

# The Case for a Gravitational de Sitter Gauge Theory

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IN HOMAGE TO PAULO LEAL FERREIRA ON  
 THE OCCASION OF HIS 70TH BIRTHDAY

## Abstract

With the exception of gravitation, the known fundamental interactions of Nature are mediated by gauge fields. A comparison of the candidate groups for a gauge theory possibly describing gravitation favours the Poincaré group as the obvious choice. This theory gives Einstein's equations in a particular case, and Newton's law in the static non-relativistic limit, being seemingly sound at the classical level. But it comes out that it is not quantizable. The usual procedure of adding counterterms to make it a consistent and renormalizable theory leads to two possible theories, one for each of the two de Sitter groups,  $SO(4,1)$  and  $SO(3,2)$ . The consequences of changing from the Poincaré to the de Sitter group, as well as the positive aspects, perspectives and drawbacks of the resulting theory are discussed.

## 1 Introduction

General Relativity is the widely accepted theory of gravitation. Besides its consistency and beauty, it has accumulated an impressive amount of experimental successes, which seems to establish its validity beyond any possibility of doubt. It was submitted to a very heavy attack some years ago [1], to which followed an equally passionate defense [2]. But the debate was, curiously enough, restricted to the soviet community. Independently of the hard core of the subject, on which we shall not take position here, it is undeniable that such controversies are highly to be praised. No theory can be accepted as eternal, or incapable of improvement and any honest, good-willed attack is always healthy. A consequence of the polemics has been a reappraisal of the accepted theory by its very supporters, which was a very positive point. Some weak points, well known to experts, have become of widespread knowledge. Even true believers of a theory accept that alternative models help its

understanding and provide references for experimental improvements. Some alternative theories of the past (Brans-Dicke, for example) have played such a "sparring" role.

Are there additional reasons to look for alternative theories? From the experimental point of view, there is none. General Relativity has an impressive record of experimental successes. We could estimate that it has been verified, in those cases in which it was possible, to around 0.1%. This is far less precise than the analogous score for some other interactions, but experiments are exceedingly difficult. From the theoretical point of view, there is no imperative, compelling reason. There are a few reasons nevertheless.

Some theoretical *defects* of General Relativity sprang out in the controversy alluded to. To start with, let us talk of two of them. First, in order to have things well-defined, even the gravitational field should be asymptotically vanishing. This means that, in particular, the space sector of spacetime should be asymptotically flat and would lead to trouble with one of the two favoured universe models of Cosmology: it would be consistent with the open Friedmann universe, but not with the closed Friedmann universe. A second difficulty is that General Relativity is not really a field theory. The argument runs as follows: when talking about a field, we must be able to say where it is. The usual means for that is to calculate its energy density: the field is there where the energy density is different from zero. General Relativity attributes no well-defined energy density to the gravitational field. It is true however that an analogous difficulty is present in gauge theories. There, also the "charge" density of the gauge field is ill-defined. Thus, for example, the color density of the gluon field has no physical meaning. Energy (the energy-momentum tensor) is the "charge" for gravitation and, just as for the gauge charges in general, only the total charge of gauge field plus source fields has a covariant meaning [3]. There is also the possibility of interpreting General Relativity as a metric field superposed to a Minkowski background. This would not change the problem of the energy, but would make it still more similar to the gauge charge problem. By the way, a frequent argument against General Relativity is that the energy-momentum tensor is the Noether current for translations, and that the remaining local symmetries of spacetime (that is, the Lorentz transformations) are not contemplated.

There are a number of further arguments against General Relativity. A very common one, which is weak to the point of being almost wrong, would run more or less as follows: the desired goal is an unified theory, and this would require similar theories. Well, amongst the 4 interactions nowadays taken as fundamental, the electromagnetic and weak interactions are described by the Weinberg-Salam gauge model, and the strong interactions are described by the SU(3) gauge chromodynamics. And these theories are also very successful!

Only gravitation stays apart from the 3 gauge-ruled interactions. It should then, *unification oblige*, have also a gauge formulation. Now, the truth is that unification can be conceived even if gravitation remains *different*. In this case, there are two possibilities: (i) it remains different, but it is quantizable, or (ii) it is so different that it is not quantizable. A favorite possibility of the last type is "induced gravitation" in the sense of Zeldovich and Sakharov [4], which conceives gravitation as a kind of elastic property of spacetime, coming from the vacua of the other interactions. It would be necessary, in order to support this idea, to show that such vacua do induce a curvature on spacetime. Unfortunately, despite gigantic efforts, this has not been shown up to now. This vision of gravitation as an effective interaction would allow it to remain different from the other and, furthermore, to remain essentially classical.

The very success of General Relativity, on the other hand, teaches two important things to those in search of alternative models: (i) of course, the alternative model should give the same well-verified results, and (ii) gravitation does exhibit a privileged relationship to spacetime. All this has led to two main kinds of alternative models. The first one is the higher-order curvature Lagrangians, like  $\mathcal{L} = R + R^2$ . The differences with respect to the Lagrangian  $R$  are known of old [5], but are not observable with present-day resources. Though improving renormalization, these Lagrangians require successive addition of terms (called counterterms) to account for divergences at higher perturbation orders. This means that we should have actually something like  $\mathcal{L} = R + R^2 + R^3 + R^4 + \dots$ . And here we notice that we work with a beloved prejudice, whose main justification lies in simplicity: we suppose the Lagrangian to be a polynomial in the field and their derivatives. Why not things like  $\mathcal{L} = 1/(1 - R)$ , for example? Non-polynomial Lagrangians have been fashionable in the seventies, but seem to have been abandoned by now. The second kind of alternative models is super-gravity [6], which adds a particle of opposite statistics to each known particle, as well as improves renormalizability for the lower order perturbative graphs, but fails at higher orders. There is still another point of view: since gravitation is different, let us quantize it differently! This is the banner of the so called "Quantum Gravity" scheme. Recently, Ashtekhar proposed another approach [7], a version of the Hamiltonian formalism in which the gravitational variables appear as gauge variables, with the advantage that some of the gauge constraints are automatically satisfied.

We go now to the main point of our paper. There is another frequent argument, which is wrong but is relevant because it calls attention to an important point: the incompatibility of General Relativity with quantum requirements. It is attributed to Bohr and Rosenfeld the statement: every field must be quantized, since otherwise it would be possible to violate the uncertainty principle.

This is a rather loose version of what is really said in the famous Bohr- Rosenfeld paper [8]. As Rosenfeld himself pointed out [9], the arguments, there applied to the electromagnetic field, do not apply to gravitation. Whether gravitation is to be quantized or not, the answer is to be given by experiment.

The fact remains, however, that General Relativity is not (at least perturbatively) renormalizable. Supported on their natural affinity to renormalization, a large number of gauge models for gravitation have been proposed. Our intention here will be to present the case for one of them, the Gravitational de Sitter Gauge Model. Because the subject is very wide [10], we shall be a bit naïve and adopt a rather assertive style, even at the risk of seeming dogmatic. In broad brushstrokes, the case is summed up in the following points: (i) gravitation is deeply related to spacetime itself, much more so than other nowadays known interactions; (ii) gauge theories describe suitably the other interactions and, despite the above discussion about the weakness of this argument, it seems natural for us to look for alternatives inspired by the gauge scheme; (iii) the natural group to be considered is the Poincaré group; (iv) a gauge theory for the Poincaré group is plagued with a deadly illness: it has no action functional; (v) if "quantized" in a way that dispenses with the action functional, it has no well-defined vertices, a problem that can be solved by a method inspired by renormalization theory, that is, by the addition of counterterms; (vi) once this is done, the resulting theory is non-renormalizable at first, but addition of new counterterms turn it into a renormalizable theory; (vii) the resulting model, once the counterterms have been added, is a gauge theory for one of the de Sitter groups. The gauge theory for a de Sitter group appears consequently as a smoothed, renormalizable Poincaré gauge theory.

As the crux of the problem lies in the question of renormalizability, we start by a brief discussion of the subject in section 2. Dimensional considerations lead to one of the main arguments favoring de Sitter gauge theories: it is very difficult to conceive a renormalizable theory with the energy-momentum as source current. With the exception of gravitation, the known fundamental interactions of Nature are mediated by gauge fields. The general, formal characteristics of gauge theories are summed up in section 3, followed by a comparison of the candidate groups for a gauge gravitation theory. The Poincaré group comes out as the obvious choice. We consequently analyse the gauge theory for the Poincaré group in the following section, together with a short discussion of the bundle of linear frames which appears as the geometric background. The theory gives Einstein's equations in a particular case, and Newton's law in the static non-relativistic limit, being seemingly sound at the classical level. But it comes out that it is not quantizable. As described in section 5, the procedure of adding counterterms to make it into a consistent and renormalizable theory leads to two possible theories, one for each of the two de

Sitter groups,  $SO(4,1)$  and  $SO(3,2)$ . The consequences of changing from the Poincaré to the de Sitter group, as well as the positive aspects, perspectives and drawbacks of the resulting theory, are outlined in the final section.

## 2 Renormalizability

A theory which does not bow to the renormalizability requirement is unacceptable from the quantum point of view: it will attribute infinite values to finite quantities. There seems to be theories which are renormalizable even if not perturbatively so, but this is too involved a subject to be considered here. We speak here of perturbative, order by order renormalizability. This is a very restrictive condition. Amongst all the polynomial models, there are only three types of renormalizable theories:

- (1) scalar theories with interaction term of type  $\lambda\phi^4$ ;
- (2) scalar-fermion interactions of Yukawa type:  $g\bar{\Psi}\Psi\phi$  for scalars and  $g\bar{\Psi}\gamma^5\Psi\phi$  for pseudo-scalars;
- (3) minimal coupling as given by gauge theories for reasonable groups (like  $SU(N)$  and  $SO(N)$ ).

An essential characteristic coming out from the detailed examination of the problem is embodied in a simple rule-of-thumb: in order that the theory be renormalizable, the coupling constants must be non-dimensional.

Elementary dimensional analysis is of help here. One uses a system of units in which  $\hbar = c = 1$ , that is, they are non-dimensional:  $[\hbar] = [c] = 0$ . One counts the dimension in terms of the mass:  $[m] = 1$ . This means that some usual quantities have dimensions like  $[\partial_\mu] = [E] = [T^{-1}] = [L^{-1}] = 1$ . Comparing the different terms in the free Lagrangians, one finds that the main fields have the following dimensions: bosons,  $[\phi] = [A_\mu] = 1$ ; fermions,  $[\Psi] = [\bar{\Psi}] = 3/2$ ; field strengths of gauge theories,  $[F_{\mu\nu}] = 2$ . A very general result is that all Noether source currents have  $[J] = 3$ .

A disturbing fact appears in General Relativity, in which the Noether source current has dimension  $[T_{\mu\nu}] = [E/V] = 4$ . This is clearly related to the fact that the usual transformation generators are dimensionless, except those of translations, which have  $[\partial_\mu] = 1$ . It is true that also the dimensions of the basic fields are anomalous in General Relativity:  $[R] = 2$  is a correct field strength, but the metric  $g_{\mu\nu}$  has  $[g_{\mu\nu}] = 0$ ! The final trouble comes really from the energy-momentum "irregularity": the coupling constant  $k$  is seen, from  $[R_{\mu\nu}] \sim k[T_{\mu\nu}]$ , to have dimension  $[k] = -2$ . The source current, being the Noether current associated to translations, enforces a non-vanishing dimension for the coupling constant  $k$  in General Relativity. It is anyhow hard to imagine a renormalizable theory with the energy-momentum as source current, which is itself a non-renormalizable tensor. In the perturbative series,

each term contains Feynman integrals in the momenta and must be dimensionless. Each vertex produces a factor  $f$ , here with  $[f] = [L^2]$ . In order to compensate this dimension, some factor  $p^2$  turns up in the integrand and the terms become more and more divergent as the order of perturbation increases [11]. The measure of divergence is given by the superficial degree of divergence  $\hat{w}_v$  for each vertex, which is given by  $\hat{w}_v = \delta_v + (3/2)f_v + b_v$ , where  $\delta_v$  is the number of derivatives in the internal lines incident at the vertex  $v$ ,  $f_v$  is the number of incident fermion lines, and  $b_v$  is the number of boson lines. If, for all vertices,  $\hat{w}_v > 4$ , the theory is non-renormalizable; if, for all vertices,  $\hat{w}_v \leq 4$ , then the theory is renormalizable; if, for all vertices,  $\hat{w}_v < 4$ , the theory is super-renormalizable. It turns out that renormalizability may be checked by inspection of the interaction terms in the Lagrangian. In last resort, what happens is that the constants in the Lagrangian are corrected order by order. One says then that the infinities are absorbed in the constants. These constants include the wavefunction normalizations, the masses, and the coupling constants. If they are enough to absorb all the infinities, the theory is renormalizable. Actually, if we find that a given theory is non-renormalizable, it can eventually be "repaired": it may happen that it becomes renormalizable once one or more terms ("counterterms") are added to the primitive Lagrangian. The theory is renormalizable when the number of necessary counterterms is finite. The physical starting Lagrangian must contain all the necessary counterterms.

Let us look at the simplest example, the electrodynamics of mesons (such as  $\pi^+$  and  $\pi^-$  mesons). The free Lagrangian density would be

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi + m\phi\phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

The interaction is added through the gauge prescription, by which ordinary derivatives are replaced by covariant ones:  $\partial^\mu \rightarrow \partial^\mu - ieA^\mu$ . The Lagrangian density becomes

$$\mathcal{L} = -[\partial_\mu + ieA_\mu]\phi^*[\partial^\mu - ieA^\mu]\phi + m\phi\phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

We then proceed to obtain the quantized, renormalized theory. A strange thing happens then: a theory as above is not consistent. In order to renormalize the graphs with four external meson legs and internal photon loops, it is necessary to add a counterterm  $+\lambda|\phi\phi^*|^2$  to the above Lagrangian. This is why one always starts with the meson Lagrangian with the term  $\lambda\phi^4$ : one knows it will be necessary. It is a fascinating point that, in order to interact correctly through the exchange of photons, the mesons are forced to interact also between themselves, and in that particular prescribed way.

What happens in General Relativity is that new counterterms must be added at each order in perturbation, so that actually an infinite number of

counterterms is required for the whole perturbative series. And, we repeat, this trouble comes from the "irregular" dimension of the energy-momentum tensor. A fact which, by the way, suggests that any theory with this tensor as a source will have the same kind of problem. Notice that some people argue that non-renormalizability may ultimately be a good thing for General Relativity. The requirements of field theory supposes Minkowski space at arbitrarily short distances, but there could be a natural cut-off given by the Planck scale [12].

Notice that no interaction mediated by a vector meson is renormalizable *unless* the meson is a gauge boson. In this case, the gauge symmetry may be spontaneously broken, so as to endow the intermediate bosons with mass while preserving renormalizability. All that said, it seems reasonable, when looking for alternative theories for gravitation, to go after a gauge model, knowing nevertheless that that model should have a privileged relationship to spacetime.

### 3 Gauge theories

What finally makes gauge theories [13] so especial? They have some really nice properties:

- (i) they embody an automatic prescription for taking symmetries into account. But attention: such symmetries are particle-classifying, the elementary particles are placed in multiplets of the symmetry group;
- (ii) they have a natural affinity with renormalization; as said above, this is a real ace!
- (iii) they have a very rigid structure, basically geometrical in character; they possess much more symmetry than that included in the gauge group: duality symmetry, conformal symmetry, BRST symmetry, etc.

Given a classifying group  $G$  (a group in whose multiplets the elementary particles can be coherently accommodated) with Lie algebra  $G'$  generated by operators  $J_a$  satisfying  $[J_a, J_b] = f^c_{ab} J_c$ , the gauge potential will have the form  $A_\mu = J_a A^a_\mu$ , and the field strength will be  $F_{\mu\nu} = J_a F^a_{\mu\nu} = J_a (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu)$ . We know perfectly the geometry behind it:  $A$  is a connection and  $F$  is its curvature, satisfying automatically the Bianchi identity

$$\left( \delta^a_c \partial_\mu + f^a_{bc} A^b_\mu \right) \tilde{F}^{c\mu\nu} = 0,$$

where  $\tilde{F}^c_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{c\rho\sigma}$  is the dual of  $F$ . A change of gauge is a transformation  $A \rightarrow A' = U(A + d)U^{-1}$ , leading to  $F' = dA' + A' \wedge A' = U(dA + A \wedge A)U^{-1} = U F U^{-1}$ . The field strength  $F$ , whose components are the measurable quantities, is covariant under gauge transformations. This means that they have, as they should, a covariant physical meaning. The basic

dynamics is given by the Yang-Mills equation

$$\left(\delta^a_c \partial_\mu + f^a_{bc} A^b_\mu\right) F^{c\mu\nu} = J^{a\nu},$$

where  $J^{a\nu}$  is the source Noether current. Given the structure constants  $f^a_{bc}$  of the group, one can always write directly the field equations, one for each group generator. We profit here to make it clear what we mean by a gauge theory: it is a theory whose basic dynamics is governed by the Yang-Mills field equations.

In the sourceless case,  $J^{a\nu} = 0$ , the Yang-Mills equation is just the expression of the Bianchi identity written for the dual of  $F$ . This is the duality symmetry. Both equations are invariant if we change the space time metric  $g_{\mu\nu}$  by multiplying it by a function:  $g_{\mu\nu} \rightarrow f(x)g_{\mu\nu}$ . This is the conformal symmetry, deeply related to the renormalizability of the theory. This is of course no place for a detailed exposition of gauge theories. We only quote these items to give an idea of how rigid they are, so much so that you cannot change anything without breaking the whole structure and losing their good properties. The background structure [14] is well-known: its a principal fiber bundle, with spacetime as the base space and the gauge group as the fiber. The bundle space is locally a direct product of spacetime by the group. The connection  $A$  takes vector fields on the bundle space into the Lie algebra of the gauge group. It represents the field mediating the interaction between source fields.

Each source field is in an associated bundle, structure similar to the principal bundles but with a representation (a multiplet to which the source field belongs) replacing the group. There are in principle infinite connections on each bundle, amongst which the Yang-Mills equation, with suitable boundary conditions, chooses one. This fixes the gauge field of the problem under consideration.

An important question is related to another dear prejudice: universality. Though experimental evidence is still lacking, there is a widespread belief that gravitation concerns all elementary particles. There would be no particle that does not feel gravitation. The fact that the Poincaré group classifies all the elementary particles is not enough to ensure such property. Given a gauge model, particles insensible to the field are classified in the singlet representations, which are one-dimensional and whose dynamics will automatically "vanish". Scalar fields are singlets of the Lorentz group, though not of the translation group. This is one reason for the importance of translations — they would "explain" universality. But there is another origin for universality, holding for any group acting on spacetime: the so called kinematic representations [15]. All these groups have a representation in terms of vector fields (that is, derivatives) on spacetime, which act through the arguments of



the wavefunctions. This is true for translations, rotations, boosts, conformal transformations, and dilatations. Given any Lie group, it is an easy task to build up a formal gauge theory by writing down the corresponding Yang-Mills equations. The presence of these kinematic representations, however, changes the scheme a lot. And they will, we repeat, be at work for any group acting on spacetime.

There will be another, deep problem concerning gravitation and gauge theories. The latter have mediating fields of spin 1. The interaction will consequently reverse sign when one of the interacting particles is changed into its antiparticle [16]. This affects another beloved prejudice: that matter and antimatter have the same, attractive, gravitational interaction. We shall see later how this problem may come to be circumvented.

The arguments listed above support the idea that, if we are to look for an alternative theory of the gauge type [17], a group classifying the elementary particles must be involved, which should be intimately related to spacetime itself. Let us then briefly review the main groups acting on spacetime:

- (i) the conformal group: it is deeply related to the causality structure [18], as it contains the transformations preserving the light cones; it should be somehow broken, as its representations can only accommodate particles of vanishing masses; in other words, it does not really classify the known elementary particles;
- (ii) the de Sitter groups: we will come back to them later;
- (iii) the Lorentz group: it has been studied by Yang [19], Camenzind [20], Carmeli [21], and many others; it does not really classify the elementary particles — it would account for spin but leaves momentum out of the game;
- (iv) Poincaré group [22]: it classifies the elementary particles, giving them both spin and momentum; it has a very clear relationship with spacetime, and is the obvious natural candidate; it is however a non-semisimple group and we shall see that this leads to a lot of trouble;
- (v) the translation group: in certain aspects, it has been used [23] to rephrase General Relativity, but it does not classify the particles — it takes only momentum into account.

Thus, the Poincaré group, which is both the classifying group in what concerns spacetime and the basic local group of Physics, appears as the natural candidate for a gauge model for gravitation. Indeed, it has been the most studied group [24], and we shall use it as the starting point of our analysis. It will have apparently unsolvable problems with quantization. Actually the best argument for the de Sitter group is that it comes out of the whole analysis as the "corrected" Poincaré group, in a sense to be made clear in the following.

## 4 The Poincaré group

We have been saying that gauge theories have as background a principal fiber bundle, with gauge group as the fiber and spacetime as the base space. This is actually the way of doing geometry in (rather) modern language [25]. And we are, repeating again, looking for a gauge theory somehow linked to spacetime, much more so than the usual gauge models. Now, there is an important fact. Every differentiable manifold  $M$  (here we are thinking of spacetime, of course) has a principal fiber bundle naturally attached to it, the bundle of linear frames. The set of linear frames at a point of the manifold is isomorphic to the real linear group  $GL(m, R)$  of real  $m \times m$  matrices (with  $m = \dim M$ ), so that the natural bundle is a principal fiber bundle with this group. As we are speaking of Minkowski spacetime, whose main characteristic is the Lorentz metric, a special role will be reserved to the sub-bundle of pseudo-orthogonal frames (that is, of the frames which are orthogonal according to the Lorentz metric).

Let us be a bit more precise. A linear frame at point  $p$  of the base space is chosen as follows. Take the tangent space at  $p$ ,  $T_p M$ . It is isomorphic to the Euclidean space  $E^m$ . On  $E^m$  there is a canonical frame, formed by the unit one-dimensional vectors  $\delta_k$ , having 1 at the  $k$ -th entry and zero everywhere else. We choose a basis, or frame, by transplanting this one to  $T_p M$ . Formally, a linear frame  $\{b_k\}$  is given by a mapping  $b : E^m \rightarrow T_p M$ ,  $b(\delta_k) = b_k$ . Thus, different frames correspond to different choices  $b$  of which  $b_k$  correspond to the unit vector  $\delta_k$ . In strict relation to these vectors there is a canonical basis for the group Lie algebra, given by "unit" matrices  $\Delta^{\alpha\beta}$ . These matrices have 1 at the  $\alpha - \beta$  entry and zero everywhere else. Now, given any metric  $\eta$  (here the Lorentz metric), we define matrices  $J_{\alpha\beta} = \eta_{\alpha\gamma} \Delta^{\gamma\beta} - \eta_{\beta\gamma} \Delta^{\gamma\alpha}$ . Then the  $J_{\alpha\beta}$ 's generate the Lie algebra of the orthogonal group of  $\eta$  (notice that from this notation comes the use of double indices for spacetime geometrical objects, at a difference with geometrical objects related to other bundles). In our case, the  $J_{\alpha\beta}$ 's will generate the Lorentz group. The frame vectors are then taken to be orthogonal, and a sub-bundle results, the bundle of Lorentzian frames. A linear connection will be the 1-form  $\Gamma = \Delta^{\alpha\beta} \Gamma_{\alpha}^{\beta} = \Delta^{\alpha\beta} \Gamma_{\alpha}^{\beta}{}_{\mu} dx^{\mu}$ . A connection defines a covariant derivative, which acts on any object in a well-defined way. A Lorentz connection will be  $\Gamma = J_{\alpha\beta} \Gamma^{\alpha\beta} = J_{\alpha\beta} \Gamma^{\alpha\beta}{}_{\mu} dx^{\mu}$ , and its curvature, which is its own covariant derivative, will be  $F = \frac{1}{2} J_{\alpha\beta} F^{\alpha\beta} = \frac{1}{2} J_{\alpha\beta} F^{\alpha\beta}{}_{\mu} dx^{\mu}$  with

$$F^{\alpha\beta}{}_{\mu\nu} = \partial_{\mu} \Gamma^{\alpha\beta}{}_{\nu} - \partial_{\nu} \Gamma^{\alpha\beta}{}_{\mu} + \Gamma^{\alpha}{}_{\epsilon\mu} \Gamma^{\epsilon\beta}{}_{\nu} - \Gamma^{\alpha}{}_{\epsilon\nu} \Gamma^{\epsilon\beta}{}_{\mu}.$$

From these expressions comes the Bianchi identity,

$$\partial_{\mu} \tilde{F}^{\alpha\beta\mu\nu} - \Gamma^{\alpha}{}_{\gamma\mu} \tilde{F}^{\gamma\beta\mu\nu} + \tilde{F}^{\alpha}{}_{\gamma}{}^{\mu\nu} \Gamma^{\gamma\beta}{}_{\mu} = 0.$$

This is quite analogous to the usual geometrical background of gauge models related to "internal" symmetries, but there is here a deep difference. The bundle of linear frames, we have said, is more deeply rooted on the base manifold than any other bundle. This is shown by the presence of a property which is not present in other bundles. The property is called *soldering* and is embodied in a special  $E^m$ -valued form on the bundle, the solder form. Being  $E^m$ -valued means that it is of the form  $S = \delta_k^m S^k$ , with  $S^k$  an usual form. It is thus a mapping  $T_b GL(m, R) \rightarrow E^m$ . It establishes a direct relation between tangent spaces of the bundle manifold and tangent spaces of the base manifold. Given the projection mapping  $\pi$  of the bundle and its differential  $\pi_*$  (which maps tangent vectors), then we have  $S = b^{-1} \circ \pi_*$ . This form is canonical, in the sense that it is always there, for any differentiable manifold. Now, choosing a frame is always done by a section, a mapping  $\sigma : M \rightarrow \text{bundle}$ . Differential forms on the bundle are brought back to  $M$  by the pull-back  $\sigma^*$  of  $\sigma$ , the dual of its differential. Each Lorentz frame (tetrad, vierbein, fourleg) is given by  $\sigma : M \rightarrow \text{bundle}$ ,  $\sigma : p \rightarrow \{h^\alpha(p)\}$ , with  $\pi \circ \sigma(p) = p$ . Now, it so happens that, if  $\sigma$  chooses the frame  $h^a$ , then its pull-back of the solder form gives back precisely  $h^a : \sigma^*(S^\alpha) = h^\alpha = h^\alpha_\mu dx^\mu$ . The presence of the solder form allows one, if given a metric  $\eta$  on  $R^m$ , to "transform" it into a metric  $g$  on the base manifold  $M$ , by  $g(X, Y) = \eta(b^{-1}X, b^{-1}Y)$ . This is the deep reason for the usual way of writing a metric in terms of the tetrads, as the last expression is just  $g_{\mu\nu} = \eta_{\alpha\beta} h^\alpha_\mu h^\beta_\nu$ .

The presence of the solder form has an important consequence: given a connection, there exists another natural characteristic of it, besides the curvature. It is its torsion, which is the covariant derivative of the solder form. As the latter is expressed on  $M$  by the tetrad, the torsion appears as the covariant derivative of the tetrad fields,

$$T^\alpha_{\mu\nu} = \partial_\mu h^\alpha_\nu - \partial_\nu h^\alpha_\mu + \Gamma^\alpha_{\epsilon\mu} h^\epsilon_\nu - \Gamma^\alpha_{\epsilon\nu} h^\epsilon_\mu.$$

And it turns out that also an extra Bianchi identity holds,

$$\partial_\mu \tilde{T}^{\alpha\mu\nu} - \Gamma^\alpha_{\gamma\mu} \tilde{T}^{\gamma\mu\nu} + \tilde{F}^\alpha_{\gamma}{}^{\mu\nu} h^\gamma_\mu = 0.$$

The Bianchi identity is half the field equations of a gauge theory (for example, the first pair of Maxwell's equations). We are thus to expect that, if we build up a gauge model related to the bundle of frames, we have some extra field equations. Thus, the special "tight-bound" relation of the linear bundle to the base manifold engenders torsion, and this is submitted to an extra Bianchi identity. Torsion is nevertheless quite absent in other bundles, like those related to usual (internal) gauge theories. Notice that "absent" is quite different from "null". The fact that  $T = 0$  has deep consequences in geometry (that is the general definition of a Riemannian space, one with vanishing torsion). It remains for us that the presence of  $T$  will, already from the

start, establish a difference for any gauge theory related to the geometry of spacetime.

Another step is the following: the space  $\mathbf{R}^4$ , endowed with the Lorentz metric, becomes  $\mathbf{E}^{3,1}$  and can be identified to the translation group  $\mathbf{T}^{3,1}$ . The whole thing can be then rewritten in terms of the bundle of the affine linear bundles. Once reduced by the imposition of Lorentz-orthogonality, the bundle is a principal bundle with the Poincaré group as the structure group (gauge group). One might call it the bundle of the affine orthogonal frames.

Define then the gauge potentials  $\Gamma = J_\alpha^\beta \Gamma^\alpha_{\beta\mu} dx^\mu$  for the Lorentz sector, and  $B = T_\alpha B^\alpha_\mu dx^\mu$  for the translational sector. Accordingly, an extra field strength will appear:

$$\tau^\alpha_{\mu\nu} = \partial_\mu B^\alpha_\nu - \partial_\nu B^\alpha_\mu + \Gamma^\alpha_{\epsilon\mu} B^\epsilon_\nu - \Gamma^\alpha_{\epsilon\nu} B^\epsilon_\mu.$$

Given then the Poincaré group, with its structure constants, one writes directly the vacuum Yang-Mills equations:

$$\partial_\mu F^{\alpha\beta\mu\nu} - \Gamma^\alpha_{\gamma\mu} F^{\gamma\beta\mu\nu} + F^\alpha_{\gamma}{}^{\mu\nu} \Gamma^{\gamma\beta}_\mu = 0;$$

$$\partial_\mu \tau^{\alpha\mu\nu} - \Gamma^\alpha_{\gamma\mu} \tau^{\gamma\mu\nu} + F^\alpha_{\gamma}{}^{\mu\nu} B^\gamma_\mu = 0.$$

Because translation generators have dimensions, the corresponding fields have rather strange dimensions themselves:  $[B] = 0$  and  $[\tau] = 1$ . The relation between the translational gauge potential and the tetrads is given by [26]:

$$h^\alpha_\mu = \frac{\partial x^\alpha}{\partial x^\mu} + k B^\alpha_\mu.$$

Thus,  $B$  is the non-trivial, anholonomous part of the tetrad. There is more here than a mere coordinate transformation. The torsion relates to the field strengths  $F$  and  $\tau$  by  $T = \tau - Fx$ . The Yang-Mills equations become

$$\partial_\mu F^{\alpha\beta\mu\nu} - \Gamma^\alpha_{\gamma\mu} F^{\gamma\beta\mu\nu} + F^\alpha_{\gamma}{}^{\mu\nu} \Gamma^{\gamma\beta}_\mu = 0; \quad (1)$$

$$\partial_\mu T^{\alpha\mu\nu} - \Gamma^\alpha_{\gamma\mu} T^{\gamma\mu\nu} + F^\alpha_{\gamma}{}^{\mu\nu} h^\gamma_\mu = 0. \quad (2)$$

It is remarkable that the above equations are just the Bianchi identities written for the dual fields, so that duality symmetry is respected.

These equations would give an answer to the issue referred to in the introduction concerning the complete treatment of the local spacetime symmetries. The Noether current for the translational invariance appears as the source in the torsion equation, whereas the Noether current for the rotational and boost invariance, the relativistic angular momentum density, comes up as the source for the curvature equation. The presence of a dynamical equation for the torsion gives the theory an advantage over the theories of Einstein-Cartan type.

Equations (1) and (2) have been proposed directly by Popov and Daikhin [27], but for them the tetrads, and not their non-trivial parts were supposed to be the translational gauge potentials. There are great difficulties [28] with this interpretation. The tetrads, as we have seen, are always there. There is no way of annullating them, there would be no way to describe the absence of gravitational field. The theory has some very good points:

- (i) in the sourceless static spherically symmetric case, one finds Newton's law with  $L = \sqrt{4\pi G}$ ;
- (ii) if we put by hand  $T = 0$ , the second equation gives  $F^\alpha{}_\gamma{}^{\mu\nu} h^\gamma{}_\mu = 0$ , which is the vacuum Einstein's equation;
- (iii) technically, as gauge fields have spin 1, particle-particle interaction has opposite sign to particle-antiparticle interaction; but, the indices  $\alpha, \mu$ , etc in  $B_\mu^\alpha$  are all related to spacetime, so that we can possibly find a way of taking  $B$  as a spin 2 field [29].

There are, however, differences with respect to usual gauge theories. For example, the Poincaré group is non-semisimple, which implies that there is no Lagrangian yielding its Yang-Mills equations. Moreover, as it acts on spacetime, it has kinematic representations, which will respond for the universality of gravitation. And finally, it is worth mentioning that the fields have abnormal dimensionalities:  $[B] = [h] = 0$  and  $[\tau] = [T] = 1$ .

## 5 Quantization: from Poincaré to de Sitter

Well, the goal is to obtain a quantum theory! Thus, we take the equations and try to quantize them. We would hope to be able to use all the so well lubricated machinery of gauge theories: partition function, Faddeev-Popov, etc. We start by looking for the Lagrangian density. And then, we have the first shock: the above Yang-Mills equations have no Lagrangian density, they do not come from any action through an extremal principle [30]. This is a negative point, but it should be remembered that at least one basic equation of classical Physics is in the same case, the Euler (and the Navier-Stokes) equation of fluid dynamics.

Whether or not an equation or set of equations come from an action functional is the object of the Helmholtz-Vainberg theorem, which can be put in a very simple form if we generalize exterior differential calculus to functionals [31]. We can show that, for the Yang-Mills equation to have a Lagrangian, the gauge group must be in one of the following cases: (i) a semisimple group; (ii) an abelian group; (iii) a direct product of an abelian and a semisimple group. As the Poincaré group is in neither of these cases, the equations have no Lagrangian. We are thus condemned to try to quantize the theory without resource to any of the usual Lagrangian-based techniques. It is possible in

principle to quantize a theory directly from the field equations, as in the formalism of Källén-Yang-Feldman [32]. One expects difficulties, of course. For example, it will be impossible to use the Faddeev-Popov trick, so that ghosts should be introduced in the rather non-systematic way of Feynman's "polish paper" [33]. Actually, one does not even have to face these problems — a second shock is waiting for us before that: the theory is not quantizable! It presents a sickness, which we can call vertex inconsistency, or perturbative incoherence. There exists a general procedure [34] to check whether or not a set of field equations leads to a coherent theory (as a curiosity, if you try to take the Navier-Stokes equation as a model field theory equation, it is incoherent: it leads to a theory which cannot be quantized). This general formalism shows that every Lagrangian theory is automatically consistent, but that there are in principle also non-Lagrangian theories which are consistent. But if we apply it to Yang-Mills equations for non-semisimple groups, we find that they are never consistent. Well, what interests us here is the bad result: the Yang-Mills equations for the Poincaré group cannot be quantized.

We should not, however, minimize the arguments favouring a Poincaré group gauge theory for gravitation. The group is *the* classical group for relativistic kinematics, it classifies all elementary particles, etc. On the other hand, one sees that such a theory is essentially classical, it cannot be given a quantum version. Is there a way out of the above difficulty? Actually, there is: the same general procedure which tells whether a set of field equations leads or not to a coherent theory provides a systematic way to "patch the theory up". It allows to obtain the minimum terms which should be added to an inconsistent theory in order to make of it a good theory. The method is analogous to that of renormalization: one adds extra terms so that the overall theory becomes tractable. The difference here is that terms are to be added to the field equations.

The simplest form to get a consistent theory is to drop all terms coupling  $B$  to  $\Gamma$  in eqs.(1-2). The result is a consistent Lagrangian theory, which is actually a gauge model for the direct product between Lorentz and translation groups. However, as a vector field, the gauge potential  $B$  should couple to the Lorentz gauge potential  $\Gamma$ . This can be achieved by considering  $B$  as a source field, and then replacing all ordinary derivatives by covariant ones. The result is

$$\partial_\mu F^{\alpha\beta\mu\nu} - \Gamma^\alpha_{\gamma\mu} F^{\gamma\beta\mu\nu} + F^\alpha_{\gamma}{}^{\mu\nu} \Gamma^{\gamma\beta}_{\mu} = L^{-2} \tau^{\alpha\mu\nu} B^\beta_{\mu} \quad (3)$$

$$\partial_\mu \tau^{\alpha\mu\nu} = \Gamma^\alpha_{\gamma\mu} \tau^{\gamma\mu\nu}. \quad (4)$$

The factor  $L^{-2}$  has to be introduced for dimensional reasons. These equations can be derivable from the Lagrangian  $\mathcal{L} = -\frac{1}{4}(F^2 + L^{-2}\tau^2)$ , from which we can see that the sources appearing in eqs.(3-4) are respectively the spin density and the spin energy-momentum tensor of the  $B$  field.

The next step is to find a way of introducing the term  $F^\alpha{}_\gamma{}^{\mu\nu}B^\gamma{}_\mu$  into eq.(4) in order to go as close as possible to the eqs.(1-2) of the Poincaré gauge model. This can be achieved by adding the term  $-(1/2L^2)F_{\alpha\beta}{}^{\mu\nu}B^\alpha{}_\mu B^\beta{}_\nu$  to the Lagrangian, which will contribute correctly to give eq.(2), as well as will give a further contribution to eq.(1). The new Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}{}^{\mu\nu} \left( F^{\alpha\beta}{}_{\mu\nu} + 2L^{-2}B^\alpha{}_\mu B^\beta{}_\nu \right) - 4L^{-2}\tau_\alpha{}^{\mu\nu}\tau^\alpha{}_{\mu\nu}, \quad (5)$$

which represents a rather complicated, but consistent theory.

Once we arrive at this point, we can examine the renormalization question. The situation is found to be quite analogous to the meson electrodynamics case: as it stands, the theory is nonrenormalizable, but it so happens that a counterterm of quartic type,  $(-1/4L^4)(B^\alpha{}_\mu B^\mu{}_\alpha B^\beta{}_\nu B^\nu{}_\beta)$  added to the Lagrangian (5) makes the theory renormalizable. The Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4} \left[ \left( F + 2L^{-2}BB \right)^2 - 4L^{-2}\tau^2 \right]. \quad (6)$$

And here comes the main point. We have said that only a few theories are perturbatively renormalizable, and we have arrived at one of them. It should be of one of the types given previously. Once we think in that way, it comes not as a surprise that the above theory — which is a Poincaré gauge theory corrected so as to be quantizable, and corrected so as to be renormalizable — is also a gauge theory. In effect, if we redefine the fields by absorbing the abnormal dimensions:  $\Gamma^{\alpha 5}{}_\mu := L^{-1}B^\alpha{}_\mu$ ,  $F^{\alpha 5}{}_{\mu\nu} := L^{-1}\tau^\alpha{}_{\mu\nu}$ , and put  $F^{\alpha\beta}{}_{\mu\nu} := F^{\alpha\beta}{}_{\mu\nu}(old) + L^{-2}(B^\alpha{}_\mu B^\beta{}_\nu - B^\alpha{}_\nu B^\beta{}_\mu)$ , we find that the above Lagrangian is that of a gauge theory for semi-simple group, actually a de Sitter group! In this way, the de Sitter gauge theory comes up as the quantizable, renormalizable Poincaré gauge theory [35].

## 6 Final comments

There are, of course, some problems. One of them concerns the non-compact character of the de Sitter group, and to the consequent unboundedness of the Hamiltonian. This problem is plausibly solved through the choice of convenient boundary conditions [36]. A second problem concerns interpretation of the length parameter  $L$ . Instead of the usual geometrical picture of a Minkowski space tangent to each point of spacetime, the tangent spaces are now de Sitter spaces. In this case,  $L$  is a length parameter attached to any de Sitter space, its pseudo-radius. The usual picture is here reversed: spacetime itself is Minkowski space, while the tangent spaces are the curved de Sitter spaces. In the case of DS(3,2), the covering space is topologically  $E^4$  and the tangent spaces are at least diffeomorphic to a flat space.

In order to accept a de Sitter group as the group of relativistic kinematics in replacement to the Poincaré group, one should accept that it, and not Poincaré, classifies the elementary particles. This is acceptable for present-day knowledge, provided  $L$  is large enough. The contact with usual Minkowski results could then be obtained through the use of stereographic coordinates. The definitions of energy and momentum would be changed, though in each case the departures from their usual meanings would be of order (at least)  $L^{-1}$ .

The de Sitter approach has many fine points, coming mainly from formal aspects [37]. One of them, noticed by Dirac a long time ago, refers to the  $\gamma$  matrices. The de Sitter group puts  $\gamma_5$  on an equal footing with other  $\gamma$ 's, as its generators in the spinor representation are  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$  and  $\sigma_{5\nu} = \frac{i}{2}[\gamma_5, \gamma_\nu]$ . This means that chiral symmetry has a foot in the theory and suggests a partial symmetry breaking. Another one is purely geometrical: the de Sitter groups coincide with the bundles of Lorentzian frames on a de Sitter space:  $DS(3,2) = SO(3,2)/SO(3,1)$  and  $DS(4,1) = SO(4,1)/SO(3,1)$ .

There are prospective effects concerning both cosmology and particle physics. Let us only touch two of them.

1. We have said that the Poincaré theory leads to Newton's law in the appropriate limit. From the additional terms appearing in the Yang-Mills equations of the de Sitter theory, we should expect some modification of this law at distances of the order of  $L$ . Changing Newton's law would alter the interpretation of data concerning the missing mass problem in galaxies and clusters. This would suggest values of cosmological scale for the universal constant  $L$ .
2. If one of the de Sitter groups is to replace the Poincaré group as the basic, kinematic classifying group, a lot is changed. The concept of energy, as said, must be reformulated. There is more: both CP and CPT will be universally violated in their present-day formulation. By "universally violated" we mean that all interactions will feel the same effects, so that in CP (always in its usual formulation) there will be no reason to privilege the  $K^0 \bar{K}^0$  system. Also the "new CPT" will differ from the usual operation. Nevertheless, the violation should be less than something, as no universal violation has been observed. Since the differences are proportional to factors of  $L^{-1}$ ,  $L$  can always be taken large enough to make them negligible. Or, if experiments come to detect such an effect, they will also fix the value of  $L$ . The  $K^0 \bar{K}^0$  system, which allows very high precision measurements of the mass splitting, seems to be a good candidate.



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