Omega_\alpha))$$

(4,17a) $$(\theta(u)U)(\Omega_1,\dots,\Omega_\alpha) = \theta(u)(U(\Omega_1,\dots,\Omega_\alpha))$$

(4,18a) $$(i(u)U)(\Omega_1,\dots,\Omega_\alpha) = i(u)(U(\Omega_1,\dots,\Omega_\alpha))$$

(note that $r(g)$, $g \in G$, preserves the ghost number, the degree of the form, and the tensor type).

<u>We then have the following commutation rules</u>

(4,19) $$i(u)s + s\, i(u) = 0, \qquad u \in L$$

(4,20) $$\theta(u)s - s\,\theta(u) = 0, \qquad u \in L$$

(4,21) $$r(g)s - s\, r(g) = 0, \qquad g \in G$$

(4,22) $$r(g)d - d\, r(g) = 0, \qquad g \in G$$

(4,23) $$r(g)\Delta - \Delta r(g) = 0, \qquad g \in G$$

(4,24) $$i(u)i(v) + i(v)i(u) = 0, \qquad u,v \in L$$

(4,25) $$\theta(u)\theta(v) - \theta(v)\theta(u) = \theta([u,v]), \qquad u,v \in L$$

(4,26) $$\theta(u)i(v) - i(v)\theta(u) = i([u,v]), \qquad u,v \in L$$

(4,27) $$\theta(u)d - d\,\theta(u) = 0, \qquad u \in L$$

(4,28) $$\theta(u)i(u) - i(u)\theta(u) = 0, \qquad u \in L$$

(4,29) $$i(u)d + d\, i(u) = id_{S(L)}\otimes\theta(u)\otimes id_{\Phi^*}$$
$$i(u)\Delta + \Delta i(u) = \theta(u) - Adu\otimes id_{\Lambda_0}\otimes id_{\Phi^*}$$

<u>Moreover, we have</u>

(4,30) $$\frac{d}{dt}\Big|_{t=0} r(e^{tu}) = \theta(\xi_u), \qquad u \in L,$$

<u>and the following invariance, resp. derivation properties in $({}^*\Lambda,\times)$</u>:

(4,31) $$r(g)(U\times V) = (r(g)U)\times(r(g)V), \qquad U,V \in {}^*\Lambda,\ g \in s$$

(4,32) $$\theta(u)(U\times V) = (\theta(u)U)\times V + U\times\theta(u)V, \qquad U,V, \in {}^*\Lambda,\ u \in L,$$

(4,33) $$i(u)(U\times V) = (i(u)U)\times V + (-1)^n U\times i(u)V, \qquad U \in {}^n\Lambda,\ V \in {}^*\Lambda,\ u \in L$$

<u>and in $({}^*\Lambda_1,[\wedge])$</u>:

(4,34) $$r(g)[A\wedge B] = [r(g)A\wedge r(g)B], \qquad A,B \in {}^*\Lambda,\ g \in G$$

(4,35) $$\theta(u)[A\wedge B] = [\theta(u)A\wedge B] + [A\wedge\theta(u)B], \qquad A,B \in {}^*\Lambda,\ u \in L$$

(4,36) $$i(u)[A\wedge B] = [i(u)A\wedge B] + (-1)^n[A\wedge i(u)B], \qquad A \in {}^n\Lambda,\ B \in {}^*\Lambda,\ n \in L$$

Properties (4,19), (4,20) and (4,21) are due to the fact that s acts "internally" whilst i(u), $\theta(u)$ and r(g) act "externally"[22]. Property (4,22) through (4,29) follow from the corresponding properties in $^*\Lambda$, resp. $^*\Lambda_1$ via tensoring with id_{Φ^*} (immediate from [1.3](ii), (iii)). We check the derivation properties (in particular the fact that properties (4,33), (4,35) hold w.r.t. the total grading). With $U = \tau\otimes\varphi \in \Lambda^p\otimes\Phi^\alpha$ $U' = \tau'\otimes\varphi' \in \Lambda^{p'}\otimes\Phi^{\alpha'}$, we have $U \times U' = (-1)^{\alpha p'}(\tau\times\tau')\otimes(\varphi\wedge\varphi')$, hence

$$(4,37)\qquad \begin{aligned}\theta(u)(U\times U') &= (-1)^{\alpha p'}((\theta(u)\tau)\times\tau' + \tau\times\theta(u)\tau')\otimes(\varphi\wedge\varphi')\\ &= ((\theta(u)\tau)\otimes\varphi) \times (\tau'\otimes\varphi') + (\tau\otimes\varphi) \times ((\theta(u)\tau')\otimes\varphi')\\ &= (\theta(u)U\times U' + U\times\theta(u)U'\end{aligned}$$

and

$$(4,38)\qquad \begin{aligned}i(u)\ (U\times U') &= (-1)^{\alpha p'}(i(u)\tau\times\tau' + (-1)^p\tau\times i(u)\tau')\otimes(\varphi\wedge\varphi')\\ &= (-1)^{\alpha p'+\alpha p'}(i(u)\tau\otimes\varphi') \times (\tau'\otimes\varphi)\\ &\qquad + (-1)^{\alpha p'+p+\alpha(p'+1)}(\tau\otimes\varphi)\times(i(u)p\otimes\varphi')\\ &= i(u)U\times U' + (-1)^{p+\alpha}U\otimes i(u)U'\end{aligned}$$

and with $A = \lambda\otimes\varphi \in \Lambda_1^p\otimes\Phi^\alpha$, $A' = \lambda'\otimes\varphi' \in \Lambda_1^{p'}\otimes\Phi^{\alpha'}$, thus $A\times A' = (-1)^{\alpha p'}(\lambda\times\lambda')\otimes(\varphi\wedge\varphi')$.

---

22) This implies commutativity of i(u), $\theta(u)$ and r(g) with δ_ρ, turned into anticommutativity for the odd grade i(u) by passing from δ_ρ to s.

$$(4,39)\qquad \begin{aligned}\theta(u)[A\wedge A'] &= (-1)^{\alpha p'}([\theta(u)\lambda\wedge\lambda'] + [\lambda\wedge(\theta(u)\lambda']) \otimes (\varphi\wedge\varphi')\\ &= [\theta(u)\lambda)\otimes\varphi\wedge\lambda'\otimes\varphi'] + [\lambda\otimes\varphi\wedge\theta(u)\lambda'\otimes\varphi']\\ &= [\theta(u)A\wedge A'] + [A\wedge\theta(u)A']\end{aligned}$$

and

$$(4,40)\qquad \begin{aligned}i(u)[A\wedge A'] &= (-1)^{\alpha p'}([i(u)\lambda\wedge\lambda'] + (-1)^p[\lambda\wedge i(u)\lambda']) \otimes (\varphi\wedge\varphi')\\ &= (-1)^{\alpha p'+\alpha p'}[i(u)\lambda\otimes\varphi\wedge\lambda'\otimes\varphi'] + (-1)^{\alpha p'+p+\alpha(p'+1)}[\lambda\otimes\varphi\wedge i(u)\lambda'\otimes\varphi']\\ &= [i(u)A\wedge A'] + (-1)^{p+\alpha}[A\wedge i(u)A']\end{aligned}$$

[4.5]. The Ad-equivariant double complex (A_k^{**},d,δ), GCDAs $(^*A,\Delta,\times)$ and $(^*A_0,\Delta,\times)$ and DGL $(^*A_1,\Delta,[\wedge])$.

With $p,\alpha,k \in \mathbb{N}$ we now consider the fixpoints sets

$$(4,41)\qquad A_k^{p\alpha} = \{U \in \mathbf{A}_k^{p\alpha},\ r(g)U = U \text{ for all } g \in G\} = A_k^p\otimes\Phi^\alpha$$

equivalently, for G connected,

$$(4,41a)\qquad A_k^{p\alpha} = \{U \in \mathbf{A}_k^{p\alpha},\ \theta(u)U = 0 \text{ for all } u \in L\}$$

with the bigraded spaces

$$(4,42)\qquad A_k^{**} = \bigoplus_{p,\alpha} A_k^{p\alpha},\quad A^{**} = \bigoplus_{p,\alpha} A^{p\alpha}$$

($A^{p\alpha} = \bigoplus_k A_k^{p\alpha}$) and the graded spaces

$$(4,43)\qquad {}^*A_k = \bigoplus_n {}^nA_k,\ {}^*A = \bigoplus_n {}^nA$$

where n is the "total grading"

(4,44) $$ {}^{n}A_k = \bigoplus_{p+\alpha=n} A_k^{p\alpha}, \quad {}^{n}A = \bigoplus_{p+\alpha=n} A^{p\alpha} $$

We now have that (A_k^{**}, d, δ), $k \in \mathbb{N}$ resp. (A^{**}, d, δ) are sub-double complexes of the double complexes $(\Lambda_k^{**}, d, \delta)$ resp. $(\Lambda^{**}, d, \delta)$; that $({}^{*}A, \Delta, \times)$, resp. $({}^{*}A_0, \Delta, \times)$ are sub-GCDAs of the GCDAs $({}^{*}\Lambda, \Delta, \times)$, resp. $({}^{*}\Lambda_0, \Delta, \times)$; and that $({}^{*}A_1, \Delta, [\wedge])$ is a sub-DGL of the DGL$({}^{*}\Lambda, \Delta, [\wedge])$. These facts immediately result from the commutation of $r(g)$ and d, δ cf. (4,21), (4,22)); the commutation of $r(g)$ and $\Delta, \times$ (cf. (4,23), (4,31)); and the additional commutation of $r(g)$ with $[\wedge]$ (cf. (4,34)), $g \in G$. Moreover, with hA_0^{**} the set of invariant horizontal real differential forms on P:

(4,45) $$ \begin{cases} hA_0^{**} = \bigoplus_{p,\alpha} hA_0^{p\alpha} \\ hA_0^{p\alpha} = \{F \in A_0^{\alpha p};\ i(u)F = 0 \text{ for all } u \in L\} \end{cases} $$

and

(4,46) $$ {}^{n}A_0 = \bigoplus_{p+\alpha=n} A_0^{p,\alpha} $$

we have that (hA_0^{**}, d, δ) is a sub-bicomplex of (A_0^{**}, d, δ) and $(h^{*}A_0, \Delta, \times)$ is a sub-GCDA of $(A_0, \Delta, \times)$ (this follows from the facts that hA_0^{**} is stable under both s and d, owing to (4,19) and (4,29) (remember that $\theta(u)$ vanishes on A^{**}).

Remark. On Λ^{**} we have also an action (Θ, I) of the Lie algebra $\mathscr{X}$, given as follows: for $U \in \Lambda^{*}$, Ω, $\Omega_1, \ldots, \Omega_j \in \mathscr{X}$:

(4,47) $$ (\Theta(\Omega)U)(\Omega_1, \ldots, \Omega_p) = \rho(\Omega)(U(\Omega_1, \ldots, \Omega_p)) + \sum_{i=1}^{\alpha} U(\Omega_1, \ldots, \Omega_{i-1}, [\Omega, \Omega_i], \Omega_{i-1}, \ldots, \Omega_\alpha) $$

(4,48) $$ (I(\Omega)U)(\Omega_1, \ldots, \Omega_{p-1}) = U(\Omega_1, \Omega_1, \ldots, \Omega_{p-1}) $$

For a proof of these facts see e.g. [1] Theorem [1.8](i) (iii).

[4.6]. Alternative definition of $({}^{*}\Lambda, \Delta, \times)$.

We now describe a different way of looking at ${}^{*}\Lambda$, which will be technically useful and has also some conceptual interest.

Since Λ^{*} was defined as the tensor product $S(L) \otimes \Lambda_0^{*}$, the space ${}^{*}\Lambda$ is the double tensor product[23)]

(4,49) $$ \begin{cases} {}^{*}\Lambda = S(L) \underset{\mathbb{R}}{\otimes} \Lambda_0^{*} \underset{\mathcal{C}^\infty(M)}{\otimes} \Phi^{*} \\ \Lambda_k^{p\alpha} = S_k(L) \underset{\mathbb{R}}{\otimes} \Lambda_0^{p} \underset{\mathcal{C}^\infty(M)}{\otimes} \Phi^{\alpha} \end{cases} $$

on which the product $\times$ is defined as follows: we have

(4,50) $$ (f \otimes \eta \otimes \varphi) \times (g \otimes \xi \otimes \psi) = (-1)^{\alpha q} (f \vee g) \otimes (\eta \wedge \xi) \otimes (\varphi \wedge \psi), \quad \begin{cases} f, g \in S(L) \\ \eta \in \Lambda_0^{*},\ \xi \in \Lambda_0^{q} \\ \varphi \in \Phi^{\alpha},\ \psi \in \Phi^{*} \end{cases} $$

---

23) Written without parentheses according to associativity of tensor product.

This can be written alternatively

$$(f\otimes(\eta\otimes\varphi)) \times (g\otimes(\rho\otimes\psi)) = (f\vee g)\otimes((\eta\otimes\varphi)\wedge(\xi\otimes\psi)) \tag{4,50a}$$

where the wedge product of the r.h.s. is the skew product of the GCDAs ${}^*\Lambda_0$ and Φ^*, naturally included in ${}^*\Lambda$ as ${}^*\Lambda_0$ (cf. (A,17) in Appendix A). We could thus have constructed $({}^*\Lambda,\Delta,\times)$ by assembling the algebras $(S(L),\vee)$, $(\Lambda_0^*,d,\wedge)$ and $(\Phi^*,\delta,\wedge)$ as follows: first build the skew tensor product $({}^*\Lambda_0,\wedge)$ of graded commutative algebras

$${}^*\Lambda_0 = \Lambda_0^*\otimes\Phi^*, \tag{4,51}$$

with corresponding grading

$${}^n\Lambda_0 = \bigoplus_{p+\alpha=n} \Lambda_0^p \underset{C^\infty(M)}{\otimes} \Phi^\alpha \tag{4,52}$$

and skew tensor product[24)]

$$(\eta\otimes\varphi)\wedge(\xi\otimes\psi) = (-1)^{\alpha q}(\eta\wedge\xi)\otimes(\varphi\wedge\psi) \quad \begin{cases} \eta \in \Lambda_0^*, & \xi \in \Lambda_0^q \\ \varphi \in \Phi^\alpha, & \psi \in \Phi^*. \end{cases} \tag{4,53}$$

then build the tensor product

$${}^*\Lambda = S(L) \underset{\mathbb{R}}{\otimes} {}^*\Lambda_0 \tag{4,54}$$

---

24) Δ in Λ_0^* is not the skew tensor product of derivatives d and δ. The latter is $d+s_0 = \Delta-\rho\wedge$, see below.

with product

$$(f\otimes S)\times(g\otimes T) = (f\vee g)\otimes(S\wedge T), \quad \begin{cases} f,g \in S(L) \\ S,T, \in {}^*\Lambda_0 \end{cases} \tag{4,55}$$

We obtain in this way the graded commutative algebra $({}^*\Lambda,\wedge)$ embedded as such in the graded commutative algebra$({}^*\Lambda,\times)$. We recapitulate our operators: they are all totally split tensorially

$$r(s) = \mathrm{Ad}s\otimes R_s^*\otimes \mathrm{id}_{\Phi^*}, \quad s \in G \tag{4,56}$$

$$\rho(\Psi) = \mathrm{id}_{S(L)}\otimes\rho(\Psi)\otimes\mathrm{id}_{\Phi^*}, \quad \Psi \in \mathfrak{D} \tag{4,57}$$

$$\rho(\Omega) = \mathrm{id}_{S(L)}\otimes\rho(\Omega)\otimes\mathrm{id}_{\Phi^*}, \quad \Omega \in \mathfrak{X} \tag{4,58}$$

$$d = \mathrm{id}_{S(L)}\otimes d\otimes\mathrm{id}_{\Phi^*} \tag{4,59}$$

$$s_0 = \mathrm{id}_{S(L)}\otimes(-1)^p\otimes\delta \tag{4,60}$$

$$i(u) = \mathrm{id}_{S(L)}\otimes i(u)\otimes\mathrm{id}_{\Phi^*}, \quad u \in L \tag{4,61}$$

$$\theta(u) = \mathrm{Ad}u\otimes\mathrm{id}_{\Lambda_0^*}\otimes\mathrm{id}_{\Phi^*} + \mathrm{id}_{S(L)}\otimes\theta(u)\otimes\mathrm{id}_{\Phi^*}, \quad u \in L \tag{4,62}$$

(4,63)[25)]

$$P = P\otimes\mathrm{id}_{\Lambda_0^*}\otimes\mathrm{id}_{\Phi^*}$$

---

25) See below [4.7] for a study of P on ${}^*\Lambda$.

except for the operator $\rho\wedge$ in

(4,64) $$\Delta = d + s_0 + \rho\wedge$$

given by

(4,65) $$\rho\wedge = \mathrm{id}_{S(L)} \otimes \rho\wedge$$

with

(4,66) $$(\rho\wedge(\alpha\otimes\varphi))(\Omega_0,\dots,\Omega_\alpha) = \sum_{i=1}^{\alpha} (-1)^i \varphi(\Omega_0,\dots,\hat{\Omega}_i,\dots,\Omega_\alpha)\rho(\Omega)\alpha, \quad \begin{cases} \alpha \in \Lambda^* \\ \varphi \in \Phi^\alpha \\ \Omega_0,\dots,\Omega_\alpha \in \mathcal{X} \end{cases}$$

Note that the product $[\wedge]$ on ${}^*\Lambda_1 = L\otimes\Lambda_0^*$ is given by

(4,67) $$[(u\otimes S)\wedge(v\otimes T)] = [u,v]\otimes(S\wedge T), \quad \begin{cases} u,v \in L \\ S,T \in {}^*\Lambda_0 \end{cases}$$

[4.7]. Covariant derivatives. A-connections.

Appendix B applied to the DGL $({}^*\Lambda_1,\Delta,[\wedge])$ yields a linear assignment, to each $B \in {}^1\Lambda_1$, of his "covariant derivative"

(4,68) $$\mathfrak{D}^B = \Delta + [B\wedge\cdot]$$

with square

(4,69) $$(\mathfrak{D}^B)^2 = [\mathfrak{F}^B\wedge\cdot]$$

where the <u>"curvature" of B</u>, $\mathfrak{F}^B \in {}^2\Lambda_1$, given by

(4,70) $$\mathfrak{F}^B = \Delta B + \tfrac{1}{2}[B\wedge B]$$

fulfills a <u>"Bianchi identity"</u>

(4,71) $$\mathfrak{D}^B\mathfrak{F}^B = 0$$

Now <u>we have a unique graded derivation (again denoted $\mathfrak{D}^B$) of $({}^*\Lambda,\times)$ restricting to $\mathfrak{D}^B$ on ${}^*\Lambda_1$, and to Δ on ${}^*\Lambda_0$</u> (in fact the latter is the sum of Δ and of the unique graded derivation restricting to $[B\wedge\cdot]$ on ${}^*\Lambda_1$ and to zero on ${}^*\Lambda_0$)[26]. $\mathfrak{D}^B$ <u>is given by</u>[27]

(4,72) $$\begin{cases} \mathfrak{D}^B(A_1\times\dots\times A_k) \\ \quad = \sum_{i=1}^{k} (-1)^{\sum_{k=1}^{i-1} n_k} A_1\times\dots\times A_{i-1}\times\mathfrak{D}^B A_i\times A_{i+1}\times\cdot\times A_k \\ A_i \in {}^{n_i}\Lambda_1 \end{cases}$$

$\mathfrak{D}^B$ thus defined on ${}^*\Lambda$ fulfills[28]

(4,73) $$(\mathfrak{D}^B)^2 = [\mathfrak{F}^B\wedge\cdot]$$

(4,74) $$\theta(u)\cdot\mathfrak{D}^B - \mathfrak{D}^B\cdot\theta(u) = [\theta(u)B\wedge\cdot], \quad u \in L$$

(4,75) $$i(u)\circ\mathfrak{D}^B + \mathfrak{D}^B\cdot i(u) = \theta(u) + [i(u)B\wedge\cdot] - \mathrm{id}_{\Lambda_0^*}\otimes\mathrm{Ad}u\otimes\mathrm{id}_{\Phi^*}, \quad u \in L$$

---

26) Generally, for $A \in {}^p\Lambda_1$, p even (resp. p odd), there is a unique derivation (resp. graded derivation) of $({}^*\Lambda,\times)$ restricting to $[A\wedge\cdot]$ on ${}^*\Lambda_1$ and to zero on ${}^*\Lambda_0$.

27) one has also the same formula with $[B\wedge\cdot]$ instead of $\mathfrak{D}^B$.

28) with the understanding that $[A\wedge\cdot]$ is the extension described in footnote 7 above.

Proof. (4,73) both sides are derivations, coinciding on ${}^*\mathcal{A}_0$ and ${}^*\mathcal{A}_1$.

(4,74) (resp. (4,75)): both sides are graded derivations (resp. derivations) vanishing on ${}^*\mathcal{A}_0$ and coinciding on ${}^*\mathcal{A}_1$; as results by combining (4,27) with (4,32) rewritten as (4,76) (resp. (4,29) with (4,38) rewritten as (4,77))

(4,76) $$\theta(u)\circ[B\wedge\cdot] - [B\wedge\cdot]\circ\theta(u) = [\theta(u)B\wedge\cdot], \quad B \in {}^1\mathcal{A}_1$$

(4,77) $$i(u)\circ[B\wedge\cdot] + [B\wedge\cdot]\circ i(u) = [i(u)B\wedge\cdot], \quad u \in L$$

Let us call $\mathcal{A}$-connections the elements of ${}^1\mathcal{A}_1$ fulfilling

(4,78) $$i(u)A = u \in {}^0\mathcal{A}_0$$

Let A be a $\mathcal{A}$-connection: we have the commutation rules

(4,79) $$\theta(u)\mathfrak{D}^A - \mathfrak{D}^A\theta(u) = 0, \quad u \in L$$

(4,80) $$i(u)\mathfrak{D}^A + \mathfrak{D}^A i(u) = \theta(u), \quad u \in L$$

showing that $\mathfrak{D}^A$ preserves ${}^*\mathcal{A}$ and $h_*{}^*\mathcal{A}$. Moreover, in that case $\mathfrak{F}^A$ belongs to $h^2\mathcal{A}_1$. (in fact $\mathfrak{D}^A$ preserves ${}^*\mathcal{A}$ for all $A \in {}^*\mathcal{A}_1$).

Proof. if $A \in {}^*\mathcal{A}_1$, $\theta(u)A = 0$, ensuring (4,79), which in turn implies $\mathfrak{D}^A{}^*\mathcal{A} \subset {}^*\mathcal{A}$. If in addition $i(u)A = u$ (requiring that $A \in {}^1\mathcal{A}_1$), we have $[(i(u)A\wedge\cdot] = [u\wedge\cdot] = id_{\Lambda_0^*}\otimes Adu\otimes id_{\Phi^*}$, hence (4.75) then reduces to (4,80), which implies $\mathfrak{D}^A h^*\mathcal{A} \subset h^*\mathcal{A}$.

We end up this section with a discussion of the "descent from ${}^*\mathcal{A}_k$ to ${}^*\mathcal{A}_0$ by means of polynomials $P \in S_k^*(L)$.

[4.8].

Let P be an Ad-equivariant polynomial of degree k on L: $P \in I_k(L)$. Setting

(4.81) $$P(\tau\otimes\varphi) = P(\tau)\otimes\varphi, \quad \rho \in \Lambda^*,\ \varphi \in \Phi^*,$$

in other terms

(4,81a) $$(PU)(\Omega_1,\dots,\Omega_\alpha) = P(U(\Omega_1,\dots,\Omega_\alpha)), \quad U \in \Lambda^{*\alpha},\ \Omega_1,\dots,\Omega_\alpha \in \mathfrak{X}$$

defines a linear map: $P: {}^*\mathcal{A}_k \to {}^*\mathcal{A}_0$ which preserves the ghost number and the degree of form, commutes with the action of G:

(4,82) $$P\circ r(s) = r(s)\circ P, \quad s \in G$$

(hence maps ${}^*\mathcal{A}_k$ into $\mathcal{A}_0$) and fulfills

(4,83) $$P\circ s = s\circ P$$

(4,84) $$P\circ d = d\circ P$$

(4,85) $$P\circ\Delta = \Delta\circ P, \quad u \in L$$

(4,86) $$P\circ\theta(u) = \theta(u)\circ P, \quad u \in L$$

(4,87) $$P\circ i(u) = i(u)\circ P, \quad u \in L$$

and

(4,88) $$P\circ\mathfrak{D}^A = \Delta\circ P, \quad A \in {}^1\mathcal{A}_1$$

(4,89) $$P\circ[A\wedge\cdot] = 0, \quad A \in {}^1\mathcal{A}_1$$

Proof. (4,83) through (4,87) result at sight from (4.59) through (4,66) (remember that $P\cdot Adu = 0$ since $P \in I_k(L)$). (4,88) follows from (4,85) and (4,89) which immediately follows from the

<u>Lemma.</u> Let $u \in L$, $f \in S(L)$, $S,T \in \Lambda_0$ we have

(4,90) $$[(u\otimes S)\wedge(f\otimes T)] = \mathrm{Adu}(f)\otimes(S\wedge T)$$

indeed $P\circ \mathrm{Adu} = 0$ then implies $P\circ[(u\otimes\bullet)\wedge\bullet] = 0$.

Proof of the Lemma. Denote the r.h.s. of (4.91) by $\delta_{uS}(f\otimes T)$, we want to have $\delta_{uS} = [(u\otimes S)\wedge\bullet]$. This holds on Λ_1 by definition cf. (4,68), and trivially on Λ_0. To prove that it holds everywhere, it thus suffice to check that δ_{uS} is a derivation of the same type as $[(u\otimes S)\wedge\bullet]$[29]. We have ideed, for $S \in {}^n\Lambda_0$, $T \in {}^n\Lambda_0$

(4,91)
$$\begin{aligned}(\delta_{uS}(f\otimes T))\times(f'\otimes T') &+ (-1)^{mn}(f\otimes T)\times\delta_{uS}(f'\otimes T')\\
&= (\mathrm{Adu}(f)\otimes(S\wedge T))\times(f'\otimes T') + (-1)^{mn}(f\otimes T)(\mathrm{Adu}(f')\otimes(S\wedge T'))\\
&= (\mathrm{Adu}(f)\vee f')\otimes(S\wedge T\wedge T') + (-1)^{mn}(f\vee \mathrm{Adu}(f'))\otimes(S\wedge T\wedge T')\\
&= (\mathrm{Adu}(f)]f'+f\mathrm{Adu}(f'))(S\wedge T\wedge T')\\
&= \mathrm{Adu}(f\vee f')\otimes S\wedge T\wedge T' = \delta_{uS}((f\otimes T)\times(f'\otimes T'))\end{aligned}$$

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29) cf. (B,8) in Appendix 8.

§5. The BRS relations.

We now exhibit the BRS relations as realized geometrically within the double complex (Λ_1^{**},d,s) equipped with the product $[\wedge]$.

As we noticed in section 1, the set $\mathfrak{A}$ of connection one-forms is contained in Λ_1^1, embedded in Λ_1^{**} as $\Lambda_1^{10} = \Lambda_1^1\otimes\mathbb{R}$.

On the other hand, let ω be the "tautological form" on $\mathcal{X}$ (equal to the identity map).

(5,1) $$\omega(\Omega) = \Omega, \quad \Omega \in \mathcal{X}$$

Remembering that we identified $\mathcal{X}$ with Λ_1^0, we see that ω as given by (5,1) appears as an element of $\Lambda_1^{01} = \Lambda_1^0\otimes L$. Hence Λ_1^{**} accomodates both a and ω

(5,2) $$\begin{cases} a \in \Lambda_1^{10}\\ \omega \in \Lambda_1^{01}\end{cases}$$

We therefore may consider sa, $s\omega$, $d\omega$, $[a\wedge\omega]$ and $[\omega\wedge\omega]$ in $\Lambda_1^{**} \subset \Lambda^{**}$: in this sense we then have the <u>BRS relations</u>

(5,3) $$\begin{cases} sa = -d\omega - [a\wedge\omega]\\ s\omega = -\frac{1}{2}[\omega\wedge\omega]\end{cases}$$

Proof of these relations by definition (cf. [4,7]), $sa \in \Lambda^{**}$ is given by

(5,4) $$(sa)(\Omega) = \rho(\Omega)a = -d\Omega-[a\wedge\Omega], \ \Omega \in \mathcal{X}$$

(cf. (2,12)), whilst, also by definition (cf. (4,6a), (5,1):

(5,5) $$(d\omega)(\Omega) = d(\omega(\Omega)) = d\Omega, \quad \Omega \in \mathcal{X}$$

and (cf. (4,12a))

$$[a\wedge\omega](\Omega) = [a\wedge\omega(\Omega)] = [a\wedge\Omega], \tag{5,6}$$

proving the first line in (4,3). As for the second, we have, by definition of $s\omega \in \Lambda_1^{0\,2}$

$$\begin{aligned} s\omega(\Omega_0,\Omega_1) &= -(\rho(\Omega_0)\omega(\Omega_1) - \rho(\Omega_1)\omega(\Omega_0) - \omega([\Omega_0,\Omega_1])) \\ &= -(\rho(\Omega_0)\Omega_1 - \rho(\Omega_1)\Omega_0 - [\Omega_0,\Omega_0]) \\ &= -([\Omega_0\wedge\Omega_1] - [\Omega_1\wedge\Omega_0] - [\Omega_0,\Omega_1]) \\ &= -[\Omega_0\wedge\Omega_1] \end{aligned} \tag{5,7}$$

(we used (4,7) and (2,13)). On the other hand

$$\begin{aligned} [\omega\wedge\omega](\Omega_0,\Omega_1) &= [\omega(\Omega_0)\wedge\omega(\Omega_1)] - [\omega(\Omega_1)\wedge\omega(\Omega_0)] \\ &= [\Omega_0\wedge\Omega_1] - [\Omega_1\wedge\Omega_0] = 2[\Omega_0\wedge\Omega_1] \end{aligned} \tag{5,8}$$

proving the second line in (5,3).

<u>Remark.</u> The "ghost" ω arising in the physical literature is an anticommuting "field" $\omega(x)$ with values linear maps $\mathcal{X}\longrightarrow L$. Specifically, assuming the principal bundle $\mathbb{P}$ trivial (so that $\mathcal{X}$ consists of smooth maps Ω: $M\longrightarrow L$) and choosing a base e_α in L (with dual base ϵ^α in the dual L^* of L), we have, for $x \in M$

$$\omega(x) = \sum_\alpha \omega^\alpha(x)e_\alpha \tag{5,9}$$

i.e.,

$$\langle\omega(x),\Omega\rangle = \sum_\alpha \langle\omega^\alpha(x),\Omega\rangle e_\alpha, \qquad \Omega \in \mathcal{X}, \tag{5,9a}$$

where $\omega^\alpha(x) \in \mathcal{X}^*$ is the dual base of the "base" $\delta_x\otimes e_\alpha$ of $\Omega = \mathcal{C}^\infty(M)\otimes L$ in the

following sense

$$\langle\omega^\alpha(x),\delta_y\otimes e_\beta\rangle = \delta^\alpha_\beta\,\delta(x-y), \qquad x,y \in M. \tag{5,10}$$

In the physical literature, the operator s is defined by requiring

$$s\omega^\alpha(x) = -\tfrac{1}{2}f^\alpha_{\beta\gamma}\omega^\beta(x)\otimes\omega^\alpha(x), \qquad x \in M \tag{5,11}$$

where the $f^\alpha_{\beta\gamma}$ are the structure constants of L:

$$[e_\beta,e_\gamma] = \sum_\alpha f^\alpha_{\beta\gamma}e_\alpha \tag{5,12}$$

these relations implying

$$s\omega(x) = -\tfrac{1}{2}[\omega(x),\omega(x)], \qquad x \in M, \tag{5,13}$$

in the sense that one has, for $\Omega_0,\Omega_1, \in \mathcal{X}$:

$$(s\omega(x))(\Omega_0,\Omega_1) = \tfrac{1}{2}[(\omega(x))(\Omega_0),\omega(x)(\Omega_1)]. \tag{5,13a}$$

The relation between our $\omega = \mathrm{id}_{\mathcal{X}}$ and the above "ghost" is given as follows: one has

$$(\omega(x))(\Omega) = \omega(\Omega)(x) = \Omega(x), \qquad x \in M,\ \Omega \in \mathcal{X}. \tag{5,14}$$

Indeed, for $\Omega = f\otimes u$, $f \in \mathcal{C}^\infty(M)$, $u \in L$, one has, from (5,9a) and (5,10)

$$\begin{aligned} (\omega(x))(f\otimes u) &= \sum_\alpha(\omega^\alpha(x))(f\otimes u)e_\alpha \\ &= \sum_\alpha(\omega^\alpha(x))(\int f(y)\delta_y dy \otimes \sum u^\beta e_\beta)e_\alpha \end{aligned} \tag{5,15}$$

$$= \sum_{\alpha} \int dy f(y) \sum_{\beta} u^{\beta} \delta^{\alpha}_{\beta} \delta(x-y)$$

$$= \sum_{\alpha} f(x) \cdot u^{\alpha} e_{\alpha} = f(x)u$$

$$= (f \otimes u)(x).$$

Furthermore, if one extends the correspondence (5,14) to elements $\psi \in \Lambda^{\alpha 0}$ by requiring that

(5,16) $$(\psi(x))(\Omega_1,\ldots,\Omega_\alpha) = \psi(\Omega_1,\ldots,\Omega_\alpha)(x)$$

one obtains (5,13) by equating the evaluation of both sides of (5,7) for $x \in M$.

§6. Cohomology of $\mathcal{G}$ with values in local functionals of connection one-forms.

This section describes the cohomology which accomodates anomalies as elements of the first cohomology group. This cohomology arises from a construction analogous to the one described in section 4, but using a different representation space of the Lie algebra $\mathcal{G}$ of the gauge group $\mathcal{D}$. We denote by $\Gamma^{loc}(\mathfrak{A})$ the set of maps from $\mathfrak{A}$ into the reals obtained as follows: $\Upsilon \in \Gamma^{loc}(\mathfrak{A})$ whenever

(6,1) $$\Upsilon(a) = \int_M \bar{\Upsilon}(a),$$

where C is a smooth q-chain in M and $\bar{\Upsilon}$ is a map

(6,2) $$a \in \mathfrak{A} \longrightarrow \Lambda^q(M,\mathbb{R})$$

such that, for all $a,a' \in \mathfrak{A}$

(6,3) $$\mathrm{Supp}(\bar{\Upsilon}(a)-\bar{\Upsilon}(a')) \subseteq \mathrm{Supp}(a-a'),$$

where Supp refers respectively to $\Lambda^q(M,\mathbb{R})$ in the left hand side and $hA^1(P,L)$ in the right hand side (due to Ad-equivariance, the M-support is well defined on $hA^1(P,L)$). We further define a representation ρ_Γ of $\mathcal{D}$ on $\theta^{loc}(\mathfrak{A})$ (and an accompanying representation of $\mathcal{G}$), by setting

(6,4) $$(\rho_\Gamma(\psi)\Upsilon)(a) = \Upsilon(\rho(\psi^{-1})a)$$

---

30) We call such maps Υ <u>local</u> <u>functionals</u> <u>on</u> $\mathfrak{A}$. Note that our definition of locality encompasses the case where a and a', instead of being connection one-forms, are sections of a smooth fiber bundle over M (not necessarily vectorial), the support on the left hand side staying unchanged, whilst the support on the right hand side is now defined as the closure of the complement of the set $\{x \in M;\ a(x) = a'(x)\}$.

$$(\rho_r(\Omega)\gamma)(a) = \frac{d}{dt}\Big|_{t=0} \gamma(\rho(e^{-t\Omega})a) \tag{6,5}$$

We now consider the cohomology of $\mathcal{X}$ with values in the representation space $\Gamma^{loc}(\mathfrak{A})$ (another example of the general procedure already encountered in section 4). On the space

$$\begin{cases} \Gamma^* = \Lambda(\mathcal{X},\Gamma^{loc}(\mathfrak{A})) = \Gamma^{loc}(\mathfrak{A})\otimes\Phi^* = \bigoplus_\alpha \Gamma^\alpha \\ \Gamma^\alpha = \Lambda^\alpha(\mathcal{X},\Gamma^{loc}(\mathfrak{A})) = \Gamma^{loc}(\mathfrak{A})\otimes\Phi^\alpha \end{cases} \tag{6,6}$$

direct sum of the $\Gamma^{loc}(\mathfrak{A})$-valued alternate multilinear forms on $\mathcal{X}$, we consider the coboundary operator δ_r given as follows: for $\gamma \in \Gamma^\alpha$

$$(\delta_r\gamma)(\Omega_0,\ldots,\Omega_\alpha) \tag{6,7}$$
$$= \sum_{i=0}^{\alpha}(-1)^i\rho(\Omega_i)\gamma(\Omega_0,\ldots,\hat{\Omega}_i,\ldots,\Omega_\alpha) + \sum_{0\leq i<j\leq\alpha}(-1)^{i+j}\gamma([\Omega_i,\Omega_j],\Omega_0,\ldots,\hat{\Omega}_i,\hat{\Omega}_j,\ldots,\Omega_\alpha)$$

One has $\delta_r^2 = 0$, (Γ^*,δ_r) then defines a cohomology denoted $H^*(\mathcal{X},\Gamma^{loc}(\mathfrak{A}))$. According to Wess-Zumino compatibility anomalies are element of $H^1(\mathcal{X},\Gamma^{loc}(\mathfrak{A}))$.

§7. The homotopy formula.

This section describes an algorithm (analogue to the usual Cartan Chern Weil homotopy formula) which provides a means to classify anomalies [12].

With $a \in \mathfrak{A}$ and ω considered as belonging to ${}^1\!A$ (cf. (5,2)) we set[31)]

$$A = a + \omega \tag{7,1}$$

and consider $\mathcal{F}^{tA}$ and $\mathcal{D}^{tA}$ as defined in (4,71), (4,69) for the element tA of ${}^1\!A$, $t \in [0,1]$. We then have the limiting values

$$\begin{cases} \mathcal{F}^{0A} = \mathcal{F}^0 = 0 \\ \mathcal{F}^{1A} = \mathcal{F}^A = F^a = da + \frac{1}{2}[a\wedge a] \end{cases} \tag{7,2}$$

and the relation

$$\mathcal{D}^{tA}A = \frac{d}{dt}\mathcal{F}^{tA} \tag{7,3}$$

Proof. The first line of (7,2) is obvious. The second follows from the BRS relations: indeed

$$\mathcal{F}^A = (d+s)(a+\omega) + \frac{1}{2}[a+\omega\wedge a+a+\omega] \tag{7,4}$$
$$= F^a + s\omega + \frac{1}{2}[\omega\wedge\omega] + sa + d\omega + [a\wedge\omega]$$

where we used the fact that $[a\wedge\omega] = [\omega\wedge a]$ in the GDL $(A_1^{**},\Delta,[\wedge])$

---

31) A is thus a A-connection in the sense of [4,7].

On the other hand we have

$$\mathcal{F}^{tA} = t\Delta a + \tfrac{1}{2}t^2[A\wedge A] \tag{7,5}$$

where (6,3)

$$\frac{d}{dt}\mathcal{F}^{tA} = \Delta A + [tA\wedge A] = \mathcal{D}^{tA}A \tag{7,6}$$

From (4,53), (6,3), the Bianchi identity (4,52), and the fact that ($^*A,\Delta,\times$) is a GCDA, we have that

$$\begin{aligned} &\mathcal{D}^{tA}(A\times\mathcal{F}^{tA}\times\mathcal{F}^{tA}\times\ldots\times\mathcal{F}^{tA}) \\ &= \frac{d}{dt}\mathcal{F}^{tA}\times\mathcal{F}^{tA}\times\ldots\times\mathcal{F}^{tA} \\ &= \frac{1}{k}\frac{d}{dt}(\mathcal{F}^{tA}\times\mathcal{F}^{tA}\times\ldots\times\mathcal{F}^{tA}) \end{aligned} \tag{7,7}$$

Applying P on the left and integrating w.r.t. t from 0 to 1, we obtain, using (4,88) and (7,2)

$$k\Delta\int_0^1 P(A\times\mathcal{F}^{tA}\times\ldots\times\mathcal{F}^{tA})dt = P(F^a\times\ldots\times F^a) \tag{7,8}$$

(i.e.)

$$\Delta Q^{2k-1} = P((F^a)^{\times k}) \tag{7,8a}$$

with

$$Q^{2k-1} = k\int_0^1 P(A\times\mathcal{F}^{tA}\times\ldots\times\mathcal{F}^{tA})dt. \tag{7,9}$$

Writing Q^{2k-1} as sum of homogeneous elements for ghost number and degree of form:

$$Q^{2k-1} = Q^{0,2k-1} + Q^{1,2k-2} + Q^{2,2k-3} + \ldots + Q^{2k-1,0}, \tag{7,10}$$

relation (6,8a) yields

$$\begin{cases} \text{component } 2k,0 & : dQ^{2k-1,0} = 0 \\ \text{————— } 2k-1,1 & : sQ^{2k-1,0} + dQ^{2k-2,1} = 0 \\ \text{————— } 2k-2,2 & : sQ^{2k-2,1} + dQ^{2k-3,2} = 0 \\ \quad\vdots & \\ \text{————— } 0,2k & : sQ^{0,2k-1} = F^a \end{cases}$$

We are particularly interested in the third relation which furnishes a means of computing anomalies. Assume the principal bundle P to be trivial, and let $\sigma: M \longrightarrow P$ be a smooth section of P. Applying the pull back by σ, which commutes with d, we obtain choosing k so that $2k-2 = d$, the dimension of M, the following vanishing d-form M

$$\sigma^*((sQ^{2k-2,1})(\Omega_0,\Omega_1)) + d(\sigma^*Q^{2k-3,2}(\Omega_0,\Omega_1)) = 0, \quad \Omega_0,\Omega_1 \in \mathcal{X}. \tag{7,12}$$

If we assume M compact without boundary (euclidean situation), Stokes theorem then implies

$$\int_M \sigma^*((sQ^{2k-2,1})(\Omega_0,\Omega_1)) = 0, \quad \Omega_0,\Omega_1 \in \mathcal{X} \tag{7,13}$$

Let

$$\begin{aligned} \alpha(\Omega,a) &= \int_M \sigma^* Q^{2k-2,1}(\Omega) \\ &= \int_M \sigma^* \int_0^1 Qt^{2k-2,1}(\Omega)dt \end{aligned} \tag{7,14}$$

with

$$Q_t = A\times(\mathcal{F}^{tA})^{\times(k-1)}, \quad A = a+\omega. \tag{7,15}$$

We now show that (7,13), which also reads (cf. 4,84)

$$\int_M \sigma^* \int_0^1 P(sQ_t^{2k-2})(\Omega)dt = 0, \tag{7,13a}$$

in fact implies the "Wess-Zumino compatibility"

$$(\delta_r \alpha)(\Omega_0, \Omega_1, a) = 0, \quad \Omega_0, \Omega_1 \in \mathcal{L}, \tag{7,16}$$

in other words the fact that the cohomology class $[\alpha]$ of α is an element of the cohomology required for anomalies:

$$[\alpha] \in H^1(\mathcal{L}, \Gamma^{loc}(\alpha)). \tag{7,16a}$$

For checking (7,16) we note that one has

$$Q_t^{2k-1,1} = \omega \times F^{ta} + (k-1)t(t-1)[a\wedge\omega]\times a) \times (F^{ta \times (k-2)} \tag{7,17}$$

this stemming from

$$\begin{cases} A^{0,1} = \omega, \\ (\mathcal{F}^{At})^{1,1} = t(t-1)[a\wedge\omega] \\ (\mathcal{F}^{At})^{0,2} = F^{ta} = fF^a + \frac{1}{2}t(t-1)[a\wedge a] \end{cases} \tag{7,18}$$

From this follows

$$\begin{aligned} Q_t^{2k-2,1}(\Omega) &= (\Omega \times F^{ta} + (k-1)t(t-1)[a\wedge\Omega]\times a) \times (F^{ta})^{\times(k-2)} \\ &= Q_t^{2k-2,1}(\Omega, a) \end{aligned} \tag{7,19}$$

hence

$$\begin{aligned} &\rho(\Omega')Q_t^{2k-2,1}(\Omega) \\ &= Q_t^{2k-2,1}([\Omega',\Omega],a) + \frac{d}{d\lambda}\Big|_{\lambda=0} Q_t^{2k-2,1}(\Omega, \rho(e^{-\lambda\Omega'})a) \end{aligned} \tag{7,20}$$

Therefore

$$\begin{aligned} (sQ_t^{2k-2,1})(\Omega_0,\Omega_1) &= Q_t^{2k-2,1}([\Omega_0,\Omega_1],a) \\ &\quad + \frac{d}{d\lambda}\Big|_{\lambda=0} Q_t^{2k-1,1}(\Omega_1, \rho(e^{-\lambda\Omega_0})a) - Q_t^{2k-1,1}(\Omega_0, \rho(e^{-\lambda\Omega_1})a)) \\ &= -\delta_r Q_t^{2k-2,1}(\Omega, a) \end{aligned} \tag{7,21}$$

Relation (6,14) then results from the fact that δ_r commutes with the operation $\int_M \sigma^* \int_0^1 dtP$, as follows by linearity.

[7.2]. **Remark.**

From (6,8a) it follows that ΔQ^{2k-1} is a basic form:

$$\theta(u)\Delta Q^{2k-1} = i(u)\Delta^{2k-1} = 0, \quad u \in L. \tag{7,22}$$

This raises the question of whether this also holds for $Q^{2k-2,1}$, which would yield an intrinsic d-form q' on the base such that $Q^{2k-2,1} = \pi^* q'$ (cf. 2,7) without recourse to a section σ^* of P (so without the assumption that the latter is trivial). However this does not arise[32], the form $Q^{2k-2,1}$ is by definition Ad–equivariant

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32) No more that in the case of the Chern–Simons form TP. What we have here is an analogy to the Chern–Simons situation, with the replacement $a \longrightarrow A$ (cf. [9]).

(7,23)
$$\theta(u)Q^{2k-2,1} = 0, \quad u \in L,$$

however $Q^{2k-2,1}$ <u>is</u> <u>not</u> <u>horizontal</u>. Indeed, we conclude from (7,17), taking account of

(7,24)
$$\begin{cases} i(u)\omega = 0 \\ i(u)a = u1 \\ i(u)F^{ta} = i(u)(tF^{a} + \frac{1}{2}t(t-1)[a\wedge a]) \\ \qquad\qquad = t(t-1)[u1\wedge a] \end{cases}$$

that one has

(7,25)
$$\frac{i(u)Q_t^{2k-2,1}}{t(t-1)}$$
$$= (-\omega\times[u1\wedge a] + (k-1)[u1\times\omega]\times a + (k-1)[a\wedge\omega]\times a) \times (F^{ta})^{\times(k-1)}$$
$$+ (k-2)(\omega\times F^{ta}+(k-1)t(t-1)[a\wedge\omega]\times a)\times[u1\wedge a]\times(F^{ta})^{\times(k-1)}$$

In particular, for $k = 2$

(7,26)
$$i(u)Q_t^{2k-2,1} = -\omega\times[u1\times a] + [u1\wedge\omega]\times a + [a\wedge\omega]\times a$$
$$= [u1\wedge(\omega\times a)] - 2\omega\times[u1\wedge a] + [a\wedge\omega]\times\omega$$

Upon application of P the first term r.h.s. vanishes (cf. (4,90), but the two following terms persist, yielding a non vanishing result.

It is easy to find a substitute for Q^{2k-1} whose terms $Q^{g,2k-1-g}$ are basic, horizontal, replacing the family tA by the family $tA + (1-t)\mathring{a}$, where $\mathring{a}$ is a fixed backgound connection on P, which then need not be assumed to be trivial [8]. The algebraic constructions of this paper have to be generalized as will be done in forthcoming article [14].

<u>Appendix</u> <u>A.</u> <u>Graded</u> <u>differential</u> <u>algebras.</u>

[A.1]. <u>Definitions.</u> (i) A GDA (<u>graded</u> <u>differential</u> <u>algebra</u>) is a graded real vector space $\mathfrak{a} = \bigoplus_{n\in\mathbb{N}} \mathfrak{a}^{n}$, equipped with a bilinear product $\mathfrak{a}\times\mathfrak{a}\longrightarrow\mathfrak{a}$; and a linear operator $d: \mathfrak{a}\longrightarrow\mathfrak{a}$ (the derivative), with the properties

(A,1)
$$\mathfrak{a}^{p}\cdot\mathfrak{a}^{q} \subset \mathfrak{a}^{p+q}, \quad p,q \in \mathbb{N}$$

(A,2)
$$d\mathfrak{a}^{p} \subset \mathfrak{a}^{p+1}, \quad p \in \mathbb{N}$$

(A,3)
$$d(ab) = Da\cdot b + (-1)^{p}da\cdot b, \quad a \in \mathfrak{a},\ b \in \mathfrak{a}^{p},\ p \in \mathbb{N}$$

(A,4)
$$d^{2} = 0.$$

(ii) A GCDA (<u>graded-commutative</u> <u>differential</u> <u>algebra</u>)[33] is a GDA $\mathfrak{a}$ with an associated and "graded-commutative" product:

(A,5)
$$a\cdot(bc) = (a\cdot b)c, \quad a,b,c \in \mathfrak{a}$$

(A,6)
$$b\cdot a = (-1)^{p\cdot q}a\cdot b, \quad a \in \mathfrak{a}^{p},\ b \in \mathfrak{a}^{q},\ p,q \in \mathbb{N}$$

(iii) A DGL[33] (<u>differential</u> <u>graded</u> <u>Lie</u> <u>algebra</u>) is a GDA with product [1] a "graded Lie bracket":

(A,7)
$$b\cdot a = -(-1)^{pq}a\cdot b,\ a \in \mathfrak{a}^{p},\ b \in \mathfrak{a}^{q}$$

(A,8)
$$(-1)^{pr}a\cdot(b\cdot c) + (-1)^{qp}b\cdot(c\cdot a) + (-1)^{rq}c\cdot(a\cdot b) = 0;$$
$$a \in \mathfrak{a}^{p},\ b \in \mathfrak{a}^{q},\ c \in \mathfrak{a}^{r}$$

---

33) In concrete examples as those encountered in the text, it is natural to denote the product of a GCDA by a wedge-like symbol, and that of a DGL by a bracket-like symbol.

(iv) Let L be a Lie algebra and $\mathfrak{A}$ be a GDA. An <u>action of L on $\mathfrak{A}$</u> is a pair (θ, i) of linear maps from L to the linear operators of $\mathfrak{A}$ with the properties[34]

(A,9) $$\begin{cases} \theta(u)\mathfrak{A}^p \subset \mathfrak{A}^p, & p \in \mathbb{N} \\ \theta(u)(a \cdot b) = \theta(u)a \cdot b + a \cdot \theta(u)b, & a,b \in \mathfrak{A},\ u \in L \end{cases}$$

(A,10) $$\begin{cases} i(u)\mathfrak{A}^p \subset \mathfrak{A}^{p-1}, & p \in \mathbb{N},\ i(u)|_{\mathfrak{A}^0} = 0, \quad u \in L \\ i(u)(a \cdot b) = i(u)a \cdot b + (-1)^p a \cdot i(u)b, & a \in \mathfrak{A}^p,\ b \in \mathfrak{A} \end{cases}$$

and

(A,11) $$i(u)^2 = 0, \quad u \in L$$

(A,12) $$\theta([u,v]) = \theta(u)\theta(v) - \theta(v)\theta(u), \quad u,v \in L$$

(A,13) $$\theta(u)i(v) - i(v)\theta(u) = i([u,v]), \quad u,v \in L$$

(A,14) $$\theta(u) = i(u)d + di(u), \quad u \in L$$

Note that these properties imply

(A,15) $$\theta(u)d = d\theta(u) \ (= di(u)d), \quad u \in L$$

(A,16) $$\theta(u)i(u) = i(u)\theta(u), \quad u \in L$$

(In fact (A,12) follows from (A,14) and the fact that $d^2 = 0$)

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34) (A,9) states the fact that $\theta(u)$, $u \in L$, is a 0-grade derivation of $\mathfrak{A}$; and (A,10) that $i(u)$, $u \in L$, is a -1-grade graded derivation of $\mathfrak{A}$.

[A.2]. <u>Skew tensor products of GDAs.</u>

Let $(\mathfrak{A} = \bigoplus_{p \in \mathbb{N}} \mathfrak{A}^p, d, \cdot)$ and $\Psi = (\bigoplus_{\alpha \in \mathbb{N}} \Psi^\alpha, \delta, \circ)$ be two real[35] GDAs. Their <u>skew product</u> as graded algebras is the usual tensor product of vector sapces

(A,17) $$\Pi = \mathfrak{A} \otimes \Psi$$

equipped with the bilinear skew product determined by

(A,18) $$(a \otimes \varphi) \Delta ((b \otimes \psi) = (-1)^{\alpha q}(a \cdot b) \otimes (\varphi \circ \psi), \quad \begin{cases} a \in \mathfrak{A}, & b \in \mathfrak{A}^{\varphi} \\ \varphi \in \Psi^\alpha, & \psi \in \Psi \end{cases}$$

and the grading

(A,19) $$\Pi^n = \sum_{p+\alpha=n} \mathfrak{A}^p \otimes \Psi^\alpha$$

We furthermore consider the following operators on Π:

(A,20) $$D = d + \sigma$$

where[36]

(A,21) $$d = d \otimes 1_\Psi$$

(A,22) $$\sigma = (-1)^\partial \otimes \delta$$

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35) Or for that matter complex.

36) Using the same symbol for d acting on $\mathfrak{A}$ and $d = d \otimes 1_\Psi$ acting on Π should cause no confusion.

where $(-1)^{\partial}$ denotes the grading in relation of $\mathfrak{A}$

(A,23) $$(-1)^{\partial} = \begin{cases} \mathrm{id} \text{ on } \mathfrak{A}^{2p} \\ -\mathrm{id} \text{ on } \mathfrak{A}^{2p+1}, \quad p \in \mathbb{N} \end{cases}$$

and $\otimes$ r.h.s. of (A,21), (A,22) denotes a standard tensor product of linear operators:[38]

(A,24) $$(A \otimes B)(a \otimes \psi) = (Aa) \otimes (B\psi), \quad a \in \mathfrak{A},\ \psi \in \Psi$$

We recall that these definitions imply the following facts:

(i) With the product (A,18) and the grading (A,19) Π is a graded algebra, i.e. $\Pi^n \cdot \Pi^m \subset \Pi^{n+m}$, $n,m \in \mathbb{N}$.

(ii) d and σ are graded derivations of Π of grade 1[37]. Moreover d and σ anticommute:

(A,25) $$d\sigma + \sigma d = 0$$

Consequently, D is a graded derivation of Π of grade -1 and vanishing square

(A,26) $$D^2 = 0,$$

making $(\Pi, \delta, \cdot)$ a GDA.

(iii) If $(\mathfrak{A}, d, \cdot)$ and $(\Psi, \delta, \circ)$ are GCDA, (Π, D, Δ) is a GCDA.

(iv) If $(\mathfrak{A}, d, \cdot)$ is a DGL and $(\Psi, \delta, \circ)$ is a GCDA, (Π, D, Δ) is a DGL.

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37) A linear operator L on Π is of grade r, $r \in \mathbb{N}$ whenever $L\Pi^n \subset L\Pi^{n+r}$, $n \in \mathbb{N}$ ($\Pi^k = \{0\}$, $k < 0$). It is a graded derivation wherever $L(\alpha \cdot \beta) = (L\alpha) \cdot \beta + (-1)^n \alpha \cdot L\beta$, $\alpha \in \Pi^p$, $\beta \in \Pi$.

38) In fact, (A,21) and (A,22) could be written $d = d \otimes 1_{\Psi}$, $\sigma = 1_{\mathfrak{A}} \otimes \delta$, with $\otimes$ a graded tensor product of operators.

Proof. (i): obvious.

(i) We have, for $a \in \mathfrak{A}^p$, $b \in \mathfrak{A}^q$, $\varphi \in \Psi^{\alpha}$, $\psi \in \Psi^{\beta}$

(A,27) $$\begin{aligned} d((a\otimes\varphi)\Delta(b\otimes\psi)) &= (-1)^{\alpha q} d((a \cdot b)\otimes(\varphi \circ \psi)) \\ &= (-1)^{\alpha q}(da \cdot b + (-1)^p a \cdot db)\otimes(\varphi \circ \psi) \\ &= (-1)^{\alpha q + \alpha q}(da\otimes\varphi)\Delta(b\otimes\psi) + (-1)^{\alpha q + p + \alpha(q+1)}(a\otimes\varphi)\Delta(db\otimes\psi) \\ &= (d(a\otimes\varphi))\Delta(b\otimes\psi) + (-1)^{p+\alpha}(a\otimes\varphi)\Delta d(b\otimes\psi) \end{aligned}$$

(A,28) $$\begin{aligned} \sigma((a\otimes\varphi)\Delta(b\otimes\psi)) &= (-1)^{\alpha q}\sigma((a \cdot b)\otimes(\varphi \circ \psi)) \\ &= (-1)^{\alpha q+p+q}(a \cdot b)\otimes(\sigma\varphi \circ \psi + (-1)^{\alpha}\varphi \circ \sigma\psi) \\ &= (-1)^{\alpha q+p+q+(\alpha+1)p}(a\otimes\sigma\varphi)\Delta(b\otimes\psi) + (-1)^{\alpha q+p+q+\alpha+\alpha q}(a\otimes\varphi)\Delta(b\otimes\psi) \\ &= (\sigma(a\otimes\varphi))\Delta(b\otimes\psi) + (-1)^{p+\alpha}(a\otimes\varphi)\Delta\sigma(b\otimes\psi) \end{aligned}$$

(A,29) $$\begin{aligned} d\sigma(a\otimes\varphi) &= (-1)^p d(a\otimes\delta\varphi) = (-1)^p(da\otimes\delta\varphi) \\ &= (-1)^{p+p+1}\sigma(da\otimes\varphi) = -\sigma d(a\otimes\varphi) \end{aligned}$$

(iii) We have, for a,b,φ,ψ as above and $c \in \mathfrak{A}^r$, $\theta \in \Psi^{\gamma}$

(A,30) $$\begin{aligned} (a\otimes\varphi)\Delta((b\otimes\psi)\Delta(c\otimes\theta)) &= (-1)^{\beta r}(a\otimes\varphi)\Delta((b \cdot c)\otimes(\psi \circ \theta)) \\ &= (-1)^{\beta r+\alpha(q+r)}(a \cdot b \cdot c)\otimes(\varphi \circ \psi \circ \theta) \\ &= (-1)^{\beta r+\alpha(q+r)+r(\alpha+\beta)}((a \cdot b)\otimes(\varphi \circ \psi))\Delta(c\otimes\theta) \\ &= (-1)^{\alpha q+\alpha q}((a\otimes\varphi)\Delta(b\otimes\psi))\Delta(c\otimes\theta) \end{aligned}$$

(A,31) $$\begin{aligned} (b\otimes\psi)\Delta(a\otimes\varphi) &= (-1)^{\beta p}(b \cdot a)\otimes(\psi \circ \varphi) \\ &= (-1)^{\beta p+pq+\alpha\beta}(a \cdot b)\otimes(\varphi \circ \psi) \\ &= (-1)^{(\alpha+\beta)(p+q)}(a\otimes\varphi)\Delta(b\otimes\psi) \end{aligned}$$

(iv) Let $a,b,c,\varphi,\psi,\theta$, be as above. For the commutation of $a\otimes\varphi$ and $b\otimes\psi$, we have the same computation as in (A,31), with the alteration that, now, $b \cdot a = -(-1)^{pq} a \cdot b$, whence an overal minus sign, leading to (A,7). We check (A,8): we have

$$(A,32)\qquad (-1)^{(p+\alpha)(r+\gamma)}(a\otimes\varphi)\Delta((b\otimes\psi)\Delta(c\otimes\theta))$$
$$= (-1)^{(p+\alpha)(r+\gamma)+\beta r}(a\otimes\varphi)(b\cdot c)\otimes(\psi\circ\theta))$$
$$= (-1)^{(p+\alpha)(r+\gamma)+\beta r+\alpha(q+r)}(a\cdot b\cdot c)\otimes(\varphi\circ\psi\circ\theta)$$
$$= (-1)^{\alpha q+\beta r+\gamma p}(-1)^{pr+\alpha\gamma}(a\cdot b\cdot c)\otimes(\varphi\circ\psi\circ\theta)$$

Since $\alpha q+\beta r+\gamma p$ is invariant under circular permutation, the graded Jacobi identity for Π is a consequence of that for $\mathfrak{A}$, given that $(-1)^{\alpha\gamma}\varphi\circ\psi\circ\theta$ is invariant under circular permutations, a straightforward consequence of the GCDA nature of Ψ.

Remark. (i) The proof of (i) applies to the more general situation where d and δ are graded derivation of $\mathfrak{A}$, resp. Ψ, of odd grade p, resp. q. Defining d and σ on the skew product Π as in (A,21), (A,22) the latter are still mutually anticommuting derivations of Π, of respective grades p and q.

Indeed, in the proofs (A,27), (A,28), (A,29), no use is made of the fact that $d^2 = 0$, $\delta^2 = 0$, and the grade of d and δ enters only through its parity.

Appendix B. "Covariant derivatives" in DGLs.

Proposition. Let $(L = \bigoplus_{n\in\mathbb{N}} L^{(n)}, d,[\wedge])$ be a GDL. For $\alpha \in L^{(1)}$, $F^{\alpha} \in L^{(2)}$ and the map D^{α}: $L \to L$ are defined as follows:

$$(B,1)\qquad F^{\alpha} = d\alpha + \tfrac{1}{2}[\alpha\wedge\alpha]$$

$$(B,2)\qquad D^{\alpha}\lambda = d\lambda + [\alpha\wedge\lambda]$$

We then have that

(i) D^{α} is a graded derivation of L

$$(B,3)\qquad D^{\alpha}[\lambda\wedge\rho] = [(D^{\alpha}\lambda)\wedge\rho] + (-1)^{n}[\lambda\wedge D^{\alpha}\rho],\quad \lambda \in L^{(n)},\ \rho \in L,$$

whose square is given by

$$(B,4)\qquad (D^{\alpha})^{2}\lambda = [F^{\alpha}\wedge\lambda],\quad \lambda \in L$$

(ii) We have the Bianchi identity

$$(B,5)\qquad D^{\alpha}F^{\alpha} = 0.$$

(iii) The map $[\alpha\wedge\cdot]$:

$$(B,7)\qquad \lambda \in L \mapsto [\alpha\wedge\lambda] \in L$$

is a graded derivation of L; in fact D^{α} is the sum of d and $[\alpha\wedge\cdot]$ (generally for $\alpha \in L^{(p)}$ $[\alpha\wedge\cdot]$ is a derivation of order p in the sense that

$$(B,8)\qquad [\alpha\wedge[\lambda\wedge\mu]] = [[\alpha\wedge\lambda]\wedge\mu] + (-1)^{np}[\lambda\wedge[\alpha\wedge\mu]],\quad \lambda \in L^{p}],\ \mu \in L^{(q)}.)$$

Proof. (i) d is by definition a graded derivation of $([L,\wedge])$: thus the first assertion in (i) is a consequence of (iii), which in turn follows from the following special case of the graded Jacobi identity: for $\lambda \in L^{(p)}$, $\mu \in L^{(q)}$, $\alpha \in L^{(n)}$

(B,9) $$(-1)^{nq}[\alpha\wedge[\lambda\wedge\mu]] + (-1)^{np}[\lambda\wedge[\mu\wedge\alpha]] + (-1)^{pq}[\mu\wedge[\alpha\wedge\lambda]]$$

with the commutation properties

(B,10) $$[\mu\wedge\alpha] = -(-1)^{nq}[\alpha\wedge\mu]$$

(B,11) $$[\mu\wedge[\alpha\wedge\lambda]] = -(-1)^{q(n+p)}[[\alpha\wedge\lambda]\wedge\mu]$$

implying

(B,12) $$[\alpha\wedge[\lambda\wedge\mu]] = [[\alpha\wedge\lambda]\wedge\mu] + (-1)^{np}[\lambda\wedge[\alpha\wedge\mu]], \quad \lambda \in L^{(p)}.$$

We now check (A,4): we have, taking now $n = 1$

(B,13) $$(D^{\alpha})^2\lambda = d(d\lambda+[\alpha\wedge\lambda] + \alpha(d\lambda + [\alpha\wedge\lambda]) = [d\alpha\wedge\lambda] + [\alpha\wedge[\alpha\wedge\lambda]]$$

however, using (A,10)

(B,14) $$[\alpha\wedge[\alpha\wedge\lambda]] = \tfrac{1}{2}([\alpha\wedge[\alpha\wedge\lambda]] + [[\alpha\wedge\alpha]\wedge\lambda] - [\alpha\wedge[\alpha\wedge\lambda]]) = \tfrac{1}{2}[[\alpha\wedge\alpha]\wedge\lambda].$$

(ii) We have

(B,15) $$D^{\alpha}f^{\alpha} = dF^{\alpha} + \alpha\wedge F^{\alpha} = d(d\alpha + \tfrac{1}{2}[\alpha\wedge\alpha]) + [\alpha\wedge[d\alpha + \tfrac{1}{2}[\alpha\wedge\alpha]]] = \tfrac{1}{2}[d\alpha\wedge\alpha] - \tfrac{1}{2}[\alpha\wedge d\alpha] - [\alpha\wedge d\alpha] = 0.$$

where $[d\alpha\wedge\alpha] = -[\alpha\wedge d\alpha]$, due to the commutasion rule, and $[\alpha\wedge[\alpha\wedge\alpha]] = 0$, to the graded Jacobi identity.

(iv) immediate from (B,4) and (B,5).

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