REF. TUW 96-25

Algebraic characterization of gauge anomalies on a nontrivial bundle

P. John, O. Moritsch[1], M. Schweda and S.P. Sorella[#]

Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Hauptstraße 8-10, A-1040 Wien, Austria

[#] UERJ, Universidade do Estado do Rio de Janeiro,
Departamento de Física Teórica,
Instituto de Física,
Rua São Francisco Xavier, 524
20550-013, Maracanã, Rio de Janeiro, Brazil

November 1996

Abstract. We discuss the algebraic way of solving the descent equations corresponding to the BRST consistency condition for the gauge anomalies and the Chern–Simons terms on a nontrivial bundle. The method of decomposing the exterior derivative as a BRST commutator is extended to the present case.

[1]Work supported in part by the "Österreichische Nationalbank" under Contract Grant Number 5393.

1 Introduction

In order to discuss the global structure of the gauge anomalies let us recall some properties of the gauge connections defined on a principal bundle $P(M,G)$, M being an arbitrary space–time base manifold M_{2k-2} of even dimension $(2k-2)$ and G a compact Lie group[2].

Connections on $P(M,G)$ are locally represented by Lie algebra valued one–forms

$$A = A_\mu dx^\mu = A^a T^a = A^a_\mu dx^\mu T^a \ , \tag{1.1}$$

where T^a are the antihermitian generators of G

$$[T^a, T^b] = f^{abc} T^c \ , \quad Tr(T^a T^b) = \delta^{ab} \ , \tag{1.2}$$

f^{abc} being the totally antisymmetric structure constants. The associated two–form field strength is given by

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2 = dA + \frac{1}{2}\{A, A\} \ . \tag{1.3}$$

It obeys the Bianchi identity

$$DF = dF + [A, F] = 0 \ , \tag{1.4}$$

where $D = dx^\mu D_\mu$ is the covariant exterior derivative with respect to the gauge field A and $d = dx^\mu \partial_\mu$ is the nilpotent ordinary exterior space–time derivative.

Gauge transformations of $P(M,G)$ are on M locally represented by means of a $G-$valued gauge connection A_μ transforming as

$$\delta_G A_\mu = -D_\mu \Omega = -\partial_\mu \Omega - [A_\mu, \Omega] \ , \tag{1.5}$$

where Ω is a $G-$valued infinitesimal gauge parameter.

Infinitesimal diffeomorphisms on M are represented by vector fields v^μ. They form an infinite dimensional Lie algebra on M, denoted by Diff(M). The gauge field transforms under the infinitesimal diffeomorphisms as a vector, *i.e.*

$$\delta_D A_\mu = -L_v A_\mu = -v^\nu \partial_\nu A_\mu - (\partial_\mu v^\nu) A_\nu \ , \tag{1.6}$$

where L_v denotes the Lie derivative along the infinitesimal vector field v^μ. The inner derivative i_v with respect to the vector field v^μ is defined as usual by

$$i_v dx^\mu = v^\mu \ . \tag{1.7}$$

[2] A general introduction to the theory of bundles can be found in refs. [1, 2].

As it is well–known, the inner derivative i_v, the Lie derivative L_v and the exterior derivative d form an infinite dimensional graded Lie subalgebra

$$\begin{aligned} &\{i_v, i_{v'}\} = 0 \ , \quad [L_v, i_{v'}] = i_{[v,v']} \ , \\ &\{i_v, d\} = L_v \ , \quad [L_v, L_{v'}] = L_{[v,v']} \ , \\ &[L_v, d] = 0 \ , \quad \{d, d\} = 0 \ , \end{aligned} \tag{1.8}$$

with $(v, v') \ \in \mathrm{Diff}(M)$.

Let us come now to the global properties of a principle bundle. A bundle is called trivial if it has the form $P = M \times G$. Bundles over contractible base manifolds M are trivial [1, 2]. In the case of a trivial bundle the set of fields introduced so far is sufficient in order to describe the algebraic structure of the gauge anomalies. However, as shown in [3], for the case of a nontrivial bundle one has to introduce a further gauge field, called the reference gauge connection $\mathring{A}$. Such a new connection is choosen to be a fixed background field, due to the fact that the action of v on A is defined only up to a gauge transformation. With the help of the reference connection $\mathring{A}$ all vector fields v on M can be lifted to G. Again, the infinitesimal gauge transformations form an infinite dimensional Lie algebra, now denoted by $\tilde{g}$. Let U be an open trivializing subset of M, $U \subset M$. Then the elements of $\tilde{g}$ are represented on U by pairs of parameters (Ω, v)

$$\Omega \in \Lambda^0(U, g) \ , \quad v \in \mathrm{Diff}(U) \ , \tag{1.9}$$

where Λ^0 denotes the space of zero–forms with values in the Lie algebra g of G. The reference connection is locally represented by the one–form $\mathring{A}$ on U with values in g

$$\mathring{A} = \mathring{A}_\mu dx^\mu \in \Lambda^1(U, g) \ . \tag{1.10}$$

The pairs of parameters obey the commutation relations

$$\begin{aligned} &[(\Omega', 0), (\Omega, 0)] = ([\Omega', \Omega], 0) \ , \\ &[(0, v'), (0, v)] = (i_{v'} i_v \mathring{F}, [v', v]) \ , \\ &[(0, v), (\Omega, 0)] = (-L_v \Omega - [i_v \mathring{A}, \Omega], 0) \ , \end{aligned} \tag{1.11}$$

where $\mathring{F}$ is the two–form field strength associated to $\mathring{A}$

$$\mathring{F} = \frac{1}{2} \mathring{F}_{\mu\nu} dx^\mu dx^\nu = d\mathring{A} + \mathring{A}^2 = d\mathring{A} + \frac{1}{2}\{\mathring{A}, \mathring{A}\} \ , \tag{1.12}$$

which fulfills the Bianchi identity

$$\mathring{D}\mathring{F} = d\mathring{F} + [\mathring{A}, \mathring{F}] = 0 \ . \tag{1.13}$$

The reference connection $\mathring{A}$ ensures that the commutators can be patched together in a consistent way on the overlap $U \cap U'$ of two open subsets U and U' of M. This is achieved by replacing the exterior derivative d in the Lie derivative L_v by the covariant exterior derivative $\overset{\circ}{D}$. In fact, one has

$$[(0,v),(\Omega,0)] = (-\overset{\circ}{\mathcal{L}}_v\Omega, 0) \ , \tag{1.14}$$

with $\overset{\circ}{\mathcal{L}}_v = i_v\overset{\circ}{D} + \overset{\circ}{D}i_v$. Note that the ordinary Lie derivative L_v in (1.6) can be rewritten as $L_v = i_v d + d i_v$.

Different reference connections $\overset{\circ}{A}$ on P yield isomorphic Lie algebras $\tilde{g}$. Note that the pure gauge transformations $(\Omega, 0)$ form an ideal of $\tilde{g}$, while the transformations with respect to the vector fields $(0, v)$ in general do not form a subalgebra of $\tilde{g}$.

Moreover, if the bundle P is trivial, one can choose $U = M$ and $\overset{\circ}{A} = 0$. Then $\tilde{g}$ is the semi–direct product of $\Lambda^0(M, g)$ and $\mathrm{Diff}(M)$, meaning that every nontrivial bundle is locally trivial.

The affine space of all connections on P carries an affine representation $\mathcal{R}$ of $\tilde{g}$ given locally by

$$\mathcal{R}(\Omega,v)A = -D(\Omega + i_v(A - \overset{\circ}{A})) - i_v F \ , \tag{1.15}$$

where $A, \overset{\circ}{A} \in \Lambda^1(U,g)$ [3] are the local expressions on U of the connections on P and D is the covariant exterior derivative with respect to A. Finally, by definition, the fixed reference connection $\overset{\circ}{A}$ does not transform under $\tilde{g}$

$$\mathcal{R}(\Omega,v)\overset{\circ}{A} = 0 \ . \tag{1.16}$$

2 The BRST transformations

In order to introduce the nilpotent BRST operator s let us replace, as usual, the infinitesimal parameters (Ω, v) introduced in the previous section with Grassmann ghost fields. We will introduce therefore two ghost fields, c and ξ^μ, associated to the gauge and to the diffeomorphism transformations, respectively. Both ghosts have form degree zero and ghost number one.

Concerning the BRST transformations [4] of the various fields and ghosts, we proceed by making use of the so–called Maurer–Cartan horizontality conditions [5, 6]. For this purpose we define the nilpotent differential operator of total degree one[4]

$$\widetilde{d} = d - s \ , \quad \widetilde{d}^{\,2} = 0 \ , \tag{2.1}$$

where d is the exterior derivative. Furthermore, we define the generalized gauge connection [5, 6] according to

$$\widetilde{A} = A + c + i_\xi(A - \overset{\circ}{A}) \ , \tag{2.2}$$

[3] $\Lambda^1(U,g)$ denotes the space of the one–forms with values in the Lie algebra g.

[4] The total degree is defined as the sum of the form degree and of the ghost number.

with i_ξ as the inner derivative with respect to the diffeomorphism ghost ξ^μ

$$i_\xi dx^\mu = \xi^\mu \ . \tag{2.3}$$

The generalized reference connection is simply given by

$$\tilde{\mathring{A}} = \mathring{A} \ , \tag{2.4}$$

expressing the fact that $\mathring{A}$ is a fixed background field which does not transform.

The corresponding generalized field strengths of total degree two are given by

$$\widetilde{F} = \widetilde{d}\,\widetilde{A} + \widetilde{A}^2 \quad \text{and} \quad \tilde{\mathring{F}} = \widetilde{d}\,\tilde{\mathring{A}} + \tilde{\mathring{A}}^2 \ , \tag{2.5}$$

and obey the generalized Bianchi identities

$$\widetilde{D}\,\widetilde{F} = \widetilde{d}\,\widetilde{F} + [\widetilde{A}\,, \widetilde{F}\,] = 0 \quad \text{and} \quad \tilde{\mathring{D}}\,\tilde{\mathring{F}} = \widetilde{d}\,\tilde{\mathring{F}} + [\tilde{\mathring{A}}\,, \tilde{\mathring{F}}\,] = 0 \ , \tag{2.6}$$

with $\widetilde{D}$ and $\tilde{\mathring{D}}$ the generalized covariant exterior derivatives with respect to $\widetilde{A}$ and to $\tilde{\mathring{A}}$.

The Maurer–Cartan horizontality conditions can then be expressed by

$$\begin{aligned}
\widetilde{F} &= \frac{1}{2} F_{\mu\nu} \widetilde{dx}^\mu \widetilde{dx}^\nu \ , \\
\tilde{\mathring{F}} &= \frac{1}{2} \mathring{F}_{\mu\nu} \widetilde{dx}^\mu \widetilde{dx}^\nu \ ,
\end{aligned} \tag{2.7}$$

with $\widetilde{dx}^\mu$ the generalized differential of degree one

$$\widetilde{dx}^\mu = dx^\mu + \xi^\mu \ . \tag{2.8}$$

Expanding these conditions according to the ghost number and the form degree one gets the following set of nilpotent BRST transformations

$$\begin{aligned}
sc &= c^2 - \mathring{\mathcal{L}}_\xi c + \frac{1}{2} i_\xi i_\xi \mathring{F} \ , \\
sA &= D(c + i_\xi(A - \mathring{A})) - i_\xi F \ , \\
s\mathring{A} &= 0 \ , \\
sF &= [(c + i_\xi(A - \mathring{A})), F] - \mathcal{L}_\xi F \ , \\
s\mathring{F} &= 0 \ ,
\end{aligned} \tag{2.9}$$

where $\mathcal{L}_\xi = i_\xi D - D i_\xi$ and $\overset{\circ}{\mathcal{L}}_\xi = i_\xi \overset{\circ}{D} - \overset{\circ}{D} i_\xi$ are the covariant Lie derivatives with respect to A and $\overset{\circ}{A}$, respectively[5]. The above BRST transformations are completed by giving the transformation of the diffeomorphism ghost

$$s\xi^\mu = -\xi^\nu \partial_\nu \xi^\mu \ . \tag{2.10}$$

Instead of the ghost c it turns out to be more useful to use the following combination

$$\hat{c} = c + i_\xi (A - \overset{\circ}{A}) \ , \tag{2.11}$$

as the basic variable. In terms of the new variable $\hat{c}$, the BRST transformations read now

$$\begin{aligned}
s\xi^\mu &= -\xi^\nu \partial_\nu \xi^\mu \ , \\
s\hat{c} &= \hat{c}^2 - \hat{F} \ , \\
sA &= D\hat{c} - i_\xi F \ , \\
s\overset{\circ}{A} &= 0 \ , \\
sF &= [\hat{c}, F] - \mathcal{L}_\xi F \ , \\
s\overset{\circ}{F} &= 0 \ ,
\end{aligned} \tag{2.12}$$

with $\hat{F}$ defined as

$$\hat{F} = \frac{1}{2} i_\xi i_\xi F \ . \tag{2.13}$$

Notice, finally, that $\hat{F}$ has form degree zero, ghost number two and that it transforms covariantly, *i.e.*

$$s\hat{F} = [\hat{c}, \hat{F}] \ . \tag{2.14}$$

3 Decomposition formula and algebraic relations

Before going further, let us make some comments on the functional space which we shall use in the following. As already done in previous works [7, 8], we shall assume that the functional space the BRST operator will act upon is identified with the space of form–polynomials constructed out of the variables $(\xi^\mu, c, A - \overset{\circ}{A})$ and the differentials $(d\xi^\mu, dc, dA, d\overset{\circ}{A})$. Of course, on the local space of form–polynomials this set of variables can always be replaced by the equivalent basis given by $(\xi^\mu, d\xi^\mu, \hat{c}, \overset{\circ}{D}\hat{c}, (A - \overset{\circ}{A}), F, \overset{\circ}{F})$. Due to the presence of the covariant derivative $\overset{\circ}{D}$ and of the curvatures

[5] Notice that the relative sign in the definition of the Lie derivatives has now been changed due to the fact that ξ is an odd variable, carrying ghost number one. This implies that i_ξ is even and that $\mathcal{L}_\xi$, $\overset{\circ}{\mathcal{L}}_\xi$ have odd degree.

$(F, \overset{\circ}{F})$ one can easily understand the latter choice, which turns out to be more convenient to solve the BRST consistency condition. We remark also that the use of the combination $(A - \overset{\circ}{A})$ steams from the fact that it emerges rather naturally in the BRST transformations (2.12) and in the formula (2.11). Let us recall, finally, that the space of form–polynomials is the most suitable choice in order to discuss the anomalies and the Chern–Simons terms which, as it is well–known, can be written in terms of differential forms.

Let us now introduce an operator δ of total degree zero defined in the following way [7, 8]

$$\delta \xi^\mu = -dx^\mu \ , \quad \delta \varphi = 0 \quad \text{for} \quad \varphi = (c, A, \overset{\circ}{A}, F, \overset{\circ}{F}) \ . \tag{3.1}$$

¿From the above expression one sees that δ increases the form degree and decreases the ghost number by one unit respectively and that it only on the diffeomorphism ghost acts. The usefulness of the operator δ relies on the fact that it allows to decompose the covariant exterior space–time derivative $\overset{\circ}{D}$ as a BRST commutator. In fact, we have

$$[s, \delta] = -\overset{\circ}{D} \ . \tag{3.2}$$

Let us remark that the presence of the covariant derivative $\overset{\circ}{D}$ in eq.(3.2) makes all the difference with respect to the case of a trivial bundle, where only the ordinary exterior space–time derivative d appears. The operators $(s, \delta, \overset{\circ}{D})$ give rise to the following algebraic relations:

$$\begin{aligned}
&[s, \delta] = -\overset{\circ}{D} \ , \quad [\overset{\circ}{D}, \delta] = [d, \delta] = 2\mathcal{G} \ , \\
&\{s, \overset{\circ}{D}\} = \{s, d\} = s^2 = d^2 = 0 \ , \quad [\mathcal{G}, \delta] = 0 \ , \\
&\{\mathcal{G}, \overset{\circ}{D}\} = \{\mathcal{G}, d\} = 0 \ , \quad \mathcal{G}\mathcal{G} = 0 \ , \\
&\{\mathcal{G}, s\} = \mathcal{K} = \overset{\circ}{D}\overset{\circ}{D} \ , \quad [\mathcal{K}, s] = 0 \ , \\
&[\mathcal{K}, \overset{\circ}{D}] = [\mathcal{K}, d] = 0 \ , \\
&[\mathcal{K}, \delta] = 0 \ , \quad [\mathcal{K}, \mathcal{G}] = 0 \ ,
\end{aligned} \tag{3.3}$$

where the operator $\mathcal{G}$ is defined as

$$\mathcal{G}\hat{c} = \overset{\circ}{F} \ , \quad \mathcal{G}\varphi = 0 \quad \text{for} \quad \varphi = (\xi^\mu, d\xi^\mu, \overset{\circ}{D}\hat{c}, (A - \overset{\circ}{A}), F, \overset{\circ}{F}) \ . \tag{3.4}$$

It decreases the ghost number by one unit and increases the form degree by two units, hence it is an operator of total degree one.

Let us also note that in the case of a trivial bundle ($\overset{\circ}{A} = 0$) the operator $\mathcal{G}$ is absent. This is consistent with the results found in [8]. In other words, its presence is related to the global properties of the bundle.

4 Cohomology and descent equations

It is well–known that the search for the invariant Lagrangians and the anomalies corresponding to a given set of field transformations can be done in a purely algebraic way by solving the BRST consistency conditions in the space of the integrated local field polynomials.

This leads to the study of the nontrivial solutions of the following cohomology problem

$$s\Delta = 0 \quad , \quad \Delta \neq s\bar{\Delta} \ , \tag{4.1}$$

where Δ and $\bar{\Delta}$ are integrated local field polynomials. Setting $\Delta = \int \mathcal{Q}^g_n$, the condition (4.1) translates at the nonintegrated level as

$$s\mathcal{Q}^g_n + d\mathcal{Q}^{g+1}_{n-1} = 0 \ , \tag{4.2}$$

where $\mathcal{Q}^g_n$ is some local polynomial in the fields with ghost number g and form degree n, n denoting the dimension of the space–time. $\mathcal{Q}^g_n$ is said to be nontrivial if

$$\mathcal{Q}^g_n \neq s\bar{\mathcal{Q}}^{g-1}_n + d\bar{\mathcal{Q}}^g_{n-1} \ . \tag{4.3}$$

In this case the integral of $\mathcal{Q}^g_n$ on space–time identifies a cohomology class of the BRST operator s and, according to its ghost number, it corresponds to an invariant Lagrangian ($g = 0$) or to an anomaly ($g = 1$).

The nonintegrated equation (4.2), due to the algebraic Poincaré Lemma [9, 10], is easily seen to generate a tower of descent equations

$$\begin{aligned}
&s\mathcal{Q}^g_n + d\mathcal{Q}^{g+1}_{n-1} = 0 \ , \\
&s\mathcal{Q}^{g+1}_{n-1} + d\mathcal{Q}^{g+2}_{n-2} = 0 \ , \\
&\qquad \ldots\ldots \\
&\qquad \ldots\ldots \\
&s\mathcal{Q}^{g+n-1}_1 + d\mathcal{Q}^{g+n}_0 = 0 \ , \\
&s\mathcal{Q}^{g+n}_0 = 0 \ .
\end{aligned} \tag{4.4}$$

As it has been well–known for several years, that these equations can be solved by using a transgression procedure based on the so–called *Russian formula* [3, 5, 6, 11, 12, 13, 14, 15, 16, 17]. More recently an alternative way of finding nontrivial solutions of the ladder (4.4) has been proposed and successfully applied in the study of the Yang–Mills gauge anomalies [7]. The method makes use of the decomposition formula (3.2). One easily verifies that, once the decomposition has been found, successive applications of the operator δ on the zero–form $\mathcal{Q}^{g+n}_0$ which solves the last equation of the tower (4.4) give an explicit nontrivial solution for the higher cocycles.

It is a remarkable fact that solving the last equation of the tower of descent equations (4.4) is only a problem of local BRST cohomology instead of a modulo–d one. One sees then that, due to the operator δ, the study of the cohomology of s modulo d is essentially reduced to the study of the local cohomology of s. The latter can be *e.g.* systematically analyzed by using the powerful technique of the spectral sequences [18].

5 Chern–Simons terms and gauge anomalies

In order to solve the descent equations (4.4) we use the following strategy. First, we look at the general nontrivial solution of the last descent equation. Then, by using the operators δ and $\mathcal{G}$ with the help of the algebra (3.3), we solve the tower iteratively as done in [7].

Let $n = 2k$ be the dimension of the base manifold M of a nontrivial bundle $P(M, G)$ with structure group G. Introducing a covariant BRST operator according to

$$S = s - \hat{c} \ , \tag{5.1}$$

one obtains a remarkable correspondence between the form sector and the gauge sector of the theory

$$DF = dF + [A, F] = 0 \ , \quad S\hat{F} = s\hat{F} - [\hat{c}, \hat{F}] \ . \tag{5.2}$$

Let us define now the interpolating shifted gauge ghost

$$\hat{c}(t) = t\hat{c} \quad , \quad t \in [0, 1] \ , \tag{5.3}$$

with $\hat{c}(0) = 0$ and $\hat{c}(1) = \hat{c}$, and the associated ghost field strength

$$\hat{F}(t) = -s\hat{c}(t) + \hat{c}(t)\hat{c}(t) \ , \tag{5.4}$$

with $\hat{F}(0) = 0$ and $\hat{F}(1) = \hat{F}$. With the help of the interpolating generalized covariant BRST operator

$$S_t = s - \hat{c}(t) \ , \tag{5.5}$$

with $S_0 = s$ and $S_1 = S$, one finds the following identities

$$\frac{d\hat{F}(t)}{dt} = -S_t\hat{c} \quad , \quad S_t\hat{F}(t) = 0 \ . \tag{5.6}$$

Therefore, in a space–time with dimension $2k$ one has

$$\begin{aligned} Tr(\hat{F}^k) &= Tr(\hat{F}^k(1) - \hat{F}^k(0)) = Tr \int_0^1 dt \, \frac{d}{dt} \, \hat{F}^k(t) \\ &= k \, Tr \int_0^1 dt \, \frac{d\hat{F}(t)}{dt} \, \hat{F}^{k-1}(t) = -k \, Tr \int_0^1 dt \, (S_t\hat{c})\hat{F}^{k-1}(t) \\ &= -s(k \, Tr \int_0^1 dt \, \hat{c} \, \hat{F}^{k-1}(t)) \ . \end{aligned} \tag{5.7}$$

Using the nilpotency of the BRST operator and the fact that $Tr(\hat{F}^k) \neq 0$ in a space–time with dimension $2k$, one may conclude that $k\,Tr \int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t)$ is nontrivial. Since $Tr(\hat{F}^k)$ contains the product of $2k$ fermionic diffeomorphism ghosts ξ^μ, it follows:

In a space–time with dimension $2k-1$ a nontrivial solution of the local equation $s\mathcal{Q}_0^{2k-1} = 0$ on a nontrivial bundle can be represented by the integrated parametric formula

$$\mathcal{Q}_0^{2k-1} = -k\,Tr \int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t)\ . \tag{5.8}$$

Acting with the operator $\mathcal{G}$ on the last of the descent equations (4.4) and using the algebra (3.3) one finds

$$\mathcal{G}s\mathcal{Q}_0^{2k-1} = 0 = -s\mathcal{G}\mathcal{Q}_0^{2k-1} + \overset{\circ}{D}\overset{\circ}{D}\mathcal{Q}_0^{2k-1}\ , \tag{5.9}$$

which, due to the vanishing of the last term in (5.9), implies that $\mathcal{G}\mathcal{Q}_0^{2k-1}$ is BRST–closed

$$s\mathcal{G}\mathcal{Q}_0^{2k-1} = 0\ . \tag{5.10}$$

The general solution of (5.10) has the form

$$\mathcal{G}\mathcal{Q}_0^{2k-1} = s\bar{\mathcal{Q}}_2^{2k-3} + \hat{\mathcal{Q}}_2^{2k-2}\ , \tag{5.11}$$

with a possible nontrivial part $\hat{\mathcal{Q}}_2^{2k-2}$

$$s\hat{\mathcal{Q}}_2^{2k-2} = 0\ , \quad \hat{\mathcal{Q}}_2^{2k-2} \neq s\hat{\mathcal{Q}}_2^{2k-3}\ . \tag{5.12}$$

Recalling that the operator $\mathcal{G}$ acting on the variable $\hat{c}$ yields the field $\overset{\circ}{F}$, we infer that

$$\hat{\mathcal{Q}}_2^{2k-2} = Tr(\overset{\circ}{F}\mathcal{P}(\hat{c},\hat{F})) = \overset{\circ}{F}{}^a\mathcal{P}^a(\hat{c},\hat{F})\ , \tag{5.13}$$

with $\mathcal{P}$ some polynomial in the fields $\hat{c}$ and $\hat{F}$ having form degree zero and ghost number $2k-2$.

In the following we will show that $\mathcal{G}\mathcal{Q}_0^{2k-1}$ is not only BRST–closed but also BRST–exact

$$\mathcal{G}\mathcal{Q}_0^{2k-1} = s\bar{\mathcal{Q}}_2^{2k-3}\ . \tag{5.14}$$

The exactness of $\mathcal{G}\mathcal{Q}_0^{2k-1}$ will be proven by showing that the polynomial $\mathcal{P}^a$ is BRST trivial, due to the fact that the variable $\overset{\circ}{F}$ is BRST invariant. Recalling then that $\mathcal{P}^a$ depends only on $(\hat{c},\hat{F})$ due to the fact that it has form degree zero, let us first express the rigid gauge transformations of $(\hat{c},\hat{F})$ as BRST anticommutators, *i.e.*[6]

$$\{s, \frac{\partial}{\partial\hat{c}^a}\} = \delta^a_{rig.} = f^{abc}\hat{c}^b\frac{\partial}{\partial\hat{c}^c} + f^{abc}\hat{F}^b\frac{\partial}{\partial\hat{F}^c}\ , \tag{5.15}$$

[6] Here we have taken into account only the part of the BRST operator relative to the fields $\hat{c}, \hat{F}$.

It is easily checked by using the Jacobi identity that the operator $\delta^a_{rig.}$ commutes with the BRST operator

$$[\delta^a_{rig.}, s] = 0 \ . \tag{5.16}$$

Furthermore, one has

$$\delta^a_{rig.} \overset{\circ}{F}{}^b = 0 \ . \tag{5.17}$$

Decomposing now the eq.(5.13) in terms of the eigenvalues of $\delta^a_{rig.}$, we get

$$\delta^a_{rig.} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} = l (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} \ , \quad \overset{\circ}{F}{}^b \mathcal{P}^b = \sum_l (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} \ . \tag{5.18}$$

For $l \neq 0$ one obtains

$$\begin{aligned} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} &= \frac{1}{l} \delta^a_{rig.} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} \\ &= \frac{1}{l} \{ s, \frac{\partial}{\partial \hat{c}^a} \} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} \\ &= s \Big(\frac{1}{l} \frac{\partial}{\partial \hat{c}^a} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} \Big) \ , \end{aligned} \tag{5.19}$$

since from eq.(5.12) and eq.(5.18) it follows that

$$s (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(l)} = 0 \ . \tag{5.20}$$

However, contributions of the type (5.19) can be neglected, since they are BRST trivial. What remains is the term corresponding to the eigenvalue $l = 0$

$$\delta^a_{rig.} (\overset{\circ}{F}{}^b \mathcal{P}^b)_{(0)} = 0 \ . \tag{5.21}$$

The above equation states that $(\overset{\circ}{F}{}^b \mathcal{P}^b)_{(0)}$ has to be invariant under rigid gauge transformations. However, since $\overset{\circ}{F}$ is left invariant by $\delta^a_{rig.}$, eq.(5.21) implies that the polynomial $\mathcal{P}(\hat{c}, \hat{F})$ has to be invariant as well. It follows therefore that $(\overset{\circ}{F}{}^b \mathcal{P}^b)_{(0)}$ vanish. This completes our proof.

Repeating and generalizing these arguments to higher $\mathcal{G}$–cocycles[7] one sees that the identity (5.14) generates a subtower of descent equations between the operators s and $\mathcal{G}$

$$\begin{aligned} \mathcal{G} \mathcal{Q}^{2k-1}_0 &= s \bar{\mathcal{Q}}^{2k-3}_2 \ , \\ \mathcal{G} \bar{\mathcal{Q}}^{2k-3}_2 &= s \bar{\mathcal{Q}}^{2k-5}_4 \ , \\ &\ldots.. \\ &\ldots.. \\ \mathcal{G} \bar{\mathcal{Q}}^{3}_{2k-4} &= s \bar{\mathcal{Q}}^{1}_{2k-2} \ , \\ \mathcal{G} \bar{\mathcal{Q}}^{1}_{2k-2} &= \mathcal{P}^0_{2k}(\overset{\circ}{F}) \ , \end{aligned} \tag{5.22}$$

[7]In this case one has to incorporate in the rigid gauge transformations of (5.15) also the contributions of the remaining fields with nonvanishing form degree.

which end up with a polynomial $\mathcal{P}^0_{2k}(\overset{\circ}{F})$ of ghost number zero and form degree $2k$. Moreover, from eq.(3.4) it follows that $\mathcal{P}^0_{2k}(\overset{\circ}{F})$ is nothing else than the Chern form of $\overset{\circ}{F}$ in $2k$ space–time dimensions.

Equiped with this algebraic setup we are now ready to discuss the solutions of the descent equations (4.4). Before analyzing the general case of a space–time of dimension $n = 2k-1$ let us first solve, as an explicit example, the three–dimensional case ($k = 2$)

$$
\begin{aligned}
&s\mathcal{Q}^0_3 + d\mathcal{Q}^1_2 = 0 \ , \\
&s\mathcal{Q}^1_2 + d\mathcal{Q}^2_1 = 0 \ , \\
&s\mathcal{Q}^2_1 + d\mathcal{Q}^3_0 = 0 \ , \\
&s\mathcal{Q}^3_0 = 0 \ .
\end{aligned}
\tag{5.23}
$$

The nontrivial solution of the last descent equation follows from (5.8) with ($k = 2$)

$$
\mathcal{Q}^3_0 = -2\,Tr\int_0^1 dt\,\hat{c}\,\hat{F}(t) = Tr(\frac{1}{3}\hat{c}^3 - \hat{F}\hat{c}) \ . \tag{5.24}
$$

Acting with the operator δ on the last equation one gets

$$
[\delta, s]\mathcal{Q}^3_0 + s\delta\mathcal{Q}^3_0 = 0 \ , \tag{5.25}
$$

which, using the decomposition (3.2), becomes[8]

$$
s(\delta\mathcal{Q}^3_0) + d\mathcal{Q}^3_0 = 0 \ . \tag{5.26}
$$

This equation shows that $\delta\mathcal{Q}^3_0$ provides a solution for the cocycle $\mathcal{Q}^2_1$ in eqs.(5.23).

In order to find an expression for the next cocycle, we apply the operator δ on the eq.(5.26). Using then the algebraic relations (3.3), one has

$$
s(\frac{\delta^2}{2}\mathcal{Q}^3_0) - \mathcal{G}\mathcal{Q}^3_0 + d(\delta\mathcal{Q}^3_0) = 0 \ . \tag{5.27}
$$

According to (5.14), the term $\mathcal{G}\mathcal{Q}^3_0$ in (5.27) can be rewritten as

$$
\mathcal{G}\mathcal{Q}^3_0 = sTr(\overset{\circ}{F}\hat{c}) = s\bar{\mathcal{Q}}^1_2 \ , \tag{5.28}
$$

so that eq.(5.27) becomes

$$
s(\frac{\delta^2}{2}\mathcal{Q}^3_0 - \bar{\mathcal{Q}}^1_2) + d(\delta\mathcal{Q}^3_0) = 0 \ . \tag{5.29}
$$

[8]Remark that all polynomials $\mathcal{Q}$ are gauge invariant and therefore the covariant exterior derivative $\overset{\circ}{D}$ reduces to the ordinary one.

Repeating again the previous steps, for the highest level we obtain

$$s(\frac{\delta^3}{3!}\mathcal{Q}_0^3 - \delta\bar{\mathcal{Q}}_2^1) + d(\frac{\delta^2}{2}\mathcal{Q}_0^3 - \bar{\mathcal{Q}}_2^1) = 0 \ , \tag{5.30}$$

from which it follows that $\mathcal{Q}_3^0$ can be identified with $(\frac{\delta^3}{3!}\mathcal{Q}_0^3 - \delta\bar{\mathcal{Q}}_2^1)$, showing then the usefulness of the operators δ and $\mathcal{G}$ in solving the tower (5.23).

Summarizing, the algebraic solution of the three–dimensional tower of descent equations (5.23) in the case of a nontrivial bundle is given by

$$\mathcal{Q}_3^0 = \frac{\delta^3}{3!}\mathcal{Q}_0^3 - \delta\bar{\mathcal{Q}}_2^1 \ , \tag{5.31}$$

$$\mathcal{Q}_2^1 = \frac{\delta^2}{2}\mathcal{Q}_0^3 - \bar{\mathcal{Q}}_2^1 \ , \tag{5.32}$$

$$\mathcal{Q}_1^2 = \delta\mathcal{Q}_0^3 \ , \tag{5.33}$$

with

$$\mathcal{Q}_0^3 = Tr(\frac{1}{3}\hat{c}^3 - \hat{F}\hat{c}) \ , \tag{5.34}$$

$$\mathcal{Q}_1^2 = Tr(\,-(A-\mathring{A})\hat{c}^2 + (A-\mathring{A})\hat{F} + i_\xi F\hat{c}) \ , \tag{5.35}$$

$$\mathcal{Q}_2^1 = Tr((A-\mathring{A})^2\hat{c} - (A-\mathring{A})i_\xi F - (F+\mathring{F})\hat{c}) \ , \tag{5.36}$$

$$\mathcal{Q}_3^0 = Tr(\,-\frac{1}{3}(A-\mathring{A})^3 + (A-\mathring{A})(F+\mathring{F})) \ , \tag{5.37}$$

In particular, $\mathcal{Q}_3^0$ is recognized to be the three–dimensional Chern–Simons form, while $\mathcal{Q}_2^1$ yields the two–dimensional gauge anomaly which, by using the definition of the shifted gauge ghost (2.11), can be rewritten as

$$\mathcal{Q}_2^1 = -Tr(c\,\mathring{D}A + c\,d\mathring{A} + \mathring{F}i_\xi(A-\mathring{A})) \ . \tag{5.38}$$

In the case of a trivial bundle ($\mathring{A} = 0$) the anomaly (5.38) reduces to the well–known familiar expression

$$\mathcal{Q}_2^1 = -Tr(c\,dA) \ . \tag{5.39}$$

Finally, acting with the exterior derivative d on $\mathcal{Q}_3^0$ one obtains the related Chern form

$$d\mathcal{Q}_3^0 = Tr(F^2 - \mathring{F}^2) \ . \tag{5.40}$$

Let us come now to the general case of a space–time with dimension $n = 2k-1$,

$$\begin{aligned}
&s\mathcal{Q}_{2k-1}^0 + d\mathcal{Q}_{2k-2}^1 = 0 \ , \\
&s\mathcal{Q}_{2k-2}^1 + d\mathcal{Q}_{2k-3}^2 = 0 \ , \\
&\qquad ..... \\
&\qquad ..... \\
&s\mathcal{Q}_1^{2k-2} + d\mathcal{Q}_0^{2k-1} = 0 \ , \\
&s\mathcal{Q}_0^{2k-1} = 0 \ ,
\end{aligned} \tag{5.41}$$

It is straightforward to iterate the previous construction to obtain the solution of the descent equations (5.41). The latter is given by

$$\begin{aligned}\mathcal{Q}_0^{2k-1} &= -k\,Tr\int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t)\ ,\\ \mathcal{Q}_{2p}^{2k-1-2p} &= \frac{\delta^{2p}}{(2p)!}\mathcal{Q}_0^{2k-1} - \sum_{q=0}^{p-1}\frac{\delta^{2q}}{(2q)!}\bar{\mathcal{Q}}_{2p-2q}^{2k-1-2p+2q}\ ,\end{aligned} \tag{5.42}$$

for the even space–time sector and

$$\begin{aligned}\mathcal{Q}_1^{2k-2} &= \delta\mathcal{Q}_0^{2k-1}\ ,\\ \mathcal{Q}_{2p+1}^{2k-2-2p} &= \frac{\delta^{2p+1}}{(2p+1)!}\mathcal{Q}_0^{2k-1} - \sum_{q=0}^{p-1}\frac{\delta^{2q+1}}{(2q+1)!}\bar{\mathcal{Q}}_{2p-2q}^{2k-1-2p+2q}\ ,\end{aligned} \tag{5.43}$$

for the odd space–time sector and $p = 1, 2, ..., k$. The solutions (5.42) and (5.43) generalize the results of [7] to the case of a nontrivial bundle.

References

[1] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Phys. Reports* 66 (1980) 213;

[2] M. Göckeler and T. Schücker, *Differential Geometry, Gauge Theories, and Gravity*, Cambridge University Press (1987);

[3] J. Mañes, R. Stora and B. Zumino, *Comm. Math. Phys.* 102 (1985) 157;

[4] C. Becchi, A. Rouet and R. Stora, *Ann. Phys. (N.Y.)* 98 (1976) 287;
I.W. Tyutin, *Gauge Invariance in Field Theory and Statistical Physics*, Lebedev Institute, preprint FIAN no. **39** (1975);

[5] J. Thierry-Mieg, *J. Math. Phys.* 21 (1980) 2834;
Nuovo Cim. 56A (1980) 396;
Phys. Lett. B147 (1984) 430;
Classical geometrical interpretation of ghost fields and anomalies in Yang-Mills theory and quantum gravity, Symposium on Anomalies, Geometry, Topology, pag. 239, edited by W. A. Bardeen and A. R. White, World Scientific (1985);

[6] L. Baulieu, *Nucl. Phys.* B241 (1984) 557;
L. Baulieu and J. Thierry-Mieg, *Phys. Lett.* B145 (1984) 53;

[7] S.P. Sorella, *Comm. Math. Phys.* 157 (1993) 231;

[8] O. Moritsch, M. Schweda and T. Sommer, *Class. Quant. Grav.* 12 (1995) 2059;

[9] L. Bonora and P. Cotta-Ramusino, *Comm. Math. Phys.* 87 (1983) 589;

[10] F. Brandt, N. Dragon and M. Kreuzer, *Nucl. Phys.* B332 (1990) 224;

[11] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* B234 (1983) 269;

[12] M. Dubois-Violette, M. Talon and C.M. Viallet, *Comm. Math. Phys.* 102 (1985) 105;
Phys. Lett. B158 (1985) 231;
Ann. Inst. Henri Poincaré 44 (1986) 103;
M. Dubois-Violette, M. Henneaux, M. Talon and C.M. Viallet, *Phys. Lett.* B267 (1991) 81;
M. Henneaux, *Comm. Math. Phys.* 140 (1991) 1;
M. Dubois-Violette, M. Henneaux, M. Talon and C.M. Viallet, *Phys. Lett.* B289 (1992) 361;

[13] G. Bandelloni, *Nuovo Cim.* 88A (1985) 1;
Nuovo Cim. 88A (1985) 31;

[14] L. Alvarez-Gaumé and P. Ginsparg, *Ann. Phys. (N.Y.)* 161 (1985) 423;

[15] L. Bonora, P. Pasti and M. Tonin, *J. Math. Phys.* 27 (1986) 2259;

[16] F. Langouche, T. Schücker and R. Stora, *Phys. Lett.* B145 (1984) 342;
R. Stora , *Algebraic structure of chiral anomalies*; lecture given at the GIFT seminar, 1-8 june 1985, Jaca, Spain; preprint LAPP-TH-143;
T. Schücker, *Comm. Math. Phys.* 109 (1987) 167;

[17] F. Brandt, N. Dragon and M. Kreuzer, *Nucl. Phys.* B340 (1990) 187;

[18] J. A. Dixon, *Comm. Math. Phys.* 139 (1991) 495;