DFTT-74/99
December 1999

On G/H geometry and its use in M-theory compactifications

Leonardo Castellani

Dipartimento di Scienze e Tecnologie Avanzate, East Piedmont University, Italy;
Dipartimento di Fisica Teorica and Istituto Nazionale di Fisica Nucleare
Via P. Giuria 1, 10125 Torino, Italy.

Abstract

The Riemannian geometry of coset spaces is reviewed, with emphasis on its applications to supergravity and M-theory compactifications. Formulae for the connection and curvature of rescaled coset manifolds are generalized to the case of nondiagonal Killing metrics.

The example of the N^{010} spaces is discussed in detail. These are a subclass of the coset manifolds $N^{pqr} = G/H = SU(3) \times U(1)/U(1) \times U(1)$, the integers p, q, r characterizing the embedding of H in G. We study the realization of N^{010} as $G/H = SU(3) \times SU(2)/U(1) \times SU(2)$ (with diagonal embedding of the $SU(2) \in H$ into G). For a particular G-symmetric rescaling there exist three Killing spinors, implying $N = 3$ supersymmetry in the $AdS_4 \times N^{010}$ compactification of $D = 11$ supergravity. This rescaled N^{010} space is of particular interest for the AdS_4/CFT_3 correspondence, and its $SU(3) \times SU(2)$ isometric realization is essential for the $OSp(4|3)$ classification of the Kaluza-Klein modes.

e-mail: castellani@to.infn.it

1 Introduction

Coset manifolds are a natural generalization of group manifolds, and play an important role in supergravity and superstring compactifications, and in the recent AdS/CFT correspondences [1]. Indeed several of these correspondences have been investigated in the context of compactifications of supergravity theories on anti-de Sitter spaces times "internal" coset spaces G/H. Many results of the 80's have been reinterpreted and extended in the AdS/CFT framework, which has prompted in particular a renewed interest in Kaluza-Klein mass spectra of the $AdS \times G/H$ supergravity compactifications. For an exhaustive list of references on this subject we refer to the introduction of [2]. Here we will cite only the papers dealing with N^{pqr} spaces (see later).

In this note we generalize some formulae of the Riemannian geometry of coset manifolds to include interesting cases, as the N^{010} spaces in the manifest $SU(3) \times SU(2)$ invariant formulation. The general formulas of ref. [3, 4] are valid only for diagonal Killing metric, and need to be extended for nondiagonal Killing metrics. While it is true that the Killing metric can always be made diagonal by a redefinition of the group generators, it may happen that the G/H structure we want to obtain prevents such a redefinition, and that we must live with a nondiagonal Killing metric. For the geometry of the N^{pqr} coset spaces, and their use in $D = 11$ supergravity compactifications, we refer to the original papers [5, 6, 7]. Recent developments using N^{pqr} geometry to derive Kaluza-Klein mass spectra and test AdS_4/CFT_3 correspondence are found in [8, 9, 10].

We give now a short review of coset space geometry, beginning with a few definitions. A metric space is said to be homogeneous if it admits as an isometry the transitive action of a group G, transitive meaning that any two points of the space are connected via the group action. For example the unit sphere S^2 in $\boldsymbol{R}^3$ is isometric under the transitive action of $SO(3)$. The subgroup H of G which leaves a point X fixed is called the isotropy subgroup. Because of the transitive action of G, any other point $X' = gX$ ($g \in G, g \notin H$) is invariant under a subgroup gHg^{-1} of G isomorphic to H. In the S^2 example any point remains fixed under $SO(2)$ rotations around the axis passing through that point, so that $SO(2)$ is the isotropy subgroup.

It is natural to label the points X of a homogeneous space by the parameters describing the G - group element which carries a conventional X_0 (the origin) into X. However these parameters are redundant: there are infinitely many group elements g such that $X = gX_0$, due to H - isotropy. Indeed if g carries X_0 into X, any other G element of the form gH does the same, since $HX_0 = X_0$. We are then led to characterize the points of a homogeneous space by the cosets gH.

A homogeneous space is therefore a coset space G/H, i.e. the set of equivalence classes of elements of G, where the equivalence is defined by right H multiplication ($g \sim g'$ if $g = g'h$, with $g, g' \in G$ and $h \in H$). Thus the two-sphere S^2 can be considered as the coset space $SO(3)/SO(2)$. In general for an n-sphere $S^n = SO(n+1)/SO(n)$. The action of an element $g' \in G$ on the coset gH is simply given by the coset $g'gH$.

Taking G to be a Lie group (as in our S^2 example), we obtain coset *manifolds*, endowed with a Riemannian structure as we will discuss. The Lie algebra of G can be split as:

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{1.1}$$

where $\mathbb{H}$ is the Lie algebra of H and $\mathbb{K}$ contains the remaining generators, called "coset generators". The structure constants of G are defined by:

$$\begin{aligned}
[H_i, H_j] &= C_{ij}{}^k H_k & H_i \in \mathbb{H} \\
[H_i, K_a] &= C_{ia}{}^j H_j + C_{ia}{}^b K_b & K_a \in \mathbb{K} \\
[K_a, K_b] &= C_{ab}{}^j H_j + C_{ab}{}^c K_c &
\end{aligned} \tag{1.2}$$

where the index conventions are obvious.

As discussed in ref. [11] (p. 251), whenever H is compact or semisimple (even if G is not compact) one can always find a set of K_a such that the structure constants $C_{ia}{}^j$ vanish. In that case the $\mathbb{G} = \mathbb{H} + \mathbb{K}$ split, or equivalently the coset space G/H is said to be *reductive*. For this reason we will deal in this note only with reductive coset spaces. Another important observation is that when G/H is reductive the structure constants $C_{ia}{}^b$ can always be made antisymmetric in a, b by an appropriate redefinition $K_a \to N_a{}^b K_b$. The proof is simple: any representation of a compact H can be made unitary by a suitable change of basis. Since the $C_{ia}{}^b$ generate a real representation of H (namely the coset representation), this representation can be made orthogonal, and consequently the $C_{ia}{}^b$ antisymmetric [11, 12].

An element g of G is specified by $dimG$ continuous parameters, the Lie group coordinates. For example we can exponentiate the Lie algebra as:

$$g = \exp[y^a K_a] \exp[x^i H_i] \tag{1.3}$$

The G coordinates are y^a, x^i. It is clear that the cosets gH are characterized by a subset of the group coordinates, i.e. by the $dimG - dimH$ parameters y^a corresponding to the K_a generators.

Each coset, labeled by the y parameters, can be mapped into an element $L(y)$ of G, the *coset representative*. For example one can choose as coset representative:

$$L(y) = \exp[y^a K_a] \tag{1.4}$$

The whole geometry of G/H can be constructed in terms of coset representatives.

Under left multiplication by a generic element g of G, $L(y)$ is in general carried into an element of G belonging to another equivalence class, with representative element $L(y')$, i.e. into an element of the form $L(y')h$:

$$gL(y) = L(y')h, \quad h \in H \tag{1.5}$$

where y' and h depend on y and g, and on the way of choosing representatives. For example, using the representative choice (1.4), $gL(y)$ loses the $\exp(yK)$ form but can be expressed (as any element of G) as $\exp(y'K)\exp(xH)$ or $L(y')\exp(xH)$. It is clear that the geometry of G/H must be insensitive to the particular representative choice, and indeed this is so (see later).

2 Vielbeins, invariant metric, H-connection on G/H

Consider the 1-form:

$$V(y) = L^{-1}(y)dL(y) \tag{2.6}$$

generalizing the left-invariant 1-form $g^{-1}dg$ of group manifolds. $V(y)$ is Lie algebra valued and can expanded on the G generators:

$$V(y) = V^a(y)T_a + \Omega^i(y)T_i \tag{2.7}$$

The 1-form $V^a(y) = V^a_\alpha dy^\alpha$ is a covariant frame (vielbein) on G/H and $\Omega_\alpha{}^i dy^\alpha$ is called the H-connection.

Under left multiplication by a constant $g \in G$, the one-form $L^{-1}dL$ is not invariant, but transforms as:

$$V(y') = hL^{-1}(y)g^{-1}d(gL(y)h^{-1}) = hV(y)h^{-1} + hdh^{-1} \tag{2.8}$$

In particular its projection on the coset generators yields the transformation rule of the vielbein:

$$V^a(y') = (hV(y)h^{-1})^a = V^b(y)D_b{}^a(h^{-1}) \tag{2.9}$$

where the adjoint representation $D_A^{\ B}$ is defined by $g^{-1}T_A g = D_b{}^a(g)T_B$.

The infinitesimal form of (1.5) is obtained by taking:

$$g = 1 + \varepsilon^A T_A \tag{2.10}$$

Consequently, also the induced h transformation is infinitesimal:

$$h = 1 - \varepsilon^A W^i_A(y)T_i \tag{2.11}$$

and the shift in y is proportional to ε^A:

$$y'^\alpha = y^a + \varepsilon^A K_A^{\ \alpha}(y) \tag{2.12}$$

The y dependent matrix $W^i_A(y)$ defined in (2.11) is called the H-compensator, and the y-dependent differential operator

$$K_A(y) \equiv K_A^{\ \alpha}(y)\frac{\partial}{\partial y^\alpha} \tag{2.13}$$

is the Killing vector on G/H associated to the G-generator T_A. The explicit expressions for the H-compensator and the Killing vectors are simply obtained by rewriting the transformation rule (1.5) for infinitesimal g:

$$T_A L(y) = K_A(y)L(y) - L(y)T_i W_A{}^i(y) \tag{2.14}$$

After multiplying on the left by $L^{-1}(y)$ and projecting on the K and H generators we find:

$$K_A{}^\alpha(y) = D_A{}^a(L(y))V_a^\alpha(y) \tag{2.15}$$
$$W_A{}^i(y) = \Omega_\alpha{}^i(y)K_A{}^\alpha(y) - D_A{}^i(L(y)) \tag{2.16}$$

where $V_a^\alpha(y)$ is defined as the inverse of the G/H vielbein V_α^a.

The infinitesimal form of the vielbein transformation (2.9) reads:

$$V^a(y+\delta y) - V^a(y) = -\varepsilon^A W_A{}^i(y)C_{ib}{}^a V^b(y) \tag{2.17}$$
$$\delta y^\alpha = \varepsilon^A K_A{}^\alpha(y) \tag{2.18}$$

easily derived by observing that the $C_{iB}{}^A$ are the generators of the adjoint representation of H, and $C_{ij}{}^a = 0$.

For reductive algebras $C_{ib}{}^a$ can be made antisymmetric in a, b: then eq. (2.17) implies that the left action of G on $V^a(y)$ is equivalent to an $SO(N)$ rotation on $V^a(y)$ ($N = dim\ G/H$). Then the "natural" coset metric

$$g_{\alpha\beta} = \delta_{ab}V_\alpha^a V_\beta^b \tag{2.19}$$

is invariant under the left action of G. Another G left-invariant metric is obtained by replacing the Kronecker delta in (2.19) with the Killing metric $\gamma_{AB} \equiv C_{AD}{}^C C_{BC}{}^D$ restricted to G/H

$$g_{\alpha\beta} = \gamma_{ab}V_\alpha^a V_\beta^b \tag{2.20}$$

Notice that both these invariant metrics are insensitive to the choice of coset representative. Indeed replacing $L(y)$ by $L(y)h$ just rotates the vielbein as in (2.17).

Transformations that leave the metric invariant are called isometries. From the preceding discussion we know that the isometries of G/H manifolds include the left action of G. However one can study also the right action of G on the coset representative:

$$L(y)g = L(y')h \tag{2.21}$$

Then one finds that $N(H)/H$ is the right isometry group of G/H, where $N(H)$ is the normalizer of H in G, i.e. the set of elements $g \in G$ such that $gHg^{-1} = H$. One is led to conclude that the full isometry group of G/H must include $G \times N(H)/H$: however this is not always true, as argued in ref.s [3, 4]. Some left $U(1)$ Killing vectors may coincide with some right $U(1)$ Killing vectors: then the actual isometry is reduced to $G' \times N(H)/H$ where $G = G' \times$ (common $U(1)$ - factors).

3 Rescaled Riemann connection and curvature

In general the two metrics (2.19),(2.20) of the preceding Section are not the only G-invariant metrics on G/H. As discussed in various ref.s (see for example [3, 11, 4]) whenever $C_{ia}{}^b$ is block diagonal in some subspaces $S_1, S_2, ...$ of $\mathbb{K}$, then the vielbeins spanning these subspaces can be independently rescaled without loss of left G symmetry. This is easily understood from the transformation rule (2.17), which remains unaltered when the vielbeins belonging to the same subspace S_i are rescaled by a common parameter r_i. Therefore the number of rescaling parameters, i.e. the number of parameters necessary to specify the particular G-invariant metric, is equal to the number of irreducible blocks of $C_{ia}{}^{b}$. This matrix describes how H acts on the subspace $\mathbb{K}$: if it acts irreducibly, the coset is called isotropy irreducible, and only the trivial rescaling $V^a \to rV^a$ (same r for all V^a) is G-symmetric. If G/H is isotropy reducible, we have an independent parameter for each irreducible subspace S_i. These rescalings must be real and nonsingular, but are otherwise unconstrained. We derive now the expressions for the Riemann connection and the curvature corresponding to the rescaled vielbeins.

Recall the Cartan-Maurer equation for the one-form $V = L^{-1}dL$:

$$dV + V \wedge V = 0 \tag{3.22}$$

which follows immediately from the definition of V. In components the Cartan-Maurer equation becomes:

$$dV^a + \frac{1}{2}C_{bc}{}^{a}V^b \wedge V^c + C_{bi}{}^{a}V^b \wedge \Omega^i = 0 \tag{3.23}$$

$$d\Omega^i + \frac{1}{2}C_{ab}{}^{i}V^a \wedge V^b + C_{jk}{}^{i}\Omega^j \wedge \Omega^k = 0 \tag{3.24}$$

After a rescaling

$$V^a \to r_a V^a \tag{3.25}$$

the above equations become:

$$dV^a + \frac{1}{2}\frac{r_b r_c}{r_a}C_{bc}{}^{a}V^b \wedge V^c + \frac{r_b}{r_a}C_{bi}{}^{a}V^b \wedge \Omega^i = 0 \tag{3.26}$$

$$d\Omega^i + \frac{1}{2}r_a r_b C_{ab}{}^{i}V^a \wedge V^b + C_{jk}{}^{i}\Omega^j \wedge \Omega^k = 0 \tag{3.27}$$

For a G-symmetric rescaling we can replace $\frac{r_b}{r_a}C_{bi}{}^{a}$ by $C_{bi}{}^{a}$ in the first equation.

The flat coset metric will be chosen in the following to be $\eta_{ab} = \eta^{ab} = -\delta_{ab}$, yielding a G-invariant metric $g_{\alpha\beta} = \eta_{ab}V_\alpha{}^{a}V_\beta{}^{b}$.

A (torsionless) connection $B^a_{\ b}$ on G/H can be defined by the equation

$$dV^a + B^a_{\ b} \wedge V^b = 0 \tag{3.28}$$

Combining (3.28) with (3.26) yields

$$B^a_{\ b} = -\frac{1}{2}\frac{r_b r_c}{r_a}C_{bc}{}^{a}V^c - C_{bi}{}^{a}\Omega^i + K_{bc}{}^{a}V^c \tag{3.29}$$

where ${K_{bc}}^a$ is symmetric in b, c, and is determined by the requirement that $B^a{}_b$ be antisymmetric in a, b (Riemann connection):

$$B^a{}_c \eta^{cb} = -B^b{}_c \eta^{ca} \tag{3.30}$$

Then:

$$K_{bc}{}^a = \frac{r_a}{2} \eta^{ad} \left(\frac{r_c}{r_b} \eta_{be} C_{dc}{}^e + \frac{r_b}{r_c} \eta_{ce} C_{db}{}^e \right) \tag{3.31}$$

and the antisymmetric connection is given by:

$$B^a{}_b = \frac{1}{2} \left(-\frac{r_b r_c}{r_a} C_{bc}{}^a + \frac{r_a r_c}{r_b} \eta_{bg} C_{dc}{}^g \eta^{ad} + \frac{r_a r_b}{r_c} \eta_{cg} C_{db}{}^g \eta^{ad} \right) V^c - C_{bi}{}^a \Omega^i \tag{3.32}$$

This connection is G-invariant, meaning that parallel transport commutes with the G-action. Indeed the most general form of a G-invariant connection on G/H is given by

$$B^a{}_b(y) = C_{ib}{}^a \Omega^i(y) + J_c{}^a{}_b V^c(y) \tag{3.33}$$

where $J_d{}^a{}_b$ is an invariant tensor of the subgroup H [12], i.e. $\delta J_c{}^a{}_b = C_i{}^d{}_c J_d{}^a{}_b - C_i{}^a{}_d J_c{}^d{}_b + C_i{}^d{}_b J_c{}^a{}_d = 0$. The connection in (3.32) has this form, and it is not difficult to prove that the term multiplying V^c is H-invariant. In fact each of the three terms within parentheses in (3.32) is H-invariant, as one can show by using Jacobi identities and $\frac{r_a}{r_b} C_{ia}{}^b = C_{ia}{}^b$.

The Riemann curvature is defined in terms of $B^a{}_b$ by:

$$R^a{}_b \equiv dB^a{}_b + B^a{}_c \wedge B^c{}_b \equiv R^a{}_{b\ de} V^d \wedge V^e \tag{3.34}$$

Substituting the connection (3.32) in the curvature formula, using the Cartan-Maurer equations (3.26) and (3.27) for dV^a and $d\Omega^i$, and Jacobi identities for products of structure constants, we determine the curvature components:

$$R^a{}_{b\ de} = \frac{1}{4} \frac{r_d r_e}{r_c} \boldsymbol{C}_{bc}{}^a C_{de}{}^c + \frac{1}{2} r_d r_e\ C_{bi}{}^a C_{de}{}^i + \frac{1}{8} \boldsymbol{C}_{cd}{}^a \boldsymbol{C}_{be}{}^c - \frac{1}{8} \boldsymbol{C}_{ce}{}^a \boldsymbol{C}_{bd}{}^c \tag{3.35}$$

with

$$\boldsymbol{C}_{bc}{}^a \equiv \frac{r_b r_c}{r_a}\ C_{bc}{}^a - \frac{r_a r_c}{r_b}\ C_{ac}{}^b - \frac{r_a r_b}{r_c}\ C_{ab}{}^c \tag{3.36}$$

These formulae generalize those of ref. [3, 4] (holding only for diagonal Killing metric) and those of [11] (for unrescaled vielbeins). The connection B allows the definition of a covariant derivative D. For example the zero-torsion condition can be written as $DV^a = 0$. Taking the exterior derivative of the zero-torsion condition (3.28) and of the curvature definition (3.34) yields the Bianchi identities:

$$R^a{}_b \wedge V^b = 0 \tag{3.37}$$
$$dR^a{}_b + R^a{}_c \wedge B^c{}_b - B^a{}_c \wedge R^c{}_b \equiv DR^a{}_b = 0 \tag{3.38}$$

What are the symmetries of the indices in the curvature components $R^a{}_{b\ cd}$? Antisymmetry in a, b, and in c, d is manifest. Furthermore, from (3.37) we deduce:

$$R^a{}_{b\ cd} V^b \wedge V^c \wedge V^d = 0 \quad \Rightarrow \quad R^a{}_{[b\ cd]} = 0 \tag{3.39}$$

i.e. the cyclic identity. Using all these index symmetries one can also show that $R^{ab}{}_{cd} = \eta^{be} R^a{}_{e\ cd}$ is symmetric under $ab \leftrightarrow cd$ interchange.

4 The geometry of the N^{010} coset manifolds

We apply here the formulae of the preceding Section to the coset manifolds N^{010}. These coset spaces are a special case in the class of the N^{pqr} coset spaces defined by the quotient:

$$N^{pqr} = \frac{G}{H} = \frac{SU(3) \times U(1)}{U(1) \times U(1)} \tag{4.40}$$

where the p, q, r are integer and coprime, and specify how the two $U(1)$ generators M, N of H are embedded into G:

$$M = -\frac{\sqrt{2}}{RQ} \left(\frac{i}{2} rp\sqrt{3}\lambda_8 + \frac{i}{2} rq\lambda_3 - \frac{i}{2}(3p^2 + q^2)Y \right) \tag{4.41}$$

$$N = -\frac{1}{Q} \left(-\frac{i}{2} q\lambda_8 + \frac{i}{2} p\sqrt{3}\lambda_3 \right) \tag{4.42}$$

$$Z = -\frac{1}{R} \left(\frac{i}{2} p\sqrt{3}\lambda_8 + \frac{i}{2} q\lambda_3 + irY \right) \tag{4.43}$$

with

$$R = \sqrt{3p^2 + q^2 + 2r^2}, \qquad Q = \sqrt{3p^2 + q^2} \tag{4.44}$$

and Z is the remaining $U(1)$ generator in the coset. The generators of $G = SU(3) \times U(1)$ are taken to be $-\frac{i}{2}\lambda$ and $-\frac{i}{2}Y$, λ being the Gell-Mann matrices. For a detailed account of the geometry of these N^{pqr} coset manifolds we refer to the original papers [5, 6], where symmetric rescalings, connection and curvature are given explicitly. The cosets N^{pqr} for $p = 0, q = 1, r = 0$ have as isometry group $SU(3) \times SU(2)$ (coming from $G \times N(H)/H$). As already observed in [5], the N^{010} cosets can also be realized as:

$$N^{010} = \frac{SU(3) \times SU(2)}{SU(2) \times U(1)} \tag{4.45}$$

where the $SU(2)$ in the denominator is diagonally embedded in $G = SU(3) \times SU(2)$. In this formulation the full isometry of N^{010} comes from the left action of G. We now study the geometry of N^{010} realized as in (4.45).

The generators of $SU(3)$ and $SU(2)$ are taken respectively to be $-\frac{i}{2}\lambda$ and $-\frac{i}{2}\tau$, λ being the Gell-Mann matrices and τ the Pauli matrices, with commutation relations:

$$[\lambda_i, \lambda_j] = 2i\ f_{ijk}\lambda_k, \quad [\tau_m, \tau_n] = 2i\ \epsilon_{mnr}\tau_r \tag{4.46}$$

where the nonvanishing components of the completely antisymmetric structure constants f_{ijk} are $f_{123} = 1, f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, f_{458} = f_{678} = \frac{\sqrt{3}}{2}$.

The $U(1)$ in the denominator of (4.45) is given by the hypercharge $-\frac{i}{2}\lambda_8$. Thus the $\mathbb{H} + \mathbb{K}$ generators are:

$\mathbb{H}$-generators:

$$H_N = -\frac{i}{2}\lambda_8, \quad H_i = -\frac{i}{2}(\lambda_1 + \tau_1, \lambda_2 + \tau_2, \lambda_3 + \tau_3) \tag{4.47}$$

$\mathbb{K}$-generators:

$$K_a = -\frac{i}{2}(\lambda_1 - \tau_1, \lambda_2 - \tau_2, \lambda_3 - \tau_3), \quad K_{\dot{A}} = -\frac{i}{2}(\lambda_4, \lambda_5), \quad K_{\bar{A}} = -\frac{i}{2}(\lambda_6, \lambda_7) \tag{4.48}$$

The $\mathbb{H} + \mathbb{K}$ basis is reductive, and the nonvanishing structure constants are:

$$C_{HK}^{\ K}: \quad C_{N\dot{A}}^{\ \dot{B}} = \frac{\sqrt{3}}{2}\epsilon_{\dot{A}\dot{B}}, \quad C_{N\bar{A}}^{\ \bar{B}} = \frac{\sqrt{3}}{2}\epsilon_{\bar{A}\bar{B}}, \quad C_{ia}^{\ \ b} = \epsilon_{iab}, \quad C_{iA}^{\ \ B} = f_{iAB} \tag{4.49}$$

$$C_{KK}^{\ K}: \quad C_{aA}^{\ \ B} = f_{aAB}, \quad C_{AB}^{\ c} = \frac{1}{2}f_{ABc} \tag{4.50}$$

$$C_{KK}^{\ H}: \quad C_{ab}^{\ \ i} = \epsilon_{abi}, \quad C_{AB}^{\ N} = f_{AB8}, \quad C_{AB}^{\ i} = \frac{1}{2}f_{ABi} \tag{4.51}$$

$$C_{HH}^{\ H}: \quad C_{ij}^{\ \ k} = \epsilon_{ijk} \tag{4.52}$$

with $A = (\dot{A}, \bar{A}) = 4, 5, 6, 7$. The Killing metric in this basis is diagonal on the coset directions and on the H directions:

$$\gamma_{ab} = -5\delta_{ab}, \quad \gamma_{AB} = -3\delta_{AB} \tag{4.53}$$

$$\gamma_{NN} = 3, \quad \gamma_{ij} = -5\delta_{ij} \tag{4.54}$$

and has nondiagonal components along HK directions:

$$\gamma_{ia} = \gamma_{ai} = -\delta_{ai} \tag{4.55}$$

By inspection of the $C_{HK}^{\ K}$ structure constants we see that these are antisymmetric in the two coset indices, and that there are two isotropy-irreducible subspaces, spanned respectively by the vielbeins V^a $(a = 1, 2, 3)$ and V^A $(A = 4, 5, 6, 7)$. We can therefore construct G-invariant metrics depending on two independent rescaling parameters, $\alpha = r_a$ and $\beta = r_A$. Applying the general formulae of the preceding Section we find the Riemann connection 1-form :

$$B^a_{\ b} = -\epsilon_{bia}\Omega^i \tag{4.56}$$

$$B^a{}_{\dot{B}} = -\frac{\beta^4}{8\alpha}\left(\delta_{a1}\epsilon_{\dot{B}\bar{A}}V^{\bar{A}} + \delta_{a2}\delta_{\dot{B}\bar{A}}V^{\bar{A}} + \delta_{a3}\epsilon_{\dot{B}\dot{A}}V^{\dot{A}}\right) \tag{4.57}$$

$$B^a{}_{\bar{B}} = \frac{\beta^4}{8\alpha}\left(-\delta_{a1}\epsilon_{\bar{B}\dot{A}}V^{\dot{A}} + \delta_{a2}\delta_{\bar{B}\dot{A}}V^{\dot{A}} + \delta_{a3}\epsilon_{\bar{B}\bar{A}}V^{\bar{A}}\right) \tag{4.58}$$

$$B^{\dot{A}}{}_{\dot{B}} = \frac{1}{8\alpha}\left[(-4\alpha^2+\beta^2)V^3 - 4\alpha(\sqrt{3}\Omega^8 + \Omega^{11})\right]\epsilon_{\dot{A}\dot{B}} \tag{4.59}$$

$$B^{\bar{A}}{}_{\bar{B}} = \frac{1}{8\alpha}\left[(4\alpha^2-\beta^2)V^3 - 4\alpha(\sqrt{3}\Omega^8 - \Omega^{11})\right]\epsilon_{\bar{A}\bar{B}} \tag{4.60}$$

$$B^{\dot{A}}{}_{\bar{B}} = \frac{1}{8\alpha}\left[(-4\alpha^2+\beta^2)(\delta_{\dot{A}\bar{B}}V^2 + \epsilon_{\dot{A}\bar{B}}V^1) - 4\alpha(\delta_{\dot{A}\bar{B}}\Omega^{10} + \epsilon_{\dot{A}\bar{B}}\Omega^9)\right] \tag{4.61}$$

and the corresponding Riemann curvature components:

$$R^{ab}{}_{cd} = \alpha^2\delta^{ab}_{cd}, \quad R^{ab}{}_{\dot{A}\dot{B}} = \frac{1}{32}\gamma\delta^1_{[a}\delta^2_{b]}\epsilon_{\dot{A}\dot{B}}, \quad R^{ab}{}_{\bar{A}\bar{B}} = \frac{1}{32}\gamma\delta^1_{[a}\delta^2_{b]}\epsilon_{\bar{A}\bar{B}} \tag{4.62}$$

$$R^{ab}{}_{\dot{A}\bar{B}} = \frac{1}{32}\gamma(-\delta^1_{[a}\delta^3_{b]}\delta_{\dot{A}\bar{B}} + \delta^2_{[a}\delta^3_{b]}\epsilon_{\dot{A}\bar{B}}) \tag{4.63}$$

$$R^{a\dot{A}}{}_{b\dot{B}} = \frac{\beta^4}{128\alpha^2}\delta^a_b\delta^{\dot{A}}_{\dot{B}} + \frac{1}{64}\gamma\ \delta^1_{[a}\delta^2_{b]}\epsilon_{\dot{A}\dot{B}} \tag{4.64}$$

$$R^{a\bar{A}}{}_{b\bar{B}} = \frac{\beta^4}{128\alpha^2}\delta^a_b\delta^{\bar{A}}_{\bar{B}} - \frac{1}{64}\gamma\ \delta^1_{[a}\delta^2_{b]}\epsilon_{\bar{A}\bar{B}} \tag{4.65}$$

$$R^{a\dot{A}}{}_{b\bar{B}} = -\frac{1}{64}\gamma\ \delta^1_{[a}\delta^3_{b]}\delta^{\dot{A}}_{\bar{B}}, \quad R^{a\bar{A}}{}_{b\dot{B}} = \frac{1}{64}\gamma\ \delta^1_{[a}\delta^3_{b]}\delta^{\bar{A}}_{\dot{B}} \tag{4.66}$$

$$R^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}} = \beta^2\left(1 - \frac{3}{64}\frac{\beta^2}{\alpha^2}\right)\delta^{\dot{A}\dot{B}}_{\dot{C}\dot{D}}, \quad R^{\bar{A}\bar{B}}{}_{\bar{C}\bar{D}} = \beta^2\left(1 - \frac{3}{64}\frac{\beta^2}{\alpha^2}\right)\delta^{\bar{A}\bar{B}}_{\bar{C}\bar{D}} \tag{4.67}$$

$$R^{\dot{A}\dot{B}}{}_{\bar{C}\bar{D}} = \frac{\beta^2}{2}\delta^{\dot{A}\dot{B}}_{\bar{C}\bar{D}}, \quad R^{\dot{A}\bar{B}}{}_{\dot{C}\bar{D}} = \frac{\beta^2}{8}\left[\left(1 - \frac{3}{16}\frac{\beta^2}{\alpha^2}\right)\delta^{\dot{A}}_{\dot{C}}\delta^{\bar{B}}_{\bar{D}} + \epsilon_{\dot{A}\dot{C}}\epsilon_{\bar{B}\bar{D}}\right] \tag{4.68}$$

with $\gamma \equiv \beta^2(8 - \beta^2/\alpha^2)$. The Ricci tensor is:

$$R_{ab} = \left(\alpha^2 + \frac{1}{32}\frac{\beta^4}{\alpha^2}\right)\delta_{ab} \tag{4.69}$$

$$R_{AB} = \frac{3}{4}\beta^2\left(1 - \frac{1}{16}\frac{\beta^2}{\alpha^2}\right)\delta_{AB} \tag{4.70}$$

Note 4.1 : only the squares of the rescalings appear in the curvatures. On the other hand the connection depends on α and β^2: the sign of β has therefore no influence on the geometry, whereas different signs of α yield different spaces.

5 $AdS_4 \times N^{010}$ as compactification of $D = 11$ supergravity

As observed in the early eighties [13], a nontrivial solution of the $D = 11$ supergravity field equations is given by setting the gravitino curvature to zero, and taking the bosonic curvatures as:

$$R_{mn} = -24e^2\delta_{mn}, \quad R_{ab} = 12e^2\delta_{ab}, \quad F_{mnpq} = e\epsilon_{mnpq} \tag{5.71}$$

all other curvature components vanishing. The indices m, n, p, q run on 4-spacetime and a, b on the internal 7-dimensional space; R_{mn} and R_{ab} are the corresponding Ricci curvatures, in our conventions $R_{mn} = R^q{}_{m\ qn}$, and F_{mnpq} is the curl of the antisymmetric three-index tensor.

Then all spaces of the type $AdS_4\times$ (7-dimensional Einstein space) are a solution of the supergravity equations, Einstein space meaning a Riemannian manifold with Ricci tensor proportional to the metric. A classification of all 7-dimensional G/H Einstein manifolds was derived in the eighties in [14], thus providing a class of $D = 11$ supergravity solutions (for their use in the more recent G/H M-branes see [15]). The coset manifolds N^{010} studied in the preceding Section are part of this classification, although they were studied as particular instances of the N^{pqr} spaces, in the $SU(3) \times U(1)$ - isometric formulation. Two inequivalent Einstein metrics were found, and the corresponding Einstein spaces were denoted by N_I^{pqr} and N_{II}^{pqr} [5, 7, 6].

What can we say about Einstein metrics in the N^{010} cosets discussed in this paper ? As easily seen from the expression of the Ricci tensor in (4.69), (4.70) the rescalings

$$\alpha^2 = 4e^2, \quad \beta^2 = 32e^2 \tag{5.72}$$

or

$$\alpha^2 = \frac{100}{9}e^2, \quad \beta^2 = \frac{160}{9}e^2 \tag{5.73}$$

both bring the Ricci tensor in the Einstein form $R_{ab} = 12e^2\delta_{ab}$. We denote by N_I^{010} and N_{II}^{010} the corresponding Einstein coset spaces, since these coincide with the N_I^{pqr} and N_{II}^{pqr} for $p = 0, q = 1, r = 0$, as one can easily prove by comparing the Riemann curvatures.

Finally, we can investigate the supersymmetry content of the $AdS_4 \times N$ compactifications. We recall that the independent supersymmetry charges preserving the $AdS_4 \times N$ vacuum are in 1-1 correspondence with the number of spinors η satisfying the equation:

$$(d + \frac{1}{4}B^{ab}\Gamma_{ab} - eV^a\Gamma_a)\eta = 0 \tag{5.74}$$

which is just the requirement that the supersymmetry variation of the gravitino vanishes in the $AdS_4 \times N$ background (see for ex. [16, 4]). The integrability condition for (5.74) is

$$(R^{cd}{}_{ab} + 4e^2\delta^{cd}_{ab})\Gamma_{cd}\eta = 0 \tag{5.75}$$

Substituting into (5.75) the Riemann tensor of eqs. (4.62)-(4.68) with the rescalings $\alpha^2 = 4e^2, \quad \beta^2 = 32e^2$ yields four independent spinors η satisfying (5.75), while for the rescalings $\alpha^2 = \frac{100}{9}e^2, \quad \beta^2 = \frac{160}{9}e^2$ only one spinor η exists. Then one has to check whether these spinors also satisfy (5.74). Whereas the sign of α is irrelevant in the integrability condition (since the Riemann curvature does not depend on it), it becomes important in the supersymmetry variation (5.74), and we find the following:

$$N_I^{010}: \quad \alpha = 2e, \quad \beta = \pm 4\sqrt{2}\, e, \qquad N = 3 \text{ supersymmetry} \tag{5.76}$$
$$\tilde{N}_I^{010}: \quad \alpha = -2e, \ \beta = \pm 4\sqrt{2}\, e, \qquad N = 0 \text{ supersymmetry} \tag{5.77}$$
$$N_{II}^{010}: \quad \alpha = -\frac{10}{3}e, \ \beta = \pm\frac{4}{3}\sqrt{10}\, e, \quad N = 1 \text{ supersymmetry} \tag{5.78}$$
$$\tilde{N}_{II}^{010}: \quad \alpha = \frac{10}{3}e, \ \beta = \pm\frac{4}{3}\sqrt{10}\, e, \qquad N = 0 \text{ supersymmetry} \tag{5.79}$$

where we have denoted by $\tilde{N}$ the spaces obtained by reversing the orientation of N^{010}, i.e. by taking $V^a \to -V^a$ or equivalently $\alpha, \beta \to -\alpha, -\beta$. Thus changing signs in α is equivalent to reverse the orientation, since the sign of β has no influence on the geometry.

Note 5.1: in Ref. [8] the N_I^{010} space corresponds to the rescaling $\alpha = -2e$, due to a sign difference in the structure constants of G.

Note 5.2: the Killing spinors satisfying eq. (5.74) are *not* constant in the $SU(3) \times SU(2)/SU(2) \times U(1)$ realization of the N^{010} spaces. On the other hand they are constant in the $N^{pqr} = SU(3) \times U(1)/U(1) \times U(1)$ spaces of [5], where for $p = 0, q = 1$ the isometry is promoted to $SU(3) \times SU(2)$ because of the right action of $N(H)/H = SU(2)$.

References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1988) 231;

S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett **B428** (1988) 105;

E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. *2* (1988) 253;

O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, hep-th/9905111.

[2] A. Ceresole, G. Dall' Agata, R. D' Auria and S. Ferrara, *M-theory on the Stiefel manifold and 3d conformal field theories*, hep-th/9912107.

[3] L. Castellani, L.J. Romans and N.P. Warner, *Symmetries of coset spaces and Kaluza-Klein supergravity*, Annals of Phys. **157** (1984) 394.

[4] L. Castellani, R. D' Auria and P. Fre', *Supergravity and Superstrings: a geometric perspective*, World Scientific, Vol.s, 1,2,3, Singapore 1991.

[5] L. Castellani and L.J. Romans, *N=3 and N=1 supersymmetry in a new class of solutions for D=11 supergravity*, Nucl. Phys. **B238** (1984) 683.

[6] L. Castellani, *The mass spectrum in the $SU(3) \times U(1)$ compactifications of D=11 supergravity*, Nucl. Phys. **254** (1985) 266.

[7] D.N. Page and C.N. Pope, *New squashed solutions of d=11 supergravity*, Phys. Lett. **147B** (1984) 55.

[8] P. Termonia, *The complete N=3 Kaluza Klein spectrum of 11D supergravity on $AdS_4 \times N^{010}$*, hep-th/9909137.

[9] P. Fre', L. Gualtieri and P. Termonia, *The structure of N=3 multiplets in AdS_4 and the complete $OSp(3|4) \times SU(3)$ spectrum of M-theory on $AdS_4 \otimes N^{010}$* , hep-th/9909188.

[10] C. Ahn and S.J. Rey, *More CFTs and RG flows from deforming M2/M5-brane horizon*, hep-th/9911199.

[11] P. van Nieuwenhuizen, *General theory of coset manifolds and antisymmetric tensors applied to Kaluza-Klein supergravity*, in the Proceedings of the Trieste Spring School 1984, Eds. B. de Wit, P. Fayet, P. van Nieuwenhuizen, World Scientific 1984.

[12] P. van Nieuwenhuizen, *An introduction to simple supergravity and the Kaluza-Klein program*, eds. B.S. DeWitt and R. Stora, Les Houches, Session XL, Elsevier 1984.

[13] P.G.O. Freund and M.A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97** (1980) 233.

[14] L. Castellani, L.J. Romans and N.P. Warner, *A classification of compactifying solutions for D=11 supergravity*, Nucl. Phys. **B241** (1984) 429.

[15] L. Castellani, A. Ceresole, R. D' Auria, S. Ferrara, P. Fre' and M. Trigiante, *G/H M-branes and AdS(p+2) geometries*, Nucl. Phys. **B527** (1998) 142, hep-th/9803039.

[16] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza Kein supergravity*, Phys. Rep. **130** (1986) 1.