The (secret?) homological algebra of the Batalin-Vilkovisky approach

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In memory of Chih-Han Sah 1934-1997

Abstract. After a brief history of 'cohomological physics', the Batalin-Vilkovisky complex is given a revisionist presentation as homological algebra, in part classical, in part novel. Interpretation of the higher order terms in the extended Lagrangian is given as higher homotopy Lie algebra and via deformation theory. Examples are given for higher spin particles and closed string field theory.

Прежде всего, я хотел бы поблагодарить организаторов этой конференции, в первую очередь, Михаила Михайловича и Иосифа Семеновича. Я рад встретиться с русскими и украинскими математиками и физиками. Я надеюсь, что в будущем многие из вас смогут посетить Чепл Хил.

Сегодня я хотел бы обсудить связи между нашими работами на Западе и вашими работами в России, особенно работами Баталина и Вилковыского. Я думаю, что здесь я, возможно, буду рассказывать вам как использовать самовар.

Каждый любит язык своей страны. Поэтому я буду говорить по-английски.

The following exposition is based in large part on work with Glenn Barnich (ITP, Berlin and Université Libre de Bruxelles) and Tom Lada and Ron Fulp of NCSU (The Non-Commutative State University). It is closely related to several other talks at this conference.

I am particularly happy to see 'Cohomological Physics' in the title of this conference. I first referred to cohomological physics in the context of anomalies in gauge theory, cf. my work with Bonora, Cotta-Ramusino and Rinaldi [**BCRRS87, BCRRS88**], more than a decade ago. (Some people thought that phrase was a bit much.) The cohomology referred to there was that of differential forms (the de Rham complex). Differential forms were implicit in physics at least as far back as

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Gauss (1833) (cf. his electro-magnetic definition of the linking number [**Gau77**]), and more visibly in Dirac's magnetic monopole (1931) [**Dir31**], which lived in a $U(1)$ bundle over $\mathbf{R}^3 - 0$. The magnetic charge was given by the first Chern number; for magnetic charge 1, the monopole lived in the Hopf bundle, introduced that same year by Hopf [**Hop31**], though it seems to have taken some decades for that coincidence to be recognized [**GP75**]. Thus were characteristic classes (and by implication the cohomology of Lie algebras and of Lie groups) secretly introduced into physics.

Cohomological physics had a major breakthrough with the 'ghosts' introduced by Fade'ev and Popov [**FP67**]. These were incorporated into what came to be known as BRST cohomology (Becchi-Rouet-Stora [**BRS75**] and, here in Russia, Tytutin [**Tyu75**]) and which was applied to a variety of problems in mathematical physics. There the ghosts were reinterpreted by Stora [**Sto77**] and others in terms of the Maurer-Cartan forms in the case of a finite dimensional Lie group and more generally as generators of the Chevalley-Eilenberg complex [**CE48**] for Lie algebra cohomology.

Group theoretic cohomology had already appeared in the work of Bargmann [**Bar47, Bar54**] on extensions of the Galilean, Lorentz and de Sitter groups.

Although I was unaware of it at the time, deformation theory (cf. Gerstenhaber [**Ger62**]) also provided a bridge between the kind of homotopy theory I was doing and mathematical physics via deformation quantization (Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer [**BFF$^+$78a, BFF$^+$78b**]), though the cohomological aspects were of minor importance in that application originally.

What I did become aware of next, thanks to Henneaux [**Hen85**] and Browning and McMullen [**BM87**], was the cohomological reduction of constrained Poisson algebras, specifically the Batalin-Fradkin-Vilkovisky approach [**BF83, FF78, FV75**], which extended BRST by reinventing the Koszul-Tate resolution of the ideal of constraints and producing a synergistic combination of both kinds of cohomology. Here it was that I saw the esential features of a strong homotopy Lie algebra (L_∞-algebra).

On the other hand, cohomology was the essence of the variational bicomplex approach to variational problems (Vinogradov [**Vin84**],Tsujishita [**Tsu82**], Anderson [**And92, And96**]) in which Lagrangians and the Euler-Lagrange equations were central. Here we already encounter cohomological physics in the guise of differential forms on the jet bundle. As I learned here at this conference, Krasil'shchik [**Kra92**] has developed the relation of deformation theory to the variational bicomplex approach. An essential feature is the rich algebraic structure of various generalizations of Poisson brackets, including Schouten, Schouten-Nijehnhuis, Gerstenhaber and Nijenhuis-Richardson brackets.

The Batalin-Vilkovisky machinery for Lagrangian field theory is often presented in terms of fields which are functions on some manifold or sections of some bundle (cf. Henneaux's talks at this conference). The alternative approach which I will adopt for this conference reworks the machinery using the jet bundle approach. For years I've been tantalized by the idea of combining the BV machinery with the variational bicomplex and had originally planned to talk on that here, but instead, with my coauthors (Barnich, Fulp and Lada), I have been concentrating on elaborating the BV approach to Lagrangian field theory, especially the higher homotopy aspects. We work with this in terms of the Euler-Lagrange complex which occurs at the edges of the variational bicomplex . I will make some remarks about the combination of the anti-field, anti-bracket machinery with the variational bicomplex ; such a combination will be directly relevant in the context of constraints and/or symmetries. Such a combination has already appeared in works of McCloud [**McC94**] and Barnich-Henenaux [**BH96**].

Hopefully this will inspire more of you to follow this route in delving further into the variational bicomplex .

Thus we see a rather intricate interweaving of several kinds of cohomology being brought to bear on problems in physics. I will not try to describe the whole web, but rather will follow one strand with just a few comments as others intersect it. The Batalin-Vilkovisky approach [**BV83, BV84, BV85**] to quantizing particle Lagrangians and Lagrangians of string field theory involves the rubric of anti-fields as well as ghosts and an 'anti-bracket', first introduced essentially by Zinn-Justin [**ZJ75, ZJ93**] in another notation. A revisionist view of the Batalin-Vilkovisky machinery recognizes parts of it as a reconstruction of homological algebra [**HT92**] with some powerful new ideas undreamt of in that discipline. The 'standard construction' is the Batalin-Vilkovisky complex, and again we see the signature of L_∞-algebras.

The 'quantum' Batalin-Vilkovisky master equation has the form of the Maurer-Cartan equation for a flat connection, while the 'classical' version has the form of the integrability equation of deformation theory. Just as the Maurer-Cartan equation makes sense in the context of Lie algebra cohomology, so the Batalin-Vilkovisky master equation has an interpretation in terms of L_∞-algebras. Particularly interesting examples are provided by Zwiebach's closed string field theory [**Zwi93**] and higher spin particles [**Bur85, BBvD84**].

Contents

1. The jet bundle setting for Lagrangian field theory

Let us begin with a space Φ of **fields** regarded as the space of sections of some bundle $\pi : E \to M$. For expository and coordinate computational purposes, I will assume E is a trivial vector bundle and will write a typical field as $\phi = (\phi^1, \ldots, \phi^k) : M \to \mathbf{R}^k$. In terms of local coordinates, the base manifold M is locally $\mathbf{R}^n$ with coordinates $x^i, i = 1, \ldots, n$ and the fibre is $\mathbf{R}^k$ with coordinates $u^a, a = 1, \ldots, k$. We 'prolong' this bundle to create the associated jet bundle $J = J^\infty E \to E \to M$ which is an infinite dimensional vector bundle with coordinates u^a_I where $I = i_1 \ldots i_r$ is a symmetric multi-index (including, for $r = 0$, the empty set of indices, meaning just u^a). The notation is chosen to bring to mind the mixed partial derivatives of order r. Indeed, a section of J is the (infinite) jet $j^\infty\phi$ of a section ϕ of E if, for all r, we have $u^a_I \circ j^\infty\phi = \partial_{i_1}\partial_{i_2}...\partial_{i_r}\phi^a$ where $\phi^a = u^a \circ \phi$ and $\partial_i = \partial/\partial x^i$.

Definition 1.1. A **local function** $L(x, u^{(p)})$ is the pullback of a smooth function on some finite jet bundle J^pE, i.e. a composite $J \to J^pE \to \mathbf{R}$. In local coordinates, it is a smooth function of the x^i and the u^a_I, where the order $|I| = r$ of the multi-index I is less than or equal to some integer p. The **space of local functions** will be denoted $Loc(E)$.

Definition 1.2. A **local functional**

$$\mathcal{L}[\phi] = \int_M L(x, \phi^{(p)}(x))dvol_M = \int_M (j^\infty\phi)^* L(x, u^{(p)})dvol_M \tag{1.1}$$

is the integral over M of a local function evaluated for sections ϕ of E. (Of course, we must restrict M and ϕ or both for this to make sense.)

The variational approach is to seek the critical points of such a local functional. More precisely, we seek sections ϕ such that $\delta\mathcal{L}[\phi] = 0$ where δ denotes the variational derivative corresponding to an 'infinitesimal' variation: $\phi \mapsto \phi + \delta\phi$. For sections of compact support, the condition $\delta\mathcal{L}[\phi] = 0$ is equivalent to the Euler-Lagrange equations on the corresponding local function L as follows: Let

$$D_i = \frac{\partial}{\partial x^i} + u^a_{Ii}\frac{\partial}{\partial u^a_I} \tag{1.2}$$

be the total derivative acting on local functions (note that $D_i u^a_I = u^a_{Ii}$) and

$$E_a = (-D)_I \frac{\partial}{\partial u^a_I} \tag{1.3}$$

the Euler-Lagrange derivative. (Summation over repeated indices, one up, one down, is understood.) The notation $(-D)_I$ means

$$\frac{\partial}{\partial u^a} - \partial_i \frac{\partial}{\partial u^a_i} + \partial_i\partial_j \frac{\partial}{\partial u^a_{ij}} - .... \tag{1.4}$$

The Euler-Lagrange equations are then

$$E_a(L) = 0. \tag{1.5}$$

Since $\mathcal{L}$ is the integral of an n-form on J, it is not surprising that this all makes sense in the deRham complex $\Omega^*(J)$, which remarkably splits as a bicomplex (though the finite level complexes $\Omega^*(J^pE)$ do not) [**Vin84, Tsu82, And96**]. The appropriate 1-forms in the fibre directions are not the du^a_I but rather the

contact forms $\theta^a_I = du^a_I - u^a_{Ii}dx^i$. A typical basis element of $\Omega^{p,q}(J)$ is of the form $f dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge\theta^{a_1}_{J_1}\wedge\cdots\wedge\theta^{a_q}_{J_q}$ where $f \in Loc(E)$. The total differential d splits as $d = d_H + d_V$ where $d_H = dx^i D_i : \Omega^{p,q} \to \Omega^{p+1,q}$ and $d_V : \Omega^{p,q} \to \Omega^{p,q+1}$. We will henceforth restrict the coefficients of our forms to be local functions, although we will not decorate $\Omega^*(J)$ to show this.

The Euler-Lagrange derivatives assemble into an operator on forms on J:

$$E(L\ dvol_M) = E_a(L) dvol_M \wedge \theta^a \tag{1.6}$$

where we will usually take $dvol_M = d^n x := dx^1 \cdots dx^n$.

The kernel of the Euler-Lagrange derivatives is given by total divergences,

$$E_a(L) = 0 \text{ for all } a = 1, \ldots, k \Longleftrightarrow L = (-1)^k D_i j^i \tag{1.7}$$

for some local functions j^i. Equivalently,

$$E_a(L) = 0 \text{ for all } a \Longleftrightarrow L(x, u^{(p)}) dvol_M = (-1)^{i-1} d_H j^i dvol_M / dx^i \tag{1.8}$$

where if $dvol_M = dx^1 \cdots dx^n$, then $dvol_M/dx^i = dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^n$.

A Lagrangian $\mathcal{L}$ determines a **stationary surface** or **solution surface** or **shell** $\Sigma \subset J^\infty$ such that ϕ is a solution of the variational problem (equivalently, of the Euler-Lagrange equations) iff $j^\infty\phi$ has its image in Σ. The corresponding algebra of local functions, $Loc(\Sigma)$ is isomorphic to the quotient $Loc(E)/\mathcal{I}$ where the **stationary ideal** $\mathcal{I}$ consists of local functions which vanish 'on shell', i.e. when restricted to the solution surface Σ. We assume enough regularity that this ideal is generated by the local functions $D_I E_a(L)$.

2. Anti-fields as Koszul generators

To present my revisionist view of the Batalin-Vilkovisky complex, let me take you 'through the looking glass' and present a 'bi-lingual' (math and physics) dictionary.

From here on, we will talk in terms of algebra extensions of $Loc(E)$, but the extensions will all be free graded commutative. We could instead talk in terms of an extension of E or J as a super-manifold, the new generators being thought of as (super)-coordinates.

We extend $Loc(E)$ by adjoining generators of various degrees to form a free graded commutative algebra $\mathcal{BV}$ over $Loc(E)$, that is, even graded generators give rise to a polynomial algebra and odd graded generators give rise to a Grassmann (= exterior) algebra. The generators (and their products) are, in fact, bigraded (p, q); the graded commutativity is with respect to the total degree $p - q$.

We begin by adjoining, for each variable u^a_I, an 'anti-field' u^*_{aI} of bidegree $(0, 1)$ (again interpret the u^*_{aI} as the formal derivatives of the u^*_a) and thus form the Koszul complex

$$\mathcal{K} = Loc(E) \otimes \Lambda u^*_{aI}$$

where Λ denotes the free graded commutative algebra, with the derivation differential δ determined by

$$\delta u^*_{aI} = D_I E_a(L)$$

so that $H^0(\mathcal{K}) \simeq Loc(E)/\mathcal{I}$.

Notice that we could interpret the pairing of u^a with u^*_a as giving rise to a Poisson-like bracket on $\mathcal{K}$; this will be carried out in section 4 and called the 'anti-bracket' because of the signs that intervene. The Koszul complex will then play a role analogous to that of the basic Poisson algebra in the Batalin-Fradkin-Vilkovisky approach to the cohomological reduction of Hamiltonian systems with first class constraints [**Sta97, Kje96**].

3. Noether identities and Tate generators

The (left hand sides of the) Euler-Lagrange equations generate $\mathcal{I}$ as a differential ideal, that is, as an ideal over the ring of total differential operators, but this means that we may have **Noether identities** not only of the form

$$r^a E_a(L) = 0 \tag{3.1}$$

but also

$$r^{aI} D_I E_a(L) = 0, \tag{3.2}$$

where $r^{aI} \in Loc(E)$. Of course we have 'trivial' identities of the form

$$D_J E_b(L) \mu_\alpha^{bJaI} D_I E_a(L) = 0, \tag{3.3}$$

with μ_α^{bJaI} skew-commutative in the index pairs bJ and aI since we are dealing with a commutative algebra of functions.

We now assume we have a set $\{\alpha\}$ of indices such that the corresponding identities

$$r_\alpha := r_\alpha^{aI} D_I E_a(L) = 0, \tag{3.4}$$

with $r_\alpha^{aI} \in Loc(E)$, generate all the non-trivial relations in $\mathcal{I}$, as a differential ideal, that is, over the algebra of total differential operators $s^I D_I$. According to Noether [**Noe18**], each such identity corresponds to a family of **infinitesimal gauge symmetries** depending on arbitrary local functions in $Loc(E)$, i.e. infinitesimal variations that preserve the space of solutions up to total divergences, or, equivalently, vector fields tangent to Σ. (Noether considers only the dependence on functions in $Loc(M)$.) For each Noether identity indexed by α, we denote the corresponding family of vector fields by $\delta_\alpha(f)$. We denote by Ξ, the **space of gauge symmetries**, considered as a vector space but also as a module over $Loc(E)$.

For electricity and magnetism and, more generally, Yang-Mills theory, the fields are the components of the gauge potential, i.e. of a Lie algebra valued connection for a G-bundle over $M = Space-Time$. In local coordinates, $A = A_\mu^a T_a dx^\mu$ where the T_a form a basis for the Lie algebra $\mathfrak{g}$. The Lagrangian is $L = tr(F \wedge *F)$ where F is the curvature (field strength) of A, $*$ is the Hodge operator with respect to a Lorentz metric on the 4-dimensional M and tr denotes trace. The corresponding Euler-lagrange equations are (the components of)

$$D_A * F = 0,$$

where D_A is the covariant derivative corresponding to the connection A. The Noether identities are (the components of)

$$D_A D_A * F = 0$$

. The corresponding *infinitesimal* gauge symmetries are

$$A \mapsto A + d_A f$$

where f is an arbitrary $\mathfrak{g}$ valued function on M. In local coordinates,

$$A_\mu^a \mapsto A_\mu^a + \partial_\mu f^a + C_{bc}^a A_\mu^b f^c.$$

The existence of non-trivial Noether identities implies that $H^1(\mathcal{K})$ is not 0. To get rid of this unwanted cohomology, Tate [**Tat57**] directs us to adjoin further generators: for each r_α, adjoin a corresponding 'anti-ghost' C^*_α of bidegree $(0,2)$ and its 'derivatives' $C^*_{\alpha I}$ with

$$\delta C^*_{\alpha I} = D_I r^{aJ}_\alpha u^*_{aJ}.$$

In the reducible case, adjoin further variables of bidegree $(0,q)$ so that ultimately the resulting Koszul-Tate complex $\mathcal{KT}$ has

$$H^0(\mathcal{KT}) = Loc(E)/\mathcal{I} \tag{3.5}$$

$$H^i(\mathcal{KT}) = 0 \text{ for } i \neq 0. \tag{3.6}$$

4. Ghosts and the anti-bracket

Now for each r_α, further adjoin a corresponding ghost C^α and derivatives C^α_I of bidegree $(1,0)$. For all these extended jet variables, we again have the obvious actions of the D_i, e.g. $D_i C^\alpha_I = C^\alpha_{Ii}$.

The resulting algebra, due to Batalin and Vilkovisky, we denote $\mathcal{BV}$. Here is a table showing the relevant math terms and the bidegrees.

Physics Term	Math Term	Ghost Degree	Anti-ghost Degree	Total Degree
field	section	0	0	0
anti-field	Koszul generator	0	1	-1
ghost	Chevelley-Eilenberg generator	1	0	1
anti-ghost	Tate generator	0	2	-2

Note that the anti-field coordinates depend on E but the ghosts and anti-ghosts depend also on the specific Lagrangian.

Following Zinn-Justin and Batalin-Vilkovisky, this algebra in turn can be given an **anti-bracket** $(\ ,\ )$ of degree -1 which, remarkably, combines with the product we began with to produce a very strong analog of a Gerstenhaber algebra (see section 9), although this was not recognized until quite recently. (Gerstenhaber's first example occurs in [**Ger62**] but the name is more recent [**LZ93, KVZ96**].) The anti-bracket is so called because of the degree shift and the corresponding signs for the symmetry under interchange of the two entries (cf. the Whitehead product in homotopy theory). In the original field-anti-field formalism, the anti-bracket was often expressed in terms of Dirac delta-functions. In the jet bundle setting, the anti-bracket can be defined as follows: First, to simplify signs, physicists use both left and right graded derivations of graded algebras. By this is meant that θ is a left derivation if $\theta(ab) = \theta(a)b + (-1)^{|\theta||a|} a\theta(b)$ and ρ is a right derivation if $\rho(ab) = a\rho(b) + (-1)^{|\rho||b|}\rho(a)b$. In keeping with the tradition in algebraic topology, we will not decorate left derivations, but will indicate the corresponding right derivation by $\bar\theta$ defined by $\bar\theta(a) = (-1)^{|\theta||a|}\theta a$. Now define the anti-bracket $(\ ,\ ) : \mathcal{BV} \otimes \mathcal{BV} \to \mathcal{BV}$ by

$$(A,B) = E_a(A)\bar{E}^a_*(B) - E^a_*(A)\bar{E}_a(B) + E_\alpha(A)\bar{E}^\alpha_*(B) - E^\alpha_*(A)\bar{E}_\alpha(B).$$

Here the extended Euler-Lagrange derivatives $E^a_*, E_\alpha, E^\alpha_*$ are defined by the formal analogs for the variables $u^*_a, C^\alpha, C^*_\alpha$ of the formula for E_a. The only non-zero anti-brackets of generators are

$$(u^a, u^*_b) = \delta^a_b \quad \text{and} \quad (C^\alpha, C^*_\beta) = \delta^\alpha_\beta,$$

and $(A,\ )$ acts as a graded derivation.

In light of this odd symplectic pairing, it is $\mathcal{K}$ rather than $Loc(E)$ that plays the role of the basic Poisson algebra in the homological reduction of constrained Hamiltonian systems, though here it is an odd analog.

5. The Batalin-Vilkovisky complex $(\mathcal{BV}, s_\infty)$

Batalin and Vilkovisky use the anti-bracket to make their algebra into a differential graded algebra using the anti-bracket and the Lagrangian as follows:

Define an operator s_0 of degree -1 on $\mathcal{BV}$ as $(L,\ )$.

This reproduces the Koszul complex in that

$$s_0 u_a^* = \frac{\delta L}{\delta u^a} \tag{5.1}$$

as in the Koszul complex for the ideal, so that $H^{0,0} \subset Loc(E)$ is given by $\frac{\delta L}{\delta u^a} = 0$, but

$$s_0(r_\alpha^a u_a^*) = r_\alpha^a \frac{\delta L}{\delta u^a} = 0, \tag{5.2}$$

so that $H^{0,1} \neq 0$.

Now consider the extended Lagrangian $L_0 + L_1$ with

$$L_1 = u_a^*(-D)_I[r_\alpha^{aI} C^\alpha] \tag{5.3}$$

and $s_1 = (L_0 + L_1,\ )$, so that

$$s_1 C_\alpha^* = u_a^* r_\alpha^a \tag{5.4}$$

reproducing the Koszul-Tate differential. To begin to capture the Chevalley-Eilenberg differential, further extend $L_0 + L_1$ by adding a term

$$L_2 = C_\alpha^* f_{\beta\gamma}^{IJ\alpha} D_I C^\beta D_J C^\gamma, \tag{5.5}$$

where the f represent a bidifferential operator. In the special case in which the $f_{\beta\gamma}^\alpha$ and the r_α^a are constants, we have, for $s_2 = (L_0 + L_1 + L_2,\ )$,

$$s_2 C^\alpha = f_{\beta\gamma}^\alpha C^\beta C^\gamma$$
$$s_2 u^a = r_\alpha^a C^\alpha,$$

which is how the Chevalley-Eilenberg coboundary looks in terms of bases for a Lie algebra and a module and corresponding structure constants. However, in general, we may not have $(s_2)^2 = 0$ since r_α^a and $f_{\beta\gamma}^\alpha$ are, in general, functions. Batalin and Vilkovisky prove that all is not lost:

THEOREM 5.1. *$L_0 + L_1 + L_2$ can be further extended by terms of higher degree in the ghosts to L_∞ so that $(L_\infty, L_\infty) = 0$ and hence the corresponding s_∞ will have square zero.*

With hindsight, we can see that the existence of these terms of higher order is guaranteed because the antifields and antighosts provide a resolution of the stationary ideal. The pattern of the proof is one standard in Homological Perturbation Theory [**Gug82, GM74, GS86, GLS90, Hue84, HK91**]: If $s_n^2 \neq 0$, at least $\delta s_n^2 = 0$, so by the acyclicity of $\mathcal{KT}$, there exists L_{n+1} as desired. (See [**Sta97, Kje96**] for details in the corresponding Hamiltonian case.)

We refer to this complex $(\mathcal{BV}, s_\infty)$ as the **Batalin-Vilkovisky complex**. The differential $s = s_\infty$ is sometimes called the BRST operator and its cohomology the BRST cohomology. Physicists tend to write $H^*(s)$ where mathematicians would write $H^*(\mathcal{BV})$.

What is the Batalin-Vilkovisky complex computing, that is, what do the groups H^i mean for $i > 0$? Given sufficient regularity conditions on the Euler-Lagrange equations $E_a(L) = 0$, there is a nice geometric answer [**HT92, Bar95**], again analogous to the Hamiltonian case. The stationary surface Σ is foliated by the gauge orbits of the gauge symmetries. The standard complex for the longitudinal cohomology, i.e. the de Rham cohomology using all of $Loc(\Sigma)$ but differential forms only along the leaves (gauge orbits) of the foliation is given by $Alt_{Loc_E/\mathcal{I}}(\Xi, Loc_E/\mathcal{I})$ with the Chevalley-Eilenberg differential. In the regular case, the Batalin-Vilkovisky complex has replaced $Loc_E/\mathcal{I}$ by the Koszul-Tate resolution and Ξ by a corresponding resolution.

6. The BV version of the variational bicomplex

In the presence of a fixed Lagrangian with or without gauge symmetries, there are various ways of combining the variational bicomplex with the BV approach. The simplest is to start with the usual variational bicomplex and adjoin anti-fields, ghosts and anti-ghosts, etc., just as we did to $Loc(E)$. The exterior derivative acts in the variational bicomplex as before and trivially on the new variables. A more subtle way is to treat the new variables $u^*_a, C^\alpha, C^*_\alpha$ as true coordinates on a super-manifold $E\times \mathbf{R}^{j|j+k}$ with corresponding jet coordinates. Then we would have the corresponding variational bicomplex , meaning that we would have differentials du^*_a, etc. and corresponding contact forms.

An intermediate alternative is to consider only u^*_a and C^*_α as super-coordinates and leave the ghosts as just Chevalley-Eilenberg generators for computing the equivariant cohomology for the appropriate Lie algebra or L_∞-algebra.

This intermediate alternative should be compared with the techniques of rational (or real) homotopy theory (which have also made an appearance in the physics literature, for example in Vasiliev[**Vas89, Vas91**]). There one would construct a 'model' of $\Omega^*(\Sigma)$ of the form $\Omega^*(J)\otimes \Lambda Z$ for a graded vector space Z determined essentially by the same process as the Koszul-Tate resolution for $Loc(\Sigma)$.

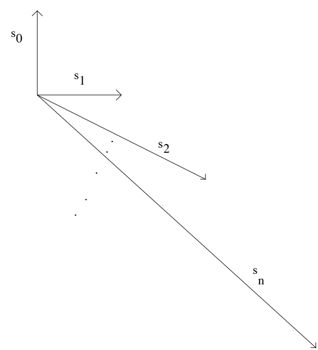
Question: Does the complex $\Omega^*(J)\otimes\Lambda(u^*_{aI}, C^*_{\alpha I})$ give a real homtopy theory model for $\Omega(\Sigma)$?

To date, none of these alternatives have been completely fleshed out, though certain computations in the physics literature may allow for one or another of these interpretations [**McC94, BH96**].

What is the significance of $(s_\infty)^2 = 0$ in our Lagrangian context or, equivalently, of the **Master Equation** $(L_\infty, L_\infty) = 0$? There are three answers: in higher homotopy algebra, in deformation theory and in mathematical physics.

7. The Master Equation and Higher Homotopy Algebra

As Henneaux has emphasized, all the structure of our problem - the Noether identities, the gauge symmetries, the reducibilities, etc. - have all been encoded in s_∞. If we expand $s = s_0 + s_1 + ...$, where the subscript indicates the change in the ghost degree p, the individual s_i do not correspond to $(L_j,\ )$ for any term L_j in L_∞ but do have the following graphical description in terms of the bigrading (p, q):



so that we see the BV-complex as a multi-complex. The differential s_0 gives us the Koszul differential δ, while s_1 gives us the Koszul-Tate δ and part of s_2 looks like that of Chevalley-Eilenberg. That is, $C^*_\alpha f^\alpha_{\beta\gamma} C^\beta C^\gamma$ describes a (not-necessarily-Lie) bracket. Indeed, if the structure functions for L_1 and L_2 are in fact constants, we could have exactly the Chevalley-Eilenberg cochain complex for a Lie algebra with coefficients in a module; in our case, the module is the Koszul-Tate resolution. But for structure functions, we need the terms of higher order. What is the significance of, for example, terms with one anti-field u^*_a, one anti-ghost C^*_α and three ghosts $C^\beta C^\gamma C^\delta$? Such terms describe a tri-linear [, ,] on gauge symemtries and so on for multi-brackets of possibly arbitrary length. Moreover, the graded commutativity of the underlying algebra of the BV-complex implies appropriate symmetry of these multi-brackets.

It is here that I began to see the shadow of a **strong homotopy Lie algebra (sh Lie algebra)** or L_∞ **algebra** [**LS93**].

A Lie algebra is a vector space V with a skew map $V \otimes V \to V$ satisfying the Jacobi identity; this is equivalent to a coderivation D of degree -1 on the graded commutative *co*algebra on V regarded as in degree 1 with $D^2 = 0$.

A graded Lie algebra is a graded vector space $V = \oplus V_i$ with graded skew maps $V_i \otimes V_j \to V_{i+j}$ satisfying the Jacobi identity with appropriate signs; this can be encoded as a coderivation D of degree -1 on the graded commutative coalgebra $\Lambda \uparrow V$ on $\uparrow V$ with $D^2 = 0$ (where $\uparrow V$ denotes the graded vector space obtained from$V$ by shifting the grading up by 1).

Certain aspects of the algebra we are concerned with are much easier to handle after this shift and extension to the graded symmetric algebra. Vasiliev has begun the study of higher spin interactions at this level [**Vas89, Vas91**]. Since the Batalin-Vilkovisky algebra includes a coderivation s on a graded commutative coalgebra, the condition that $s^2_\infty = 0$ translates to include the following identities, which are the defining identities for an L_∞-algebra [**LS93**]:

$$d[v_1, \dots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \dots, dv_i, \dots, v_n]$$

$$= \sum_{p+q=n+1} \sum_{\text{unshuffles } \sigma} \epsilon(\sigma)[[v_{i_1}, \dots, v_{i_p}], v_{j_1}, \dots, v_{j_q}], \tag{7.1}$$

where $\epsilon(i) = (-1)^{|v_1|+\cdots+|v_{i-1}|}$ is the sign picked up by taking d through $v_1, \dots, v_{i-1}$ and, for the unshuffle $\sigma : \{1, 2, \dots, n\} \mapsto \{i_1, \dots, i_k, j_1, \dots, j_l\}$ with $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$, the sign $\epsilon(\sigma)$ is the sign picked up by the v_i passing through the v_j's during the unshuffle of $v_1, \dots, v_n$, as usual in superalgebra.

Realized in the Batalin-Vilkovisky complex, these defining identities tell us, for small values of n, that d_{KT} is a graded derivation of the bracket, that the bracket may not satisfy the graded Jacobi identity but that we do have (with the appropriate signs)

$$[[v_1, v_2], v_3] \pm [[v_1, v_3], v_2] \pm [[v_2, v_3], v_2] = \tag{7.2}$$
$$-d[v_1, v_2, v_3] \pm [dv_1, v_2, v_3] \pm [v_1, dv_2, v_3] \pm [v_1, v_2, dv_3].$$

i.e. the Jacobi identity holds *up to homotopy* or, for closed forms, the Jacobi identity holds modulo an exact term given by the tri-linear bracket.

Note that the identity is the Jacobi identity if $d_{KT} = 0$. Although the other brackets are then not needed for the Jacobi identity (we have a strict graded Lie algebra), the other brackets need not vanish and indeed such brackets occur in the homology of a dg Lie algebra and are known as Massey-Lie brackets [**Ret93**]. (They are the Lie analogs of the Massey products first introduced in cohomology of topological spaces [**Mas58, UM57**].)

Furthermore, the identity has content even if only for one n is the n-linear bracket non-zero, all others vanishing. Precisely that situation has recently been studied quite independently of my work and of each other by Hanlon and Wachs [**HW95**] (combinatorial algebraists), by Gnedbaye [**Gne96**] (of Loday's school) and by Azcarraga and Bueno [**dAPB97**] (physicists). On the other hand, this is not Takhtajan's fundamental identity [**Tak94**] for the generalized Nambu n-linear 'bracket'. (This identity was known also to Flato and Fronsdal in 1992, though unpublished, and to Sahoo and Valsakumar [**SV92**].)

See M. M. Vinogradov's talk in these proceedings for a comparison of these two distinct generalizations of the Jacobi identity for n-ary brackets (including reference to Filippov [**Fil85**]).

8. Deformation Theory and the Master Equation in Field Theory

In the Lagrangian setting, we wish to deform not just the local functional, but rather the underlying local function L. In the case of electricity and magnetism (Maxwell's equations) and Yang-Mills, the relevant algebra of gauge symmetries is described by a finite dimensional Lie algebra which, moreover, holds off shell. In terms of an appropriate basis and in the notation of section 3 for families of vector fields, we have

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma \tag{8.1}$$

for structure *constants* $f^\gamma_{\alpha\beta}$ and acting on all fields, not just on solutions. This allows the extended Lagrangian to be no more than quadratic in the ghosts.

As field theories, a Yang-Mills 'particle' can be described by a field of spin 1, while the graviton can be described by a field of spin 2. Somehow (Mother Nature?) this is related to the strict Lie algebra structures just described. For higher spin particles, however, we have quite a different story, which first caught my attention in the work of Burgers, Behrends and van Dam [**Bur85, BBvD84**], though I have since learned there was quite a history before and after that and major questions still remain open. By higher spin particle Lagrangians, I mean that the fields are symmetric s-tensors (sections of the symmetric s-fold tensor product of the tangent bundle). If the power is s, the field is said to be of spin s and represents a particle of spin s. Burgers, Behrends and van Dam start with a free theory with abelian gauge symmetries and calculate all possible infinitesimal (cubic) interaction terms up to the appropriate equivalence (effectively calculating the appropriate homology group, as in formal deformation theory). They then sketch the problem of finding higher order terms for the Lagrangian, but do not carry out the full calculation. In fact, according to the folklore in the subject, a consistent theory for $s \geq 3$ will require additional fields of arbitrarily high spin s. For $s = 3$, the statement is that all higher integral spins are needed, while for $s = 4$, all higher even spins.

From the deformation theory point of view, we are deforming the pair consisting of the abelian Lie algebra and the stationary surface while constraining the stationary surface to remain of Lagrangian type. Stephen Anco [**Anc92, Anc95, Anc97**] has looked at deformations of the gauge symmetries and the Euler-Lagrange equations directly as opposed to our interpretation of Batalin-Vilkovisky as deforming the Lagrangian function. In contrast, Krasil'shchik [**Kra92**] considers deforming the stationary surface (more generally, the diffiety) without regard to gauge symmetries. Deformation theory [**Ger64**] suggests the following attack: Compute the primary obstructions and discover that all infinitesimals are obstructed. Add additional fields to kill the obstructions and calculate that indeed additional fields of arbitrarily high spin s are needed. Whether such computations are feasible to all orders remains to be seen; perhaps eventually they lie in cohomology groups which can be seen to vanish. In one memorable phrase, this would be 'doing string field theory the hard way".

In contrast, Zwiebach [**Zwi93**] does indeed have a consistent closed string field theory (CSFT), but produced in an entirely different way. Recall one of the earliest examples of deformation quantization, the Moyal bracket. Moyal was able to produce a non-trivial deformation of the commutative algebra of smooth functions on $\mathbb{R}^{2n}$, with infinitesimal given by the standard Poisson bracket, by writing down the entire formal power series.[1] Similarly, Zwiebach is able to describe the entire CSFT Lagrangian (at tree level) by giving it in terms of the differential geometry of the moduli space of punctured Riemann spheres (tree level = genus 0). In fact, Zwiebach has the following structure: a differential graded Hilbert space $(\mathcal{H}, <\ ,\ >, Q)$ related to the geometry of the moduli spaces from which he deduces n-ary operations $[\ , \dots ,\ ]$ which give an L_∞ structure. The deformed Lagrangian (still classical) and hence the Master Equation is satisfied for

$$S(\Psi) = \frac{1}{2}\langle \Psi, Q\Psi\rangle + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!}\{\Psi \dots \Psi\}.$$

The expression $\{\Psi \dots \Psi\}$ contains n-terms and will be abbreviated $\{\Psi^n\}$; it is given in terms of the brackets by $\{\Psi \dots \Psi\} = \langle \Psi, [\Psi, \dots, \Psi]\rangle$.

The field equations follow from the classical action by simple variation:

$$\delta S = \sum_{n=2}^{\infty} \frac{\kappa^{n-2}}{n!}\{\delta\Psi, \Psi^{n-1}\}$$

with gauge symmetries given by

$$\delta_\Lambda \Psi = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!}[\Psi^n, \Lambda].$$

[1] Since the conference, Kontsevich [**Kon97**] has constructed a deformation quantization of the algebra of smooth functions on any Poisson manifold with infinitesimal given by the Poisson bracket. In a remarkable tour de force, he produces the entire formal power series via a specific L_∞-map to a sub dg Lie algebra of the Hochschild cochain complex from its homology.

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9. Quantization and other puzzles

So far our description of the anti-field, anti-bracket formalism has been in the context of deformations of 'classical' Lagrangians. Batalin and Vilkovisky (as well as much of the work on BRST cohomology) were motivated by problems in quantization. (The 'finite dimensionality' of the BV complex removes some of the problems with the path integral approach to quantization.) The quantum version of the anti-field, anti-bracket formalism involves a further 'second order' differential operator Δ of square 0 on the BV complex $\mathcal{BV}$ relating the graded commutative product and the bracket - namely, the bracket is the deviation of the operator Δ from being a derivation of the product. This has led to the abstract definition of a BV-algebra as a Gerstenhaber algebra with additional structure [**LZ93**].

Definition 9.1. A **Gerstenhaber algebra** is a graded commutative algebra A with a bracket $[\ ,\ ]$ of degree -1 satisfying the graded Jacobi identity, i.e $[a,\ ]$ is a graded derivation of the bracket, and $[a,\ ]$ is also a graded derivation of the commutative product:

$$[a, bc] = [a, b]c + (-1)^{|b|(|a|+1)}b[a, c].$$

Definition 9.2. A **BV-algebra** is a Gerstenhaber algebra with an operator (necessarily of degree -1 for a bracket of degree -1) such that

$$[A, B] = \Delta(AB) - \Delta(A)B + (-1)^A\Delta(B).$$

Alternatively, a definition can be given in terms of a graded commutative algebra with an appropriate 'second order' operator Δ with the bracket being defined by the above equation [**Sch93, Akm95**].

The quantization of Zwiebach's CSFT involves further expansion of the Lagrangian in terms of (the moduli space of) Riemann surfaces of genus $g \geq 0$. Here the operator Δ is determined by the self-sewing of a pair of pants (a Riemann sphere with 3 punctures). Now Zwiebach's CSFT provides a solution of the 'quantum Master Equation' which in the context of a BV-algebra, is

$$(S, S) = 1/2i\hbar\Delta S.$$

Again we see an analog of the Maurer-Cartan equation or of a flat connection, but why?

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