PRINCIPAL BUNDLES, CONNECTIONS AND BRST COHOMOLOGY

H. García-Compeán, J.M. López-Romero, M.A. Rodríguez-Segura and M. Socolovsky

Abstract

We review the elementary theory of gauge fields and the Becchi-Rouet-Stora-Tyutin symmetry in the context of differential geometry. We emphasize the topological nature of this symmetry and discuss a double Chevalley-Eilenberg complex for it.

1 Introduction

From their appearence *gauge theories* [1] have had a large influence on both physics and mathematics. On the physical side one can date back to Maxwell-Faraday (MF) abelian gauge theory (AGT) unifying electric and magnetic phenomena (1860-70); the Einstein theory of general relativity (GR) (1915) describing the gravitational force; first attemps of Weyl (1919), and Kaluza and Klein (1919, 1926) to unify electromagnetism with gravity; the birth of non abelian gauge theories (NAGT) in 1954 with the seminal work of Yang and Mills (YM) which together with ideas from solid state physics (basically that of spontaneous symmetry breaking) led to the SU(2)$\times$U(1) Glashow -

Weinberg - Salam (1961, 1967, 1968) electroweak (EW) theory unifying the electromagnetic and the weak nuclear forces. (Notice that from the geometrical point of view the EW theory involves just two spheres: S^3 and S^1.) We should stress here that till now we have only mentioned the *classical* parts of the corresponding theories; *quantum* electrodynamics (QED) and quantum NAGT's were proved to be renormalizable *i.e.* capable of absorbing infinite quantities appearing in perturbation theory through the re-normalization of masses and coupling (interaction) constants, after the contribution of a large number of physicists, among others Dirac, Feynman, Dyson, Tomonaga, Schwinger, Salam and Ward for the case of QED (1930-50) and Faddeev, Popov, t'Hooft and Veltman for quantum NAGT's (1967, 1971-72). The theory of the strong nuclear force based on the group $SU(3)$ (unfortunately not a sphere!), quantum chromodynamics (QCD) found its place in the present context after the works of Politzer, Weinberg, Gross and Wilczek (1973), thus leading to the succesful $SU(3) \times SU(2) \times U(1)$ standard model (SM) for the electronuclear (EN) interactions. Unfortunately it does not exist at present a renormalizable theory of quantum gravity (QG); following Hawking [2] we might say that perhaps the greatest problem of theoretical physics in the last quarter of our century is the conciliation of quantum mechanics and general relativity. In this field there are at least two approaches: one is that of Ashtekar and co-workers [3] who mantain general relativity as the classical limit but construct the quantum theory after redefining the fundamental variables, so the hope exists of constructing a theory of quantum general relativity (QGR); the other is that of string theory (ST) (1970, 1974, 1984) [4] which represents a deep departure from the usual description of elementary particles since at the roots of the theory is the idea that the fundamental objects in Nature are not point-like but *string-like* *i.e.* extended objects (even at the classical level!) though extremely small (of the order of $10^{-35}m$) so that GR is modified at short distances and therefore a theory of quantum gravity should not be QGR but what we might call quantum string gravity (QSG); obviously GR as a classical theory is recovered from ST in the large distance limit.

On the mathematical side the list of applications of gauge theory is much shorter, but however of great importance: it consists in the application done by Donaldson (1983) [5] of the theory of moduli spaces of instantons (self-dual and anti-self-dual solutions to the classical YM equations based on the group $SU(2)$) to the problem of classification of closed orientable 1- connected differentiable 4-manifolds, and its relation to the same problem but for topological manifolds previously considered by Freedman (1982) [6].

From the physical point of view the basic idea behind gauge theory [1] is to extend a *global* symmetry of a Lagrangian describing a particular set of free (non-interacting) fields to a *local* symmetry *i.e.* one in which the symmetry transformations of the fields can be done in an independent way at each space-time point. One of the most beautiful and important results of this procedure is the appearance of *physical interactions* (couplings) among the originally non-interacting fields. For example starting from the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(\gamma^\mu \partial_\mu + m)\psi$ describing free electrons and positrons (ψ and $\bar{\psi}$ fields) which has a global $U(1)$ symmetry $\psi \to \psi' = e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-i\alpha}$ ($\alpha =$ const. $\in \mathbf{R}$ and γ^μ : Dirac matrices, $\mu = 0,1,2,3$), the introduction of the electromagnetic (photon) field (gauge potential) through the replacement $\partial_\mu \to D_\mu := \partial_\mu + A_\mu$, with A_μ transforming as $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + i\partial_\mu \alpha(x)$ (notice the locality $\alpha = \alpha(x)$!) leads to the interaction between the electron-positron field and the photon field *i.e.* to the term $\bar{\psi}\gamma^\mu A_\mu \psi$ in the QED Lagrangian, $\mathcal{L}_{QED} = \bar{\psi}(\gamma^\mu D_\mu + m)\psi$. Also, while global symmetries lead to conserved quantities (like electric charge), local symmetries lead to Ward-Takahashi-Slavnov-Taylor identities among Green's functions that is crucial for the proof of renormalizability.

Geometrically, classical gauge theories are *theories of connections on principal fiber bundles* (p.f.b.'s), and related concepts in associated bundles (like sections and covariant derivatives). In this framework the gauge potentials of physics are local pull-backs on the base space (typically a space-time) of connections which are differential 1-forms globally defined on the total space of the bundle satisfying a set of suitable conditions and with values in the Lie algebra of the symmetry group of the theory (the fiber of the bundle), e.g. $U(1)$ in the case of QED, $SU(2) \times U(1)$ for EW theory, $SU(3)$ for QCD, $SO(3,1)$ for GR, etc. It is interesting to mention that even a concept like spontaneous symmetry breaking has been incorporated into the bundle language, see e.g. ref. [7]. The introduction of p.f.b.'s inmediately leads us to consider problems in infinite dimensional geometry, e.g. that of the gauge group of the bundle and its Lie algebra; as we shall see in this article it is precisely the cohomology of this algebra which leads to the extension (BRST symmetry) into the quantum domain of the classical gauge symmetry.

The purpose of the present article is to review some basic ideas involved in the theory of gauge fields from the geometrical viewpoint. The material presented here is not original, except possibly the idea of extending the usual BRST complex into a double complex which could be naturally studied through the use of spectral sequences. In section 2 we define principal and associated fiber bundles, discuss the gauge group and its Lie algebra (without entering into the difficulties of the relevant analysis), comment about the idea of classification of p.f.b.'s, and discuss vector spaces of sections of suitable vector bundles which are relevant for the next section. In section 3 the concept of connection is introduced in its four different variants, together with the definitions of: curvature; gauge transformation of a connection and its relation to the local transformations in physics, which can then be understood as changes of local trivializations of the bundle; covariant derivative and parallel transport in associated bundles; and the YM function. Finally we discuss the total space of connections and its quotient by the gauge group. In section 4, we briefly discuss the Bonora and Cotta-Ramusino [19] geometrical interpretation of the (quantum) BRST symmetry and cohomology [20] of gauge theories, as the Chevalley-Eilenberg cohomology of the Lie algebra of the gauge group with coefficients in the space of functions on the space of connections. The definition of the relevant coboundary operator only depends on the principal fiber bundle in question and is independent of any particular connection (with the possible exception of a base point), thus suggesting a deep relation between the topology of fiber bundles and quantum mechanics. The definition of the usual BRST cochain complex allows an inmediate generalization into a doble complex whose total cohomology could in principle be computed through the use of spectral sequences.

2 Principal and associated bundles

In this section we shall present as much information as we need about a principal fiber bundle (or an equivalence class of principal fiber bundles) without using the notion of a connection (gauge field), which is an additional structure that we can impose on a principal bundle and which basically allows to define the concepts of parallel transport in the total bundle space and of covariant derivatives of sections (matter fields) in associated bundles (section 3), which physically lead to the concept of coupling (interaction) between the matter and the gauge fields. As we shall see in section 4 the concept of BRST cohomology in its algebraic formulation is a property of the bundle itself and does not involve any particular connection, except for the choice of a base point. In fact it depends on the total space of connections which is a natural object associated with the bundle. In this sense we can argue that the BRST cohomology is a property of the "space" where the connections live.

A smooth *principal fiber bundle* (p.f.b.) is a sextet $\xi = (P, B, \pi, G, \mathcal{U}, \psi)$ where P (total space) and B (base space) are respectively $s+r$- and s-dimensional differentiable manifolds, $P \xrightarrow{\pi} B$ (projection) is a smooth surjective function, G is an r-dimensional Lie group (structure group) which acts freely and smoothly on P through $P \times G \xrightarrow{\psi} P$, $\psi(p,g) = \psi_g(p)$ $(= pg)$, $\psi_g^{-1} = \psi_{g^{-1}}$, and transitively on fibers $G_b = \pi^{-1}(\{b\})$, $b \in B$ *i.e.* for all $p, q \in G_b$ there exists $g \in G$ such that $q = pg$ (since for any $b \in B$, G_b is diffeomorphic to G, one says that G is *the* fiber of the bundle); $\mathcal{U}$ is an atlas on ξ *i.e.* $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in J}$ with open $U_\alpha \subset B$ and $P_\alpha = \pi^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times G$, $\phi_\alpha(p) = (\pi(p), \gamma_\alpha(p))$ a diffeomorphism satisfying the condition $\pi_1 \circ \phi_\alpha = \pi_\alpha$ $(\pi_\alpha = \pi|_{P_\alpha})$; $\gamma_\alpha : P_\alpha \longrightarrow G$ is smooth and satisfies $\gamma_\alpha(pg) = \gamma_\alpha(p)g$. Two p.f.b.'s $\xi = (P, B, \pi, G, \mathcal{U}, \psi)$ y $\xi' = (P', B, \pi', G, \mathcal{U}', \psi')$ are equivalent ($\xi \cong \xi'$) if there exist smooth maps $\alpha : P \to P'$, $\beta : P' \to P$ such that: 1) $\pi' \circ \alpha = \pi, \pi \circ \beta = \pi'$, 2) α, β are G-equivariant , 3) $\beta \circ \alpha = 1_P$, $\alpha \circ \beta = 1_{P'}$. (For simplicity, in the following we shall use the notation $\xi : G \longrightarrow P \xrightarrow{\pi} B$, saying that ξ is a p.f.b. on B or a G- bundle on B.)

Given ξ' and ξ p.f.b.'s, a *bundle map* $\xi' \longrightarrow \xi$ is a triple (α, β, h) where $P' \xrightarrow{\alpha} P$ and $B' \xrightarrow{\beta} B$ are smooth functions and $G' \xrightarrow{h} G$ is a Lie group homomorphism such that $\psi \circ (\alpha \times h) = \alpha \circ \psi'$ and $\pi \circ \alpha = \beta \circ \pi'$; notice that α induces β: if $b' \in B'$, there exists $p' \in P'$ such that $b' = \pi'(p')$, then $\beta(b') = \beta \circ \pi'(p') = \pi(\alpha(p'))$; the composition of bundle maps is given by $(\alpha', \beta', h') \circ (\alpha, \beta, h) = (\alpha' \circ \alpha, \beta' \circ \beta, h' \circ h)$. If α and β are diffeomorphisms and h is a Lie group isomorphism, then ξ and ξ' are isomorphic p.f.b.'s. In particular for $P' = P$, $B' = B$ and $G' = G$ the set $\mathcal{G} = \mathcal{G}(\xi) = \{(\alpha, id_B, id_G)\}_{\alpha \in Diff(P)}$ is an infinite dimensional Lie group [8] called the *group of vertical automorphisms of* ξ or the *gauge group of* ξ [9]. Each element of $\mathcal{G}$ is represented by the following commutative diagram:

$$\begin{tikzcd}
P \times G \arrow[r, "\alpha \times id_G"] \arrow[d, "\psi"'] & P \times G \arrow[d, "\psi"] \\
P \arrow[r, "\alpha"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\
B \arrow[r, "id_B"] & B
\end{tikzcd}$$

We can give a second version of the gauge group of ξ: let $\Gamma_{eq}(P, G) = \{\gamma : P \longrightarrow G$ smooth, $\gamma(pg) = g^{-1}\gamma(p)g\}$ with the composition law $\gamma \cdot \gamma'(p) = \gamma(p)\gamma'(p)$; there is a group isomorphism $\mathcal{G}(\xi) \xrightarrow{\Sigma} \Gamma_{eq}(P, G)$ given by: $\Sigma(\alpha)(p)$ is such that $p\Sigma(\alpha)(p) = \alpha(p)$. A third version of $\mathcal{G}(\xi)$ will be discussed after the concept of associated bundle is presented.

A *section* of a p.f.b. ξ is a smooth function $B \xrightarrow{s} P$ such that $\pi \circ s = id_B$. One can prove that ξ is *trivial*, *i.e.* there exists a G-equivariant diffeomorphism $P \xrightarrow{\phi} B \times G$ such that $\pi_1 \circ \phi = \pi$ if and only if ξ has a section (given the section the global trivialization is $\phi(p) = (b, g)$ with $b = \pi(p)$ and $g \in G$ such that $p = s(b)g$; given ϕ the section is $s(b) = \phi^{-1}(b, e)$ with e the identity in G). A *local section* in ξ is a smooth function $s_\alpha : U_\alpha \to P_\alpha$ such that $\pi_\alpha \circ s_\alpha = id_{U_\alpha}$; given the atlas $\mathcal{U}$ of a p.f.b. a canonical set of local sections is $\sigma_\alpha(b) = \phi_\alpha^{-1}(b, e)$.

If ξ is a p.f.b. on B and $f : B' \to B$ is a smooth function, then the *pull-back bundle* $f^*(\xi)$ on B' has structure group G and total space $P' = f^*(P) = \{(b', p) | f(b') = \pi(p)\} \subset B' \times P$. Clearly (p_2, f, id_G) is a bundle map $f^*(\xi) \to \xi$. A well known construction due to Milnor [10] says that for any Lie group G (in fact the construction extends to any topological group G) there exists a *universal principal bundle* $\xi G : G \to EG \xrightarrow{\pi} BG$ (unique up to homotopy type) such that for any G-bundle ξ' on B' there exists a smooth function (unique up to homotopy) $B' \xrightarrow{f} BG$ with $\xi' \cong f^*(\xi G)$; in other words, if $\mathcal{B}_B(G)$ denotes the equivalence classes of p.f.b.'s on B with structure group G, and $[B, BG]$ is the set of homotopy classes of maps from B to BG, then $\mathcal{B}_B(G) \cong [B, BG]$; in particular for $G = \mathbf{Z}_2 \cong S^0$, $U(1) \cong S^1$ and $SU(2) \cong S^3$ one respectively has $BS^0 \cong \mathbf{R}P^\infty$, $BS^1 \cong \mathbf{C}P^\infty$ and $BS^3 \cong \mathbf{H}P^\infty$; and $\mathcal{B}_{S^1}(S^0) \cong \pi_1(\mathbf{R}P^\infty) \cong \pi_0(S^0) \cong \mathbf{Z}_2$, $\mathcal{B}_{S^2}(S^1) \cong \pi_2(\mathbf{C}P^\infty) \cong \pi_1(S^1) \cong \mathbf{Z}$ and $\mathcal{B}_{S^4}(S^3) \cong \pi_4(\mathbf{H}P^\infty) \cong \pi_3(S^3) \cong \mathbf{Z}$. In physical applications, $SU(2) - bundles$ on $S^4 = \mathbf{R}^4 \cup \{\infty\}$ and $U(1) - bundles$ on $S^2 = \mathbf{R}^2 \cup \{\infty\}$ are important examples, the choice of an equivalence class of such bundles being equivalent to the choice of an integer number, in physical terms the *winding number* also called the *instanton* or *monopole* number for the $SU(2)$ and $U(1)$ cases respectively.

Let X be a differentiable manifold and $G \times X \xrightarrow{\mu} X$, $\mu(g,x) = \mu_g(x)(= g \cdot x)$, $\mu_g^{-1} = \mu_{g^{-1}}$ a smooth left action of G on X. Together with the p.f.b. ξ this action induces the *associated fiber bundle* $\xi_X : X - P \times_G X \xrightarrow{\pi_X} B$ with fiber X, total space $P_X = P \times_G X = \{\langle p, x\rangle\}_{(p,x)\in P\times X}$, $\langle p, x\rangle = \{(pg, g^{-1} \cdot x)\}_{g\in G}$, base space B, projection $\pi_X(\langle p, x\rangle) = \pi(p)$, and local triviality condition given by $P_\alpha \times_G X \xrightarrow{\Phi_\alpha} U_\alpha \times X$, $\Phi_\alpha(\langle p, x\rangle) = (\pi(p), \gamma_\alpha(p) \cdot x)$ in a given atlas $\mathcal{U}$ of ξ. If $X = V$ is a real (complex) n-dimensional vector space then ξ_V is called a real (complex) vector bundle of rank n. There is a bijection between the set of equivariant functions $\Gamma_{eq}(P, X) = \{\gamma : P \to X \text{ smooth}, \gamma(pg) = g^{-1} \cdot \gamma(p)\}$ and the set of sections of $\xi_X, \Gamma(\xi_X)$: in fact $\gamma \in \Gamma_{eq}(P, X)$ induces $s_\gamma \in \Gamma(\xi_X)$ with $s_\gamma(b) = \langle p, \gamma(p)\rangle$ for any $p \in G_b$ and *viceversa*, $s \in \Gamma(\xi_X)$ induces $\gamma_s \in \Gamma_{eq}(P, X), \gamma_s(p) = x$ where $s(\pi(p)) = \langle p, x\rangle$. Pictorially,

$$\begin{tikzcd}
G \arrow[d] & \\
P \arrow[r, "\gamma"] \arrow[d, "\pi"'] & X \arrow[d, dash] \\
B & P_X \arrow[d, shift left, "\pi_X"] \\
& B \arrow[u, shift left, "s"]
\end{tikzcd}$$

If P $\xrightarrow{f}$ P is a gauge transformation of ξ *i.e.* an element of $\mathcal{G}$ (ξ) and s $\in \Gamma(\xi_X)$ then the gauge transformation of s is defined to be

$$s' := s_{f^*(\gamma_s)} \tag{1}$$

with $f^*(\gamma_s)$ given by the diagram

$$\begin{tikzcd}
& X \\
P \arrow[r, "f"'] \arrow[ur, "f^*(\gamma_s)"] & P \arrow[u, "\gamma_s"']
\end{tikzcd}$$

then s'(b) $= \langle p, \gamma_s \circ f(p) \rangle = \langle p, \gamma_s(pg) \rangle = \langle p, g^{-1} \cdot \gamma_s(p) \rangle$ with p$\in G_b$ and g $\in G$ such that f(p) = pg.

For later applications, two important bundles canonically associated to a p.f.b. ξ are the following. Let **g** be the Lie algebra of G. $G \times G \xrightarrow{Ad}$

$G, (g,h) \mapsto Ad(g,h) = A_g(h) = g \cdot h = ghg^{-1}$ and $G \times \mathbf{g} \xrightarrow{ad} \mathbf{g}, (g,v) \mapsto ad(g,v) = g \cdot v := A_{g*e}(v) = dA_g|_e(v)$ are the left *adjoint actions* of G on itself and on $\mathbf{g}$, respectively. Associated with these actions one has the *bundle of Lie groups of* ξ, $\xi_G = (P_G = P \times_G G \equiv F, B, \pi_G, G)$ and the *bundle of Lie algebras of* ξ, $\xi_{\mathbf{g}} = (P_G = P \times_G \mathbf{g} \equiv E, B, \pi_{\mathbf{g}}, \mathbf{g})$ which is a real vector bundle of rank r. It is easy to see that if $\Gamma(\xi_G) = \Gamma(F)$ is the space of sections of ξ_G with composition law $s \cdot s'(b) = \langle p, gg' \rangle$ if $s(b) = \langle p, g \rangle$ and $s'(b) = \langle p, g' \rangle$, then $\Gamma_{eq}(P,G) \xrightarrow{\mu} \Gamma(\xi_G)$ given by $\mu(\gamma)(b) := \langle p, \gamma(p) \rangle$ with $p \in G_b$ is a group isomorphism , which provides the promised third equivalent version of the gauge group: we have the isomorphism $\mathcal{G}(\xi) \xrightarrow{\mu \circ \Sigma} \Gamma(\xi_G)$. Moreover, $\Gamma(\xi_{\mathbf{g}}) = \Gamma(E)$, the space of sections of $\xi_{\mathbf{g}}$, is the *Lie algebra* of $\mathcal{G}(\xi)$ with exponential map $\Gamma(\xi_{\mathbf{g}}) \xrightarrow{Exp} \Gamma(\xi_G)$ given by $Exp(\sigma)(b) = \langle p, exp(v) \rangle$ if $\sigma(b) = \langle p, v \rangle$ and $exp : \mathbf{g} \to G$ is the usual exponential function associated with the Lie group G. We summarize this with the following picture:

$$
\begin{array}{ccc}
G & & \mathbf{g} \\
| & & | \\
F & & E \\
\pi_G \downarrow\uparrow s & \xleftarrow{Exp} & \sigma \uparrow\downarrow \pi_{\mathbf{g}} \\
B & & B
\end{array}
$$

Notice that locally (or globally for trivial bundles) $P_\alpha \times_G G \cong U_\alpha \times G$ and $P_\alpha \times_G \mathbf{g} \cong U_\alpha \times \mathbf{g}$; so $\mathcal{G}(\xi_\alpha) \cong \Gamma((\xi_G)_\alpha) \cong C^\infty(U_\alpha, G)$: G-valued smooth functions on U_α, and Lie $(\mathcal{G}(\xi_\alpha)) \cong \Gamma((\xi_{\mathbf{g}})_\alpha) \cong C^\infty(U_\alpha, \mathbf{g})$: $\mathbf{g}$-valued smooth functions on U_α.

The center of $\mathcal{G}$ is defined as the subset z of $\Gamma(\xi_G)$ such that $s(b) = \langle p, g \rangle$ with $g \in Z(G)$, the center of G; such elements are well defined since $\langle p, g \rangle = \langle ph, h^{-1}gh \rangle = \langle ph, g \rangle$. Clearly z is an invariant subgroup of $\mathcal{G}$ and one has the quotient group $\mathcal{G}/\mathrm{z}$. For later use the subgroup $\bar{z}$ of z of constant Z-valued sections of ξ_G is of interest (in particular $\bar{z} = z$ for $SU(n)$), also $\bar{\mathcal{G}} := \mathcal{G}/\bar{z}$ is well defined.

For $p = 0, 1, \ldots, s = dim\ B$ define the infinite dimensional real vector spaces $A^p := \Gamma(\bigwedge^p T^*B \otimes E)$: differential p-forms on B with values in the Lie algebra of $\mathcal{G}(\xi)$ *i.e.* if $\alpha \in A^p$ then $\alpha : \Gamma(TB) \times \ldots \times \Gamma(TB) \to \Gamma(E), (X_1, \ldots X_p) \mapsto \alpha(X_1, \ldots, X_p) : B \to E; A^0 = \Gamma(\xi_{\mathbf{g}})$ and as we shall see in the next section the affine space of connections on $\xi, \mathcal{C}(\xi)$ is modelled on A^1 and therefore $A^1 \cong T_\omega(\mathcal{C}, \omega_0)$ for any $\omega \in \mathcal{C}$ and arbitrary fixed $\omega_0 \in \mathcal{C}$ (base point). After Sobolev completion the $A^p{}'s$ become complete inner product linear spaces *i.e.* Hilbert spaces [8] with the inner products defined as follows: i) let G be a compact connected simply-connected Lie group e.g. $G = SU(2)$; then the Killing form on $\mathbf{g}$, $\mathcal{B} : \mathbf{g} \times \mathbf{g} \to \mathbf{R}, \mathcal{B}(v, w) := -tr(ad(v) \circ ad(w))$ with $ad : \mathbf{g} \to End(\mathbf{g}), ad(v)(w) = [v, w]$ the adjoint representation of $\mathbf{g}$ (ad is a Lie algebra homomorphism : $ad([v, w]) = ad(v) \circ ad(w) - ad(w) \circ ad(v))$ is a positive-definite non-degenerate symmetric bilinear form [11]; $\mathcal{B}$ induces the Killing-Cartan Riemannian metric on G, $\langle v_g, w_g \rangle_g = \mathcal{B}(L_{g^{-1}*g}(v_g), L_{g^{-1}*g}(w_g))$, where L_g is the left translation by g; ii) let B be a compact and orientable manifold; paracompactness and orientability respectively guarantee the existence of a Riemannian metric and therefore of a Hodge-* operation and of a volume form *1 on B; then the inner product on A^p is given by $\langle , \rangle_p : A^p \times A^p \to \mathbf{R}, \langle \alpha, \beta \rangle_p := -\int_B tr(\alpha \wedge * \beta)$. Locally, $-tr(\alpha \wedge * \beta) = \frac{1}{p!} g^{k_1 l_1} \ldots g^{k_p l_p} \mathcal{B}(\alpha_{k_1 \cdots k_p}, \beta_{l_1 \cdots l_p}) \times \sqrt{det(g_{ij})} dx^1 \wedge \cdots \wedge dx^s$ (repeated indices are summed from 1 to s and at each $b \in B$, $\alpha_{k_1 \cdots k_p}(b)$, $\beta_{l_1 \cdots l_p}(b) \in \mathbf{g}$). For $\alpha \in A^p$, $\| \alpha \|_p := +\sqrt{\langle \alpha, \alpha \rangle_p}$. It can be shown that these inner products are invariant under gauge transformations of ξ. If $\alpha \in A^q$ its gauge transformed under $f \in \mathcal{G}(\xi)$ is defined as follows: let $X_1, \ldots, X_q \in \Gamma(TB)$, then $\alpha(X_1, \ldots, X_q) \in \Gamma(\xi_{\mathbf{g}})$ and $\alpha(X_1, \ldots, X_q)'(b) = \langle p, A_{g^{-1}*e} \circ \gamma_{\alpha(X_1, \ldots, X_q)}(p) \rangle$ with $p \in G_b$. In particular $\| \alpha \|_q$ is gauge invariant for any $\alpha \in A^q$.

Finally, let $\Gamma^k = \Gamma(\bigwedge^k T^*P \otimes \mathbf{g})$ be the vector space of differential k-forms on P with values in $\mathbf{g}$. In applications it is useful to consider the subspace $\bar{\Gamma}^k = \bar{\Gamma}(\bigwedge^k T^*P \otimes \mathbf{g})$ consisting of the k-forms ϕ satisfying the conditions : i) $\phi_p(X_{1p}, \ldots, X_{kp}) = 0 \in \mathbf{g}$ if for some j, $X_{jp} \in V_p = T_p G_b$, the *vertical space at* p, *i.e.* the $\phi's$ are horizontal; ii) for $X_1, \ldots, X_k \in \Gamma(TP)$, $g \in G$ and $q = pg \in P$, $\psi^*_{gq}(\phi_q)(X_{1p}, \ldots, X_{kp}) = \phi_q(\psi_{g*p}(X_{1p}), \ldots, \psi_{g*p}(X_{kp})) = A_{g^{-1}*e} \circ \phi_p(X_{1p}, \ldots, X_{kp})$ *i.e.* under diffeomorphisms of P induced by the action of G on P, the $\phi's$ transform according to the adjoint representation of G. ($\bar{\Gamma}^k$ is usually called the space of $Ad - G$ invariant horizontal $k-$forms on P with values in $\mathbf{g}$.) In particular we will show in section 3 that the spaces $\bar{\Gamma}^1$ and A^1 are isomorphic.

3 Space of connections

There are four equivalent definitions of a connection on a p.f.b. or *principal connection*. To be self-contained and for practical purposes, we give here all the definitions leaving to the reader the details of the proof of their equivalence. A *connection* on a *p.f.b.* $\xi = (P, B, \pi, G, \mathbf{U}, \psi)$ is:

a) (Geometric definition) An assignment at each $p \in P$ of a vector subspace H_p (horizontal space) of T_pP with the properties: *i)* $\pi_{*p}|_{H_p} : H_p \to T_{\pi(p)}B$ is a vector space isomorphism; *ii)* $\psi_{g*p}(H_p) = H_{\psi_g(p)}$; *iii)* for all $p \in P$ there exists open $U_p \subset P$ and a set of vector fields $V_{1q}, ..., V_{sq}$ on U_p such that for all $q \in U_p, V_{1q}, ..., V_{sq}$ is a basis of H_q *i.e.* the asignment $p \mapsto H_p$ is a smooth s dimensional distribution on P. A consequence of this definition is that $T_pP = H_p \bigoplus V_p$ and so for each $v_p \in T_pP$ the decomposition $v_p = hor(v_p) \oplus ver(v_p)$ is unique.

b) (Algebraic definition) A differential 1-form on P with values in $\mathbf{g}$ *i.e.* an element $\omega \in \Gamma(T^*P \otimes \mathbf{g})$ with the following properties: i) $\psi_{g_q}^*(\omega_q) = A_{g^{-1}*e} \circ \omega_p$ if $q = pg$ (*i.e.* under the action of G on P, ω transforms according to the adjoint representation of G); ii) $\omega_p(A_p^*) = A$, for all $p \in P$ and for all $A \in \mathbf{g}$ with $A_p^* \in V_p$ and $A^* \in \Gamma(TG_b)$ the fundamental vector field associated with A and defined by $A^* : G_b \to TG_b$, $A^*(p) = (p, A_p^*)$, $A_p^* : C^\infty(G_b, \mathbf{R}) \to \mathbf{R}$, $A_p^*(f) = \frac{d}{dt} f(p\ exp\ tA)|_{t=0}$. The horizontal vector space at p is then defined as $H_p := ker(\omega_p)$ and $\omega_p(v_p) = \phi_p^{-1}(ver(v_p))$ with $\mathbf{g} \overset{\phi_p}{\to} V_p$ the canonical vector space isomorphism given by $\phi_p(A) = A_p^*$. Notice that $\omega_p(v_p) = 0$ iff $v_p \in H_p$ *i.e.* ω is a *vertical* 1- form on P.

c) (Gauge theory definition) A set $\{A_\alpha\}_{\alpha \in J}$ of $\mathbf{g}$-valued differential 1-forms on $\{U_\alpha\}_{\alpha \in J}$ with $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in J}$ an atlas for ξ, related by $A_{\alpha b} = A_{g^{-1}*e} \circ A_{\beta b} + L_{g^{-1}*g} \circ g_{\beta\alpha * b}$ for each $b \in U_\alpha \cap U_\beta$ and where $g = g_{\beta\alpha}(b)$ with $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ such that $\sigma_\alpha(b) = \sigma_\beta(b) g_{\beta\alpha}(b)$. The A_α's are the *gauge potentials* in physical applications and the U_α's are open sets in space-time; usually $U_\alpha = B$ and ξ is trivial. The restriction ω_α on P_α of the unique connection ω in ξ determined by the A_α's satisfies $A_\alpha = \sigma_\alpha^*(\omega_\alpha)$ or $A_{\alpha b} = \omega_{\sigma_\alpha(b)} \circ \sigma_{\alpha * b}$. The local relation between the A_α's is called in physics "gauge transformation"; however according to the modern geometric terminology we shall reserve that name of the elements of the global group $\mathcal{G}(\xi)$.

d) (Topological definition) A G-equivariant splitting γ of the short exact sequence (s.e.s.) of vector bundles over P, $0 \to VP \xrightarrow{i} TP \xrightarrow{\tilde{\pi}} \pi^*TB \to 0$ where $VP = \coprod_{p\in P} V_p$ is the vertical bundle of P, i is the inclusion, $\pi^*(TB) \subset P \times TB$ is the pull-back bundle induced by the projection $P \xrightarrow{\pi} B$ *i.e.*

$$\begin{tikzcd} \pi^*(TB) \arrow[r, "p_2"] \arrow[d, "p_1"'] & TB \arrow[d, "\pi_B"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

and $\tilde{\pi}$ the bundle map induced by *i)* the map

$$\begin{tikzcd} TP \arrow[r, "\pi_*"] \arrow[d, "\pi_P"'] & TB \arrow[d, "\pi_B"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

and *ii)* the factorization property of the pull-back, leading to $\pi_* = p_2 \circ \tilde{\pi}$:

$$\begin{tikzcd} TP \arrow[r, "\tilde{\pi}"] \arrow[dr, "\pi_P"'] & \pi^*(TB) \arrow[r, "p_2"] \arrow[d, "p_1"'] & TB \arrow[d, "\pi_B"] \\ & P \arrow[r, "\pi"'] & B \end{tikzcd}$$

$(\tilde{\pi}(p, v_p) = (p, \pi_*(p, v_p))$. For each $p \in P$, there exists $(\pi^*(TB))_p \xrightarrow{\gamma_p} T_pP$, a 1-1 linear transformation of vector spaces satisfying $\tilde{\pi}_p \circ \gamma_p = id_{(\pi^*(TB))_p}$

and one has the linear isomorphism $V_p \oplus (\pi^*(TB))_p \xrightarrow{\eta_p} T_pP, \eta_p(v_p \oplus w_b) = i_p(v_p) + \gamma_p(w_p)$. Clearly $(\pi^*(TB))_p$ is the horizontal space at $p \in P$ and G-equivariance of γ means the condition *ii)* of definition a). (Paracompactness of P guarantees the existence of a connection on ξ.)

Let $\mathcal{C}$(or $\mathcal{C}(\xi)$) denote the set of *all* connections on ξ, its algebraic structure and topology will be discussed later. Notice that $\mathcal{C}$ is a "natural" object associated with ξ. If ω_0 is a fixed connection (base point) $\mathcal{C}$ is denoted by $\mathcal{C}_0$ or $(\mathcal{C}, \omega_0)$. In the trivial bundle, the *product connection* on $B \times G$ is canonically defined as follows: $TP = TB \oplus TG = \coprod_{(b,g)\in B\times G} T_bB \oplus T_gG$, so $H_{(b,g)} := T_bB$ for all $g \in G$ ($V_{(b,g)} = T_gG$); the connection form ω_0 is given by $\omega_{0_{(b,g)}}(v_b \oplus v_g) = \phi^{-1}_{(b,g)}(ver(v_b \oplus v_g)) = \phi^{-1}_{(b,g)}(v_g) := L_{g^{-1}*g}(v_g) = v_e \in \mathbf{g}$ *i.e.* $\phi^{-1}_{(b,g)} = L_{g^{-1}*g}$ for all $b \in B$ and since $L_{g^{-1}*g} : T_gG \to \mathbf{g}$ is canonical we can identify $\omega_{0(b,g)}(v_b \oplus v_g) \equiv v_g$. For the trivial bundle $B \xrightarrow{\sigma} B \times G, \sigma(b) = (b, e)$ is a global canonical section and if ω is an arbitrary connection, $A_b = \omega_{(b,e)} \circ \sigma_{*b}$ does not vanish in general; however for the product connection the gauge potential vanishes: in fact, $A^0_b(v_b) = \omega_{0(b,e)} \circ \sigma_{*b}(v_b) \equiv ver(\sigma_{*b}(v_b))$ according to the previous identification and $\sigma_{*b}(v_b)$ is horizontal since $T_bB \xrightarrow{\sigma_{*b}} T_bB \oplus \mathbf{g}$, $\sigma_{*b}(v_b) = v_b \oplus 0$, so $A^0 = \sigma^*(\omega_0) = 0$.

A connection on a p.f.b. immediately leads to the concept of *horizontal lifting* of vector fields: let X be a vector field on B, then $X_b \in T_bB$ and we define the horizontal lifting $\tilde{X} \in \Gamma(TP)$ by $\tilde{X}_p = (\pi_*|_{H_p})^{-1}(X_b) \in H_p$.

Let $\omega \in \mathcal{C}$ and $f \in \mathcal{G}(\xi)$, then it is easy to prove that the pull-back $f^*(\omega)$, the *gauge transformed connection*, is also an element of $\mathcal{C}$; a little more work shows that $f^*(\omega)$ is given by

$$f^*(\omega_{f(p)}) = (L_{\gamma(p)^{-1}} \circ \gamma)_{*p} + A_{\gamma(p)^{-1}*e} \circ \omega_p \tag{2}$$

which belongs to $T^*_pP \otimes \mathbf{g}$ and where $\gamma = \Sigma(f) \in \Gamma_{eq}(P, G)$. The right hand side of (2) is given by the sum of the two diagonals in the following diagrams:

$$\begin{tikzcd} & \mathbf{g} \\ T_pP \arrow[ur] \arrow[r, "\gamma_{*p}"'] & T_{\gamma(p)}G \arrow[u, "L_{\gamma(p)^{-1}*\gamma(p)}"'] \end{tikzcd}$$

$$\begin{array}{ccc} & & \mathbf{g} \\ & \nearrow & \uparrow A_{\gamma(p)^{-1}*e} \\ T_pP & \xrightarrow[\omega_p]{} & \mathbf{g} \end{array}$$

In physical language, the formulas (1) and (2) summarize the gauge transformations of matter and gauge fields. Locally, for matrix Lie groups (2) is given by the well known formula

$$\omega' = g^{-1}\omega g + g^{-1}dg \tag{3}$$

where $g \in C^\infty(U_\alpha, G)$.

The *curvature* of the connection ω is the differential 2-form on P with values in $\mathbf{g}$ given by $\Omega = \mathcal{D}\omega := (d\omega)^{hor} \in \Gamma(\Lambda^2 T^*P \otimes \mathbf{g})$ with $(d\omega)^{hor}(X,Y) = d\omega(horX, horY)$; clearly Ω is horizontal *i.e.* $\Omega_p(X_p, Y_p) = 0$ if X_p and/or $Y_p \in V_p$ and one can prove that $\Omega = d\omega + \frac{1}{2}[\omega,\omega]_\wedge$ where $\frac{1}{2}[\omega,\omega]_\wedge(X,Y)$ denotes $\frac{1}{2}\omega^a \wedge \omega^b(X,Y)[e_a,e_b] = \frac{1}{2}(\omega^a(X)\omega^b(Y) - \omega^a(Y)\omega^b(X))\ [e_a,e_b]$ in a basis $\{e_a\}_{a=1}^{dimG}$ of $\mathbf{g}$, for a matrix Lie group $\frac{1}{2}[\omega,\omega]_\wedge(X,Y) = [\omega(X),\omega(Y)]$. A connection ω for which $\mathcal{D}\omega = 0$ is called *flat*. Clearly, the product connection on the trivial bundle is flat. Under a gauge transformation $f \in \mathcal{G}(\xi)$,

$$f^*(\Omega_{f(p)}) = A_{\gamma(p)^{-1}*e} \circ \Omega_p \in \Lambda^2 T_p^*P \otimes \mathbf{g} \tag{4}$$

which locally and for matrix groups reduces to the formula

$$\Omega' = g^{-1}\Omega g \tag{5}$$

with $g \in C^\infty(U_\alpha, G)$.

Let ω be a connection on P, Ω its curvature, $X, Y \in \Gamma(TB)$ and $\tilde{X}$, $\tilde{Y}$ their horizontal liftings. One can prove that $\Omega(\tilde{X},\tilde{Y}) \in \Gamma_{eq}(P,\mathbf{g})$, then $\tilde{\Omega}(X,Y) := s_{\Omega(\tilde{X},\tilde{Y})}$ defines an element $\tilde{\Omega} \in A^2 = \Gamma(\Lambda^2 T^*B \otimes E)$. The *Yang-Mills action* is the function $YM : \mathcal{C} \to \mathbf{R}$, $\omega \mapsto YM(\omega) :=$ $(\parallel \tilde{\Omega} \parallel_2)^2$ whose extrema (critical points in the sense of Morse theory) give the solutions of the classical equations of motion

$$\mathcal{D}^{2*}\tilde{\Omega} = 0 \tag{6}$$

(see below the definition of the covariant derivative and divergence). By gauge invariance of the inner product $\langle , \rangle_2$, YM is indeed a function on the quotient space $\mathcal{C}/\mathcal{G}$ (more precisely $\mathcal{C}'/\bar{\mathcal{G}}$, see below), whose topology is non-trivial due to the non-trivial homotopy of the gauge group $\mathcal{G}$. Finally, Ω satisfies the Bianchi identity $\mathcal{D}\Omega = (d\Omega)^{hor} = 0$; from the physical point of view one can say that the "half" of the classical equations of motion for the gauge fields has purely geometric origin (Bianchi identity), while the "other half" or "dynamical equations" (6) are a consequence of the somewhat arbitrary definition of YM.

Let $\xi_V : V - P_V \xrightarrow{\pi_V} B$ be a real vector bundle associated with $\xi : G \rightarrow P \xrightarrow{\pi} B$, ω a connection on ξ and s a section of ξ_V. As we saw in section 2, s induces $\gamma_s \in \Gamma_{eq}(P,V)$ and if X is a vector field on B then one can prove that $\tilde{X}(\gamma_s)$ is also equivariant *i.e.* $\tilde{X}(\gamma_s) \in \Gamma_{eq}(P,V)$, which induces the *covariant derivative* of s with respect to ω in the direction X

$$\nabla^{\omega}_V(X,s) := s_{\tilde{X}(\gamma_s)} \tag{7}$$

(For $f \in \Gamma_{eq}(P,V)$, $Y \in \Gamma(TP)$ and $\omega \in \mathcal{C}$, the covariant derivative of f with respect to ω in the direction Y is defined as $Df(Y) := df(horY)$, clearly if Y is horizontal $Df(Y) = Y(f)$; for $G \times V \xrightarrow{\mu} V$ a linear action and $p \in P$, $(Df)_p(Y_p) = f_{*p}(Y_p) + \tilde{\mu}_{*e}(\omega_p(Y_p))(f(p)) \in V$ with $\tilde{\mu} : G \rightarrow GL(V)$ given by $\tilde{\mu}(g) := \mu_g$.)

Diagramatically

$$
\begin{tikzcd}
 & G \arrow[d] & \\
TP \arrow[r, shift left, "\pi_p"] & P \arrow[l, shift left, "\tilde{X}"] \arrow[r, "\omega"'] \arrow[d, "\pi"'] & T^*P \otimes \mathbf{g} \\
TB \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "X"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P \arrow[r, shift left, "\gamma_s"] \arrow[r, shift right, "\tilde{X}(\gamma_s)"'] & V \arrow[d, dash] \\
 & P_V \arrow[d, shift left, "\pi_V"] \\
 & B \arrow[u, shift left, "\nabla^\omega_V(X,s)"]
\end{tikzcd}
$$

For brevity one omits V and ω in the symbol of the covariant derivative and writes $\nabla^\omega_V(X,s) = \nabla_X s$. It is easy to verify that the operator $\nabla^\omega : \Gamma(TB) \times \Gamma(P_V) \to \Gamma(P_V), \nabla^\omega(X,s) := \nabla_X s$ is a *linear connection* in ξ_V *i.e.* $\nabla_{X+X'}s = \nabla_X s + \nabla_{X'}s$, $\nabla_{fX}s = f\nabla_X s$, $\nabla_X(s+s') = \nabla_X s + \nabla_X s'$ and $\nabla_X(fs) = X(f)s + f\nabla_X s$ for any $X, X' \in \Gamma(TB), s, s' \in \Gamma(P_V)$ and $f \in C^\infty(B, \mathbf{R})$. In other words, ∇^ω is $C^\infty(B,\mathbf{R})$-linear with respect to X but satisfies the Leibnitz rule with respect to s. The *curvature* of the linear connection ∇^ω is the operator $\mathcal{R}^\omega : \Gamma(TB) \times \Gamma(TB) \to End(\Gamma(P_V)), (X,Y) \mapsto \mathcal{R}^\omega(X,Y) : \Gamma(P_V) \to \Gamma(P_V), s \mapsto \mathcal{R}^\omega(X,Y)(s) := ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(s)$ with $[\nabla_X, \nabla_Y] = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X$. $\mathcal{R}^\omega$ is $C^\infty(B,\mathbf{R})$-linear with respect to X, Y and s *i.e.* $\mathcal{R}^\omega(fX,Y)(s) = \mathcal{R}^\omega(X,fY)(s) = \mathcal{R}^\omega(X,Y)(fs) = f\mathcal{R}^\omega(X,Y)(s)$.

In terms of the previously defined covariant derivative one defines the (linear) *covariant derivative operator* with respect to ω in the associated bundle ξ_V, $d_\omega : \Gamma(P_V) \to \Gamma(T^*B \otimes P_V)$, $s \mapsto d_\omega s : \Gamma(TB) \to \Gamma(P_V), d_\omega s(X) := \nabla_X s$. One easily verifies that $d_\omega(fs) = (df)s + f d_\omega s$. (Again the full notation should be d_ω^V). From the physical point of view the object $d_\omega s$ establishes the *interaction* between the matter field s and the gauge field ω. In the following and with the purpose of practical applications we shall restrict the discussion to the case $V = \mathbf{g}$ (one could of course mantain the discussion at a general level for an arbitrary associated vector bundle ξ_V).

In the same way as the De Rham exterior derivative generalizes the concept of differential of a function, the concept of *covariant exterior differentiation* generalizes the covariant derivative of sections of $\xi_{\mathbf{g}}$. The obvious spaces which replace the spaces $\Omega^p(B)$ of differential p-forms on B are the spaces A^p defined in section 2; then one defines $\mathcal{D}^p : A^p \to A^{p+1}$ with $\mathcal{D}^0 = d_\omega$ and for $p = 1, ..., s, \mathcal{D}^p(\alpha)(X_1, ..., X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{X_i}(\alpha(X_1, ..., \hat{X}_i,$ $..., X_{p+1})) + \sum_{1\leq i<j\leq p+1}(-1)^{i+j}\alpha([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ...X_{p+1})$. Comparing with the objects defined in Appendix A which lead to the Chevalley-Eilenberg (C-E) cohomology of a Lie algebra [12], one is tempted to identify $\mathcal{C}^p = A^p$, $\mathbf{g} = \Gamma(TB)$, $V = \Gamma(E)$ and $\delta^p = \mathcal{D}^p$; however contrary to the case in De Rham theory the sequence $0 \to A^0 \xrightarrow{d_\omega} A^1 \xrightarrow{\mathcal{D}^1} A^2 \xrightarrow{\mathcal{D}^2} \dots \xrightarrow{\mathcal{D}^{s-1}} A^s \to 0$ is *not* in general a complex since $\{\mathcal{D}^p\}_{p=0}^{s-1}$ is not a coboundary operator *i.e.* $\mathcal{D}^{p+1} \circ \mathcal{D}^p$ does not vanish in general. The necessary and sufficient condition to have a coboundary and therefore a C-E cochain complex is that the operator $\delta^\omega : \Gamma(TB) \to End(\Gamma(E)), X \mapsto \delta^\omega(X) : \Gamma(E) \to \Gamma(E), s \mapsto \delta^\omega(X)(s) := \nabla_X s$ be a Lie algebra representation, but from the definition of $\mathcal{R}^\omega$, $\delta^\omega([X,Y]) = [\delta^\omega(X), \delta^\omega(Y)] - \mathcal{R}^\omega(X,Y)$ and thus we have the result that for each *flat* connection ω (if any) on a p.f.b. for which $\mathcal{R}^\omega = 0$ one has a C-E complex *i.e.* the set of operators $\{\mathcal{D}^p\}_{p=0}^{s-1}$ does indeed satisfy $\mathcal{D}^{p+1} \circ \mathcal{D}^p = 0$ and therefore a C-E cohomology $H^*_{CE}(\Gamma(TB),\ \delta^\omega,\ Lie(\mathcal{G}(\xi); \mathbf{R})$. One says that the curvature of the connection is an *obstruction* to the existence of the cohomology of the Lie algebra $\Gamma(TB)$ with coefficients in $\Gamma(E)$.

Independiently of the flatness or not of the connection ω, one can prove that each $\mathcal{D}^p$ has an *adjoint* operator with respect to the inner product $\langle , \rangle_p$ in $A^p, \mathcal{D}^{p*} : A^{p+1} \to A^p$, called *exterior covariant divergence* for $p \geq 1$ and *covariant divergence* for $p = 0$, such that for any $\alpha \in A^p$ and $\beta \in A^{p-1}, \langle \alpha, \mathcal{D}^{p-1}\beta\rangle_p = \langle \mathcal{D}^{p*}\alpha, \beta\rangle_{p-1}$. (The construction of the adjoints runs according to the following general procedure: Let V and W be vector spaces with inner products $\langle , \rangle_V$ and $\langle , \rangle_W$ which induce isomorphisms $V \xrightarrow{\mu_V} V^*$ and $W \xrightarrow{\mu_W} W^*$ given by $\mu_V(v)(v') = \langle v, v' \rangle_V$ and $\mu_W(w)(w') = \langle w, w' \rangle_W$. This is the case e.g. of Hilbert spaces in the infinite dimensional case or for arbitrary finite dimensional inner product spaces. If $V \xrightarrow{f} W$ is a linear transformation (as $A^p \xrightarrow{\mathcal{D}^p} A^{p+1}$ is), $W^* \xrightarrow{\bar{f}} V^*$ given by $\bar{f}(\gamma) = \gamma \circ f$ closes the diagram

$$\begin{array}{ccc} V & \xrightarrow{\mu_V} & V^* \\ f \downarrow\uparrow f^* & & \uparrow \bar{f} \\ W & \xrightarrow[\mu_W]{} & W^* \end{array}$$

and defines the composition $f^* = \mu_V^{-1} \circ \bar{f} \circ \mu_W$ which satisfies $\langle f(v), w \rangle_W = \langle v, f^*(w) \rangle_V$.) One has the sequence

$$0 \longleftarrow A^0 \xleftarrow{d^*_\omega} A^1 \xleftarrow{\mathcal{D}^{1*}} A^2 \xleftarrow{\mathcal{D}^{2*}} \cdots \xleftarrow{\mathcal{D}^{s-2*}} A^{s-1} \xleftarrow{\mathcal{D}^{s-1*}} A^s \longleftarrow 0$$

which again in general is not a chain complex; clearly $\{\mathcal{D}^{p*}\}_{p=0}^{s-1}$ is a *boundary* *i.e.* $\mathcal{D}^{p-1*} \circ \mathcal{D}^{p*} = 0$ iff $\{\mathcal{D}^p\}_{p=0}^{s-1}$ is a coboundary. ($\mathcal{D}^{p*}$ generalizes the codifferential d^* in De Rham theory.) At each p one defines the generalized *Laplace-Beltrami* operator $\Delta_p := \mathcal{D}^{p*} \circ \mathcal{D}^p + \mathcal{D}^{p-1} \circ \mathcal{D}^{p-1*}$ (pictorially

$$\cdots A^{p-1} \underset{\mathcal{D}^{p-1^*}}{\overset{\mathcal{D}^{p-1}}{\rightleftarrows}} A^p \underset{\mathcal{D}^{p*}}{\overset{\mathcal{D}^p}{\rightleftarrows}} A^{p+1} \cdots = \cdots A^p \cdots),$$

in particular $\Delta_0 = d^*_\omega \circ d_\omega : A^0 \longrightarrow A^0$ with inverse (if it exists) the Green function $G_\omega := \Delta_0^{-1}$.

Let $\bar{\Gamma}^1$ be the space of horizontal $Ad - G$ invariant 1-forms on P with values in $\mathbf{g}$, and $\mathcal{C}$ the set of connections on P. It is easy to show that $\bar{\Gamma}^1 \times \mathcal{C} \xrightarrow{\hat{+}} \mathcal{C}, \alpha \hat{+} \omega := \alpha + \omega$, where the sum in the right hand side is the one in $\Gamma(T^*P \otimes \mathbf{g})$, is a free and transitive action of $\bar{\Gamma}^1$ on $\mathcal{C}$; then by definition $(\bar{\Gamma}^1, \mathcal{C}, \hat{+})$ is an *affine space* and if ω_0 is a distinguished connection (base point) in $\mathcal{C}$, $\mu_0 : \bar{\Gamma}^1 \longrightarrow \mathcal{C}_0, \mu_0(\tau) := \tau \hat{+} \omega_0$ is a bijection with inverse $\mu_0^{-1}(\omega) = \omega - \omega_0$. Therefore $(\mathcal{C}_0, \tilde{+}; \mathbf{R}, \tilde{\cdot})$ with $\omega \tilde{+} \omega' := \mu_0^{-1}(\omega) + \mu_0^{-1}(\omega') \hat{+} \omega_0$ and $\lambda \tilde{\cdot} \omega :=$

$\lambda\mu_0^{-1}(\omega)\hat{+}\omega_0$ is a real infinite dimensional *vector space*. (In particular one has the "straight line" of connections through ω_1 and ω_0 for arbitrary ω_1 in $\mathcal{C}_0$ given by $\omega(t) = \omega_0 + t(\omega_1 - \omega_0) = (1-t)\omega_0 + t\omega_1$.) Notice that for the product bundle $B \times G$ the affine space of connections can be *identified* with a vector space since the product connection is canonical; this is not the case however in an arbitrary principal bundle.

Giving $\mathcal{C}_0$ the limit topology of $\mathbf{R}^\infty$ makes it a topological space of the homotopy type of a point *i.e.* contractible and therefore with zero homotopy groups. The choice of a vector space basis provides $\mathcal{C}_0$ with a global chart and makes it an infinite dimensional differentiable manifold with tangent space $T_\omega\mathcal{C}_0$ at each $\omega \in \mathcal{C}_0$ isomorphic to $\bar{\Gamma}^1$, the differentiable structure is however independent of the choice of basis and of ω_0. The function $\rho_0 : \mathcal{C}_0 \longrightarrow A^1, \omega \mapsto \rho_0(\omega) : \Gamma(TB) \longrightarrow \Gamma(E), \rho_0(\omega)(X) := s_{\omega(\tilde{X}_0)}$ where $\tilde{X}_0$ is the horizontal lifting of X by ω_0 and $s_{\omega(\tilde{X}_0)}$ the element in $Lie(\mathcal{G}(\xi))$ corresponding to the equivariant $\omega(\tilde{X}_0)$ in $\Gamma_{eq}(P, \mathbf{g})$, turns out to be a bijection, and therefore one has the composition

$$
\begin{array}{ccc}
 & & \bar{\Gamma}^1 \\
 & \mu_0 \swarrow & \downarrow {\scriptstyle \rho_0\circ\mu_0} \\
\mathcal{C}_0 & \xrightarrow[\rho_0]{} & A^1
\end{array}
$$

which establishes a 1-1 and onto linear transformation between the vector spaces $\bar{\Gamma}^1(T^*P \otimes \mathbf{g})$ and $\Gamma(T^*B \otimes E)$; the isomorphism however is not canonical (except for the trivial bundle $B \times G$) since $\rho_0 \circ \mu_0$ depends on ω_0.

One can prove that the infinite dimensional universal principal bundle $\bar{\mathcal{G}} \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C}'/\bar{\mathcal{G}}$ where $\mathcal{C}'$ is the subspace of $\mathcal{C}$ consisting of *irreducible* connections (ω is irreducible if any two points p and p' in P can be joined by a horizontal curve) is *not trivial* since $\bar{\mathcal{G}}$ has at least one non-zero homotopy group [13] (see also [14]); this leads to the *imposibility of a global gauge fixing, i.e.* of a continuous choice of a representative for each gauge equivalence class of connections in $\mathcal{C}'/\bar{\mathcal{G}}$. (The concept of irreducibility is closely related to that of parallel transport. For any two points b and b' in B and a smooth curve γ joining them with $\gamma(0) = b$ and $\gamma(1) = b'$ it can be proved that there exists a unique horizontal lifting $\bar{\gamma}_p$ of γ in P (all tangent vectors to $\bar{\gamma}_p$ are horizontal and $\pi \circ \bar{\gamma}_p = \gamma$) passing through any $p \in G_b$. Then there exists the diffeomorphism $\tau_\gamma : G_b \longrightarrow G_{b'}, \tau_\gamma(p) = \tau_\gamma(\bar{\gamma}_p(0)) := \bar{\gamma}_p(1)$ which is called the *parallel transport of* G_b *in* P *through* γ *in* B. From the physical point of view one can imagine that when a particle is classically transported through the path γ in space-time B in the presence of the fields $\{A_\alpha\}_{\alpha\in J}$ from the point b to the point b' its "internal state" changes from $\bar{\gamma}_p(0) = p$ (initial state) to $\bar{\gamma}_p(1)$ (final state). For a closed curve γ, there exists and is unique $g \in G$ such that $\bar{\gamma}_p(1) = pg$, and for the loop space of B at b, $\Omega(B, b)$ the corresponding set of parallel transports $G_b \longrightarrow G_b$ form a group Φ_b , the *holonomy group of the connection* ω *at the point* b. It can be shown [15],[16] that for each $p \in G_b$ there exists a group isomorphism $J_p : \Phi_b \longrightarrow \phi_p$ where ϕ_p is the Lie group (closed subgroup of G) given by $\phi_p = \{g \in G | \exists\gamma \in \Omega(B, b) | \tau_\gamma(p) = pg\}$, *the holonomy group of the connection* ω *with reference point* p. Also, if p and p' can be joined with an horizontal curve then $\phi_p = \phi_{p'}$ and if all points of P can be joined with horizontal curves to a given fixed point $p_0 \in P$, then $\phi_p = G$ for all $p \in P$.)

Let $\omega \in \mathcal{C}_0$ and $\eta \in A^0 = \Gamma(E) = Lie(\mathcal{G}(\xi))$, then $d_\omega\eta \in A^1$ and $(\rho_0 \circ \mu_0)^{-1}(d_\omega\eta) \in \bar{\Gamma}^1$. So $\omega' = \omega + d^0_\omega\eta$ is in $\mathcal{C}^0$ with $d^0_\omega := (\rho_0 \circ \mu_0)^{-1} \circ d_\omega$; ω' is the *Lie algebra* or *infinitesimal transformation* of ω generated by η . Notice that the "effective" covariant derivative operator d^0_ω depends on ω_0 , the distinguished element in $\mathcal{C}$; only for trivial bundles or locally for arbitrary bundles d_ω and d^0_ω can be identified and one has the usual formula $\omega' = \omega + d_\omega\eta$ which can be formally obtained from (3) as the $\mathcal{O}(t)$ term in the expansion of $g = exp(t\eta)$ at $t = 1$ considering $\mathcal{G}(\xi)$ as a group of matrices and identifying $d\eta + [\omega, \eta]$ with $d_\omega\eta$. Similarly as in finite dimensional Lie groups, for each $\eta \in Lie(\mathcal{G}(\xi))$ one has the *one parameter group* of infinitesimal transformations of $\omega, \omega(t) = \omega + td^0_\omega\eta$.

4 BRST cohomology

As we mentioned in the previous section the group of the gauge transformations on a p.f.b., $\mathcal{G}(\xi)$ has a right action on the space of connections $\mathcal{C}(\xi)$, namely $\mathcal{C} \times \mathcal{G} \xrightarrow{\Phi} \mathcal{C}$, $(\omega, f) \mapsto \Phi(\omega, f) := f^*(\omega)$ *i.e.* the action is given by the gauge transformations of the connections.

As first observed in ref. [19] this action leads to the Chevalley-Eilenberg cohomology of the Lie algebra of $\mathcal{G}(\xi)$ with coefficients in the real valued functions (0-forms) on $\mathcal{C}(\xi)$, which was identified with the Becchi-Rouet-Stora-Tyutin (BRST) cohomology [20] previously found in the context of the quantum theory of gauge fields as a consequence of the global symmetry naturally appearing at the end of the quantization procedure *via* the method of path integration [1], the generator of the symmetry precisely being the coboundary operator of the BRST complex. (Immediately after the discovery in [20], an anti- BRST symmetry was found within the same context by Curci and Ferrari [25], and it was until 1982 when Alvarez-Gaumé and Baulieu [26] made the deep assertion that the full BRST symmetry is the gauge symmetry in the perturbative quantum theory; moreover they proved that the gauge invariance of the physical S-matrix elements is equivalent to the full BRST invariance (though not gauge invariance) of the "quantum Lagrangian"; this point of view was extensively reviewed by Baulieu in [27].)

There are at least two reasons why the approach to the BRST symmetry from the geometrical point of view is interesting. First, the BRST cohomology turns out to be a property of the chosen principal bundle ξ *i.e.* of the "space" where the gauge fields live; in this sense it can be considered as a purely geometrical property, independent of any lagrangian theory that one can set up on the base space of the bundle (in particular of the YM lagrangian). Second, the BRST cohomology groups contain information about the quantum theory of any gauge theory that one can place on the principal bundle, in fact the gauge *anomalies* are contained in the cohomology groups with ghost number larger than or equal to one, as it is discussed in references [19] and [21]. (Local ghost and anti-ghost fields, scalar particles with Fermi statistics, naturally appear in the path integral quantization when one expresses the Faddeev-Popov determinant in terms of the algebraic generators of an infinite dimensional complex Grassmann algebra with involution. In the geometrical description *the* (global) ghost field is the Maurer-Cartan form of the gauge group.) This fact again suggests a deep connection between the *topology* of fiber fundles and the *quantum theory* of gauge fields (connections) as it was recently emphasized, though in an apparently different context, by Atiyah [22] and Witten [23]. (Incidentaly in the recent literature([21],[24]) there are efforts towards the explicit calculation of the BRST cohomology groups for several different concrete situations using the technique of spectral sequences.)

Using the proposition of Appendix C one makes the identifications $G =$

$\mathcal{G}(\xi)$ and $M = \mathcal{C}(\xi)$; then for each $p \in \{0, 1, 2, \ldots\}$ one has the representation of the Lie algebra of the gauge group on the differential p-forms over the space of connections given by $A^0 = Lie(\mathcal{G}(\xi)) \stackrel{\phi_p}{\to} End(\Omega^p(\mathcal{C}(\xi)))$, $\phi_p(s) = \mathcal{L}_{s^*}$ where s^* is the fundamental vector field on $\mathcal{C}(\xi)$ associated with s and $\mathcal{L}_{s^*}$ is the Lie derivative with respect to s^*, so for any real valued $f \in \Omega^0(\mathcal{C}(\xi))$, $s^*_\omega(f) = \frac{d}{dt} f \circ \Phi(\omega, (\mu \circ \Sigma)^{-1}(Exp\ ts))|_{t=0} = \frac{d}{dt} f(\omega + t d^0_\omega s)|_{t=0}$ where the last expression for $s^*_\omega(f)$ depends on the base point ω_0 in $\mathcal{C}$ (as mentioned in section 3 for trivial bundles ω_0 is canonical and d^0_ω can be identified with d_ω). Defining the spaces $\mathcal{C}^\nu_p(\xi)$ of alternating continuous (see below) functions $A^0 \times \ldots \times A^0$ (ν times)$\xrightarrow{\alpha}\Omega^p(\mathcal{C}(\xi))$ for $\nu = 1, 2, 3, \ldots$ and $\mathcal{C}^0_p(\xi) = \Omega^p(\mathcal{C}(\xi))$ for $\nu = 0$, with $\mathcal{C}(\xi)$ and A^0 respectively given suitable Sobolev k- and $(k+1)$-norm completions with integer $k \geq [\frac{dimB}{2}] + 1$ (these completions guarantee that the action $\mathcal{C} \times \mathcal{G} \to \mathcal{C}$ extends to a smooth action $\mathcal{C}_k \times \mathcal{G}_{k+1} \to \mathcal{C}_k$ [8]; see also reference [28] for the case $k = 3$, $G = SU(2)$ and B a compact 4-manifold), one has the double complex

$$
\begin{array}{ccccccc}
\mathcal{C}^0_0(\xi) & \xrightarrow{d^0_0} & \mathcal{C}^0_1(\xi) & \xrightarrow{d^0_1} & \mathcal{C}^0_2(\xi) & \xrightarrow{d^0_2} & \cdots \\
\delta^0_0 \downarrow & & \delta^0_1 \downarrow & & \delta^0_2 \downarrow & & \\
\mathcal{C}^1_0(\xi) & \xrightarrow{d^1_0} & \mathcal{C}^1_1(\xi) & \xrightarrow{d^1_1} & \mathcal{C}^1_2(\xi) & \xrightarrow{d^1_2} & \cdots \\
\delta^1_0 \downarrow & & \delta^1_1 \downarrow & & \delta^1_2 \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

with differentials
$\delta^\nu_p : \mathcal{C}^\nu_p(\xi) \to \mathcal{C}^{\nu+1}_p(\xi)$, $\delta^\nu_p(\alpha)(s_0, \ldots, s_\nu) = \Sigma^\nu_{i=0}(-1)^i \mathcal{L}_{s^*}(\alpha(s_0, \ldots, \hat{s}_i, \ldots, s_\nu))$ $+\Sigma_{0\leq i<j\leq\nu}(-1)^{i+j}\alpha([s_i, s_j], s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_\nu)$ and $d^\nu_p : \mathcal{C}^\nu_p(\xi) \to \mathcal{C}^\nu_{p+1}(\xi)$, $d^\nu_p(\alpha)(s_1, \ldots, s_\nu) = d(\alpha(s_1, \ldots, s_\nu))$ where d is the De Rham operator on the infinite dimensional manifold $\mathcal{C}$ (the double complex is like that in appendix C, except that now it extends to infinity also in the horizontal direction). The continuity condition for α is given as follows: if $\alpha \in \mathcal{C}^\nu_p(\xi)$ then the map $\alpha_{\omega,\xi_1,\ldots,\xi_p} : A^0 \times \cdots \times A^0(\nu\ times) \to \mathbf{R}$, $(s_1, ..., s_\nu) \mapsto \alpha_{\omega,\xi_1,\ldots,\xi_p}(s_1, ..., s_\nu) :=$

$\alpha(s_1, ..., s_\nu)(\xi_1, ..., \xi_p)(\omega)$ is continuous for all $\omega \in \mathcal{C}_k$ and $\xi_1, ..., \xi_p \in Vect(\mathcal{C}_k)$. The usual BRST complex [19] is the Chevalley-Eilenberg complex consisting of the first column in the previous "lattice" which defines the *BRST cohomology of the principal bundle* ξ as $H^*_{BRST}(\xi)$ $= \oplus_{\nu=0}^{\infty} H^\nu_{BRST}(\xi) = \oplus_{\nu=0}^{\infty} \frac{ker\delta_0^\nu}{im\delta_0^{\nu-1}}$ $(\delta_0^{-1} = 0)$, ν is the ghost number and the coboundary $\{\delta_0^\nu\}_{\nu=0}^\infty$ is identified with the BRST nilpotent operator appearing in the quantum theory. The columns corresponding to $p = 1, 2, \ldots$ and the corresponding differentials have been defined here in a formal way and we do not have yet a physical interpretation (if any) of them. Following Appendix B one has an associated total complex (K, D) with $K^n(\xi) = \oplus_{\nu+p=n}\mathcal{C}_p^\nu(\xi)$ and $D^n = \oplus_{\nu+p=n}(\delta_p^\nu \oplus (-1)^\nu d_p^\nu)$ and therefore a total cohomology $H^*(K, D)$ which we might call the *total BRST cohomology of a p.f.b.* ξ and denote by $\mathcal{H}^*_{BRST}(\xi)$. We believe that the possible implications and interpretations of this definition deserves further research.

In reference [19] a geometric interpretation of the above Lie algebra cohomology in terms of the vertical part of the De Rham exterior derivative on the space of irreducible connections is given; however we believe that an interpretation in terms of the topological (Eilenberg-Steenrod) cohomologies of the relevant spaces of the bundle (Lie group, total space and base space) should be more conclusive towards establishing a relationship between quantum mechanics and topology [29].

Acknowledgements

We wish to tank Drs. M. Aguilar, H. Bhaskara and M. Rosenbaum for discussions and for their teachings on several subjects related to the present work. One of us (M.S.) thanks the hospitality of the Depto. de Física de la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires where part of this work was performed, and Dr. D. Vergara for his comments on BRST symmetry. We also thank the referee for pointing out several errors and for suggestions which led to an improvement of the manuscript.

Appendix A. Cohomology of Lie algebras [12]

Let V and $\mathbf{g}$ respectively be a vector space and a Lie algebra over the field k of real or complex numbers, and $\phi : \mathbf{g} \longrightarrow End_k(V)$ a representation of $\mathbf{g}$ on V *i.e.* a k-linear map such that $\phi([A,B]) = \phi(A)\circ\phi(B) - \phi(B)\circ\phi(A)$. Define the vector spaces $\mathcal{C}^0 := V$ and for $p \in \mathbf{Z}^+, \mathcal{C}^p := \{t : \mathbf{g} \times \ldots \times \mathbf{g}(p\ times) \longrightarrow V,\ t\ \textit{multilinear totally antisymmetric (alternating)}\}$. One defines the set of linear operators $\delta^p : \mathcal{C}^p \longrightarrow \mathcal{C}^{p+1}, \delta^p(t)(v_1, \ldots v_{p+1}) := \Sigma_{i=1}^{p+1}(-1)^{i+1}\phi(v_i)(t(v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}))$
$+\Sigma_{1\le i<j\le p+1}(-1)^{i+j}t([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}))$ which satisfy
$\delta^{p+1}\circ\delta^p = 0$ *i.e.* $\{\delta^p\}_{p=0}^{\infty}$ is a coboundary. This leads to the complex $0 \longrightarrow \mathcal{C}^0 \xrightarrow{\delta^0} \mathcal{C}^1 \xrightarrow{\delta^1} \mathcal{C}^2 \xrightarrow{\delta^2} \mathcal{C}^3 \xrightarrow{\delta^3} \cdots$ with p-cocycles $Z^p = ker\ \delta^p$ and p-coboundaries $B^p = im\ \delta^{p-1}$, and one defines the *Chevalley-Eilenberg cohomology* of $\mathbf{g}$ with respect to the representation ϕ of $\mathbf{g}$ on V (or "with coefficients in V") given by the graded group (direct sum of abelian groups) $H^*_{CE}(\mathbf{g}, \phi, V; k) = \oplus_{i=0}^{\infty} H^i_{CE}(\mathbf{g}, \phi, V; k)$ with $H^i_{CE}(\mathbf{g}, \phi, V; k) = Z^i/B^i$. In particular $H^0_{CE} = \{t \in V \mid t \in ker(\phi(v)), \exists v \in \mathbf{g}\} = \cap_{v\in\mathbf{g}} ker(\phi(v))$.

Appendix B. Double Complexes and Total Cohomology

A double (cochain) complex (C, ∂, d) is a double array *i.e.* a "lattice" of abelian groups $C^{p,q}, p, q = 0, 1, 2, ...,$ with differentials (group homomorphisms) $d^{p,q} : C^{p,q} \to C^{p,q+1}$ and $\partial^{p,q} : C^{p,q} \to C^{p+1,q}$, $d^{p,q+1}\circ d^{p,q} = \partial^{p+1,q}\circ\partial^{p,q} = 0$, satisfying $\partial^{p,q+1}\circ d^{p,q} = d^{p+1,q}\circ\partial^{p,q}$ *i.e.* the commutative diagrams

$$\begin{tikzcd}
C^{p,q} \arrow[r, "d^{p,q}"] \arrow[d, "\partial^{p,q}"'] & C^{p,q+1} \arrow[d, "\partial^{p,q+1}"] \\
C^{p+1,q} \arrow[r, "d^{p+1,q}"'] & C^{p+1,q+1}
\end{tikzcd}$$

(C, ∂, d) induces a *total (simple) cochain complex* (K, D) as follows: for $n =$

$0, 1, 2, ...$ one defines the abelian groups $K^n = \oplus_{p+q=n} C^{p,q}$ and the operators $D^n = \oplus_{p+q=n} D^{p,q}$ with $D^{p,q} := \partial^{p,q} \oplus (-1)^p d^{p,q}$; then it is easy to verify that $D^n : K^n \to K^{n+1}$ are differentials *i.e.* group homomorphisms satisfying $D^{n+1} \circ D^n = 0$, thus leading to the complex

$$K^0 \xrightarrow{D^0} K^1 \xrightarrow{D^1} K^2 \xrightarrow{D^2} \cdots$$

More pictorially

where the $\circ$ dots denote the groups $C^{p,q}$.

The cohomology of the simple complex (K, D), $H^*(K, D) = \oplus_{n=0}^{\infty} H^n(K, D)$, $H^n(K, D) = ker D^n / im D^{n-1}$ $(D^{-1} = 0)$ is called the *total cohomology* of the original double complex (C, ∂, d).

A technique to compute $H^*(K, D)$ is that of *spectral sequences* (SS) [17]. A double complex has two filtrations, each having an associated SS, and both SS's converge to the total cohomology.

Appendix C. Group Actions and Double Complexes [18]

Proposition: Let G be a Lie group, M a diffentiable manifold and $M \times G \xrightarrow{\psi} M$ a right *action* of G on M. Associated with this action there exists a *double cochain complex* involving the Lie algebra of G and the differential forms on M.

Proof: *i*) Let $A \in \mathbf{g} = Lie(G)$; its fundamental vector field A^* is the vector field on M given by $A^*_x(f) = \frac{d}{dt}(f(xexptA))|_{t=0} = \frac{d}{dt}(f \circ \psi(x, exptA))|_{t=0}$ for any $x \in M$ and $f \in C^\infty(M, \mathbf{R})$.

ii) If $\Phi(A^*) = \{\phi_t\}_{t\in(-\epsilon,\epsilon)}$ is the flow of A^* then the Lie derivative of a tensor τ on M with respect to A^* is the tensor of the same type $\mathcal{L}_{A^*}(\tau) = \frac{d}{dt}\phi_t^*(\tau)|_{t=0}$ where $\phi_t^*(\tau)$ is the pull-back of τ by ϕ_t; in particular this holds for p-forms on M and then for each $p = 0, 1, 2, ..., n = dim\ M$ one has the operator $\mathcal{L}_{A^*} : \Omega^p(M) \to \Omega^p(M)$.

iii) For each fixed $p \in \{0, \ldots, n\}$ one defines the infinite set of vector spaces: $\mathcal{C}_p^\nu = \{\mathbf{g} \times \ldots \times \mathbf{g}\ (\nu\ times) \xrightarrow{\alpha} \Omega^p(M), \alpha\ alternating\}$ for $\nu \in \mathbf{Z}^+$ and $\mathcal{C}_p^0 = \Omega^p(M)$ for $\nu = 0$.

iv) The (canonical) representation $\phi_p : \mathbf{g} \longrightarrow End(\Omega^p(M))$, $\phi_p(A) := \mathcal{L}_{A^*}$ of $\mathbf{g}$ on $\Omega^p(M)$ (ϕ_p is a Lie algebra homomorphism since $\mathcal{L}_{[A,B]^*} = \mathcal{L}_{[A^*,B^*]} = [\mathcal{L}_{A^*}, \mathcal{L}_{B^*}]$) induces the infinite (cochain) complex

$$\mathcal{C}_p^0 \xrightarrow{\delta_p^0} \mathcal{C}_p^1 \xrightarrow{\delta_p^1} \mathcal{C}_p^2 \xrightarrow{\delta_p^2} \cdots$$

where $\{\delta_p^\nu\}_{\nu=0}^\infty$ is the coboundary $\delta_p^\nu : \mathcal{C}_p^\nu \to \mathcal{C}_p^{\nu+1}$, $\alpha \mapsto \delta_p^\nu(\alpha)$: $\mathbf{g}\times\ldots\times\mathbf{g}$ ($\nu+1$ *times*) $\to \Omega^p(M)$, $\delta_p^\nu(\alpha)(A_0, \ldots, A_\nu) := \Sigma_{i=0}^\nu (-1)^i \mathcal{L}_{A_i^*}(\alpha(A_0, \ldots, \hat{A}_i, \ldots, A_\nu)) + \Sigma_{1\leq i<j\leq\nu}(-1)^{i+j}\alpha([A_i, A_j], A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_\nu)$ (it holds $\delta_p^{\nu+1} \circ \delta_p^\nu = 0$) which in turn induces the p-th C-E cohomology of $\mathbf{g}$ with coefficients in $\Omega^p(M)$, $H_{CE}^*(\mathbf{g}, \Omega^p(M)) = \oplus_{\nu=0}^\infty H_{CE}^\nu(\mathbf{g},\ \Omega^p(M))$ with $H_{CE}^\nu(\mathbf{g}, \Omega^p(M)) = \frac{ker\delta_p^\nu}{im\delta_p^{\nu-1}}$.

v) Regarding the set of the previously defined C-E complexes as $n+1$ infinite (vertical) columns one defines horizontal operators $d_p^\nu : \mathcal{C}_p^\nu \to \mathcal{C}_{p+1}^\nu$, $\alpha \mapsto d_p^\nu(\alpha) : \mathbf{g} \times \ldots \times \mathbf{g}$ (ν *times*) $\to \Omega^{p+1}(M)$, $d_p^\nu(\alpha)(A_1, \ldots, A_\nu) := d(\alpha(A_1, \ldots, A_\nu))$ where d is the De Rham operator of the manifold M; the commutativity of the Lie derivative with the De Rham operator *i.e.* $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$ for all $X \in \Gamma(TM)$ implies that each vertical ladder $\{d_p^\nu\}_{\nu=0}^\infty$ is a *cochain complex homomorphism* and one has the following "lattice" of commuting diagrams:

$$
\begin{array}{ccccccccccc}
\mathcal{C}_0^0 & \xrightarrow{d_0^0} & \mathcal{C}_1^0 & \xrightarrow{d_1^0} & \mathcal{C}_2^0 & \xrightarrow{d_2^0} & \dots & \xrightarrow{d_{n-2}^0} & \mathcal{C}_{n-1}^0 & \xrightarrow{d_{n-1}^0} & \mathcal{C}_n^0 \\
\delta_0^0 \downarrow & & \delta_1^0 \downarrow & & \delta_2^0 \downarrow & & & & \delta_{n-1}^0 \downarrow & & \delta_n^0 \downarrow \\
\mathcal{C}_0^1 & \xrightarrow{d_0^1} & \mathcal{C}_1^1 & \xrightarrow{d_1^1} & \mathcal{C}_2^1 & \xrightarrow{d_2^1} & \dots & \xrightarrow{d_{n-2}^1} & \mathcal{C}_{n-1}^1 & \xrightarrow{d_{n-1}^1} & \mathcal{C}_n^1 \\
\delta_0^1 \downarrow & & \delta_1^1 \downarrow & & \delta_2^1 \downarrow & & & & \delta_{n-1}^1 \downarrow & & \delta_n^1 \downarrow \\
\mathcal{C}_0^2 & \xrightarrow{d_0^2} & \mathcal{C}_1^2 & \xrightarrow{d_1^2} & \mathcal{C}_2^2 & \xrightarrow{d_2^2} & \dots & \xrightarrow{d_{n-2}^2} & \mathcal{C}_{n-1}^2 & \xrightarrow{d_{n-1}^2} & \mathcal{C}_n^2 \\
\delta_0^2 \downarrow & & \delta_1^2 \downarrow & & \delta_2^2 \downarrow & & & & \delta_{n-1}^2 \downarrow & & \delta_n^2 \downarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}
$$

Each "square" is of the form

$$
\begin{array}{ccc}
\mathcal{C}_p^\nu & \xrightarrow{d_p^\nu} & \mathcal{C}_{p+1}^\nu \\
\delta_p^\nu \downarrow & & \downarrow \delta_{p+1}^\nu \\
\mathcal{C}_p^{\nu+1} & \xrightarrow[d_p^{\nu+1}]{} & \mathcal{C}_{p+1}^{\nu+1}
\end{array}
$$

and it holds $\delta_{p+1}^\nu \circ d_p^\nu = d_p^{\nu+1} \circ \delta_p^\nu$.

vi) $d^2 = 0$ implies $d_{p+1}^\nu \circ d_p^\nu = 0$ and therefore the above lattice of abelian groups and differentials is a *double complex* $(\mathcal{C}, \delta, d)$. $\Box$

From appendix B one has the total complex $(\mathcal{K}, \mathcal{D})$, with groups $\mathcal{K}^m = \oplus_{\nu+p=m}\mathcal{C}_p^\nu$ and coboundaries $\mathcal{D}^m = \oplus_{\nu+p=m}\mathcal{D}_p^\nu : \mathcal{K}^m \to \mathcal{K}^{m+1}$ for $m = 0, 1, 2, \dots$ with $\mathcal{D}_p^\nu = \delta_p^\nu \oplus (-1)^\nu d_p^\nu$, which in turn induces the total cohomology $H^*(\mathcal{K}, \mathcal{D})$ of the original double complex $(\mathcal{C}, \delta, d)$.

References

[1] E.S. Abers and B.W. Lee, "Gauge Theories", Phys. Rep. **C9**, 1(1973); L.D. Faddeev and A.A. Slavnov, *Gauge Fields, Introduction to Quantum Theory,* Benjamin, 1980; P. Ramond, *Field Theory, A Modern Primer*, Benjamin, 1981; J.C. Taylor, *Gauge Theories of Weak Interactions*, Camb. Univ. Press, 1979; L.H. Ryder, *Quantum Field Theory* , Cambridge Univ. Press, Cambridge, 1985; M. Göckeler and T. Schücker , *Differential Geometry, Gauge Theories, and Gravity*, Cambridge Univ. Press, Cambridge, 1987; T. Eguchi, P.B. Gilkey and A.J. Hanson, "Gravitation, gauge theories and differential geometry", Phys Rep. **C66**, 213(1980); S. Pokorski, *Gauge Field Theories*, Cambridge Univ. Press, London,1987; J. Madore, "Geometric methods in classical field theory", Phys. Rep. **C75**, 125(1981); D.D. Bleecker, "Physics from the G-bundle viewpoint", Int. J. Theor. Phys. **23**, 735(1984); C.M. Viallet, "Symmetry and functional integration", in *Physics, Geometry, and Topology*, ed. H.C. Lee, Plenum Press, New York, 1990.

[2] S.W. Hawking, *A Brief History of Time, From the Big Bang to Black Holes*, Bantam Books, New York, 1988.

[3] A. Ashtekar, *New Perspectives in Canonical Gravity*, Bibliopolis, Napoli, 1988; *Non Perturbative Canonical Gravity*, with R.S. Tate, IUCAA, Poona, India, 1990; K.V. Kuchar, "Canonical quantum gravity", UU-REL-92/12/10, to be published in 1993 Proceedings of the 13th Int. Conf. on General Relativity and Gravitation, ed. C. Kozameh; "Time and interpretations of quantum gravity", to be published in the proceedings of the 4th Canadian Conf. on General Relativity and Relativistic Astrophysics, eds. G. Kurstatter, D. Vincent and J. Williams (World Scient. Singapore, 1992).

[4] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory* (2 volumes), Cambridge Univ. Press, New York, 1987.

[5] S.K. Donaldson, "An application of gauge theory to four dimensional topology", J. Diff. Geom. **18**,269(1983);S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four Manifolds*, Clarendon Press, Oxford, 1990.

[6] M.H. Freedman, "The topology of four-dimensional manifolds", J. Diff. Geom. **17**, 357(1982).

[7] M.E. Mayer, "The geometry of symmetry breaking in gauge theories", Acta Phys. Austr.,Suppl. XXIII, 477(1981).

[8] M.S. Narasimhan and T.R. Ramadas, "The geometry of $SU(2)$ gauge fields", Comm. Math. Phys. **67**, 21(1979); P.K. Mitter and C.M. Viallet, "On the bundle of connections and the gauge orbit manifold in Yang-Mills theory", Comm. Math. Phys. **79**, 457(1981); J.Milnor, "Remarks on infinite-dimensional Lie groups" in *Relativity, Groups and Topology II*, Les Houches, Session XL, 1983, B.S. De Witt and R. Stora eds., Elsevier, 1984.

[9] M.E. Mayer and A. Trautman, "A brief introduction to the geometry of gauge fields", Acta Phys. Austr., Suppl. XXIII, 433(1981).

[10] J. Milnor "Construction of universal bundles, II", Annals of Math., **63**, 430(1956).

[11] A. Trautman, *Differential Geometry for Physicists*, Bibliopolis, Napoli, 1984.

[12] C. Chevalley and S. Eilenberg, "Cohomology theory of Lie groups and Lie algebras", Trans. Am. Math. Soc. **63**, 85 (1948).

[13] I.M. Singer, "Some remarks on the Gribov ambiguity", Comm. Math. Phys. **60**, 7(1978).

[14] M. Daniel and C.M. Viallet, "The geometrical setting of gauge theories of the Yang-Mills type", Rev. Mod. Phys. **52**, 175(1980).

[15] S. Kobayashi and N. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley, 1963.

[16] C. von Westenholtz, *Differential Forms in Mathematical Physics*, Nort-Holland, New York, 1978.

[17] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and applied Math., Wiley, New York, 1978.

[18] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, Vol. III, Academic Press, New York, 1976.

[19] L. Bonora and P. Cotta-Ramusino, "Some Remarks on BRS Transformations, anomalies and the cohomology of the Lie Algebra of the group of gauge transformations", Comm. Math. Phys. **87**, 589(1983).

[20] C. Becchi, A. Rouet and R. Stora,"Renormalization of gauge theories", Ann. Phys. **98**, 287 (1976); "Renormalization of the abelian Higgs-Kibble model", Comm. Math. Phys. **42**, 127 (1975); I. V. Tyutin, "Gauge invariance in field theory and statistical mechanics in operator formalism", Lebedev Phys. Inst., preprint no. 39 FIAN (1975) (unpublished); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton Univ. Press, Princeton, New Jersey, 1992; D. Bes and J. Kurchan, *The Treatment of Collective Coordinates in Many Body Systems. An Application of the BRST Invariance*, World Scientific, Lecture Notes in Physics, Vol. 34, 1990.

[21] J.A. Dixon, "Calculation of BRS Cohomology with spectral sequen ces", Comm. Math. Phys. **139**, 495(1990).

[22] M. Atiyah, *The Geometry and Physics of Knots*, Cambridge Univ. Press, Cambridge, 1990.

[23] E. Witten,"Topological quantum field theory", Comm. Math. Phys. **117**, 353(1988); "Quantum field theory and the Jones polynomial", Comm. Math. Phys. **121**, 351(1989); D. Birmingham, M. Blau, M. Rakowsky and G. Thomson, "Topological field theory", Phys. Rep. **C209**, 129(1991).

[24] M. Dubois-Violette *et al*, "Some results on local cohomologies in field theory", Phys. Lett. **B267**, 81 (1991); *idem* **289**, 361 (1992).

[25] G. Curci and R. Ferrari, "On a class of Lagrangian models for massive and massless Yang-Mills fields", Nuovo Cim. **32A**, 151 (1976).

[26] L. Alvarez-Gaumé and L. Baulieu, "The two quantum symmetries associated with a classical symmetry", Nucl. Phys., **B212**, 255 (1983).

[27] L. Baulieu, "Perturbative gauge theories", Phys. Rep.**C129**, 1 (1985).

[28] B. Lawson, "The theory of gauge fields in four dimensions", Regional Conference Series in Mathematics, Nr 58, American Mathematical Society, Providence, R.I., 1983.

[29] M. Aguilar, private communication.

H. García-Compeán[1], J.M. López-Romero[1], M.A. Rodríguez-Segura[1]
Departmento de Física, Centro de Investigación y Estudios Avanzados-IPN,
A.P. 14-740, 07000, México D.F.

M. Socolovsky[2]
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Circuito Exterior C.U. A.P. 70-543, 04510,México D.F.

[1]Supported by a CONACyT Graduate Fellowship. e-mail compean@fis.cinvestav.mx, mars@fis.cinvestav.mx

[2] e-mail socolovs@roxanne.nuclecu.unam.mx