Algebraic characterization of the Wess-Zumino consistency conditions in gauge theory

S.P. Sorella[1]
Département de Physique Théorique
24, quai Ernest Ansermet
CH – 1211 Genève 4 (Switzerland)

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Abstract

A new way of solving the descent equations corresponding to the Wess-Zumino consistency conditions is presented. The method relies on the introduction of an operator δ which allows to decompose the exterior space-time derivative d as a BRS commutator. The case of the Yang-Mills theories is treated in detail.

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1 Introduction

It is well known that the anomalies in gauge theories have to be non-trivial solutions of the Wess-Zumino consistency conditions [1]. These conditions, when formulated in terms of the Becchi-Rouet-Stora transformations [2], yield a cohomology problem for the nilpotent BRS operator s:

$$s\Delta = 0 \ , \tag{1.1}$$

where Δ is the integral of a local polynomial in the fields and their derivatives. An useful way of finding non-trivial solutions of (1.1) is given by the so-called $descent - equations$ technique [3, 4, 5, 6, 7, 8, 9].

Setting $\Delta = \int \mathcal{A}$, eq. (1.1) translates into the local condition

$$s\mathcal{A} + d\mathcal{Q} = 0, \tag{1.2}$$

for some $\mathcal{Q}$; d being the exterior differential on the space-time M. The operators s and d verify:

$$s^2 \ = \ d^2 \ = \ sd \ + \ ds \ = 0 \ . \tag{1.3}$$

One can easily prove that equation (1.2), due to the triviality of the cohomolgy of d [5, 7, 10], generates a tower of descent equations

$$\begin{aligned} &s\mathcal{Q} \ + \ d\mathcal{Q}^1 = 0 \\ &s\mathcal{Q}^1 \ + \ d\mathcal{Q}^2 = 0 \\ &\quad \ldots\ldots \\ &\quad \ldots\ldots \\ &s\mathcal{Q}^{k-1} \ + \ d\mathcal{Q}^k = 0 \\ &s\mathcal{Q}^k = 0 \ , \end{aligned} \tag{1.4}$$

where the $\mathcal{Q}^i$ are local polynomials in the fields.

The aim of this work is to give a systematic procedure for generating explicit solutions of the descent equations (1.4). This will be done in the class of polynomials of differential forms [5] and for any space-time dimension and ghost-number. The use of the space of polynomials of differential forms; i.e. polynomials in the variables A, dA, c, dc (A and c being respectively the gauge connection and the ghost field), is the natural choice when dealing with anomalies and Chern-Simons terms. This is due to the topological origin of the latters [11]. It is worthwhile to mention that actually, as recently proven by M. Dubois-Violette et al. [12], the use of the space of polynomials of forms is not a restriction on the generality of the solution of the consistency equations. To avoid the introduction of a reference connection [4] we will assume that the principal fiber bundle for the gauge potential is trivial, i.e. of the form $(M \times G)$ with G a compact semisimple Lie group.

The main idea which we will use to solve the descent equations consists in writing the exterior derivative d as a BRS commutator; i.e. we will be able to make the decomposition:

$$d \;=\; -[\, s \,,\, \delta \,] \,, \tag{1.5}$$

where δ is an operator which will be specified later. One easily shows that, once the decomposition (1.5) has been found, repeated applications of the operator δ on the polynomial $\mathcal{Q}^k$ which solves the last of the equations (1.4) will give an explicit solution for the higher cocycles $\mathcal{Q}^{k-1}$, ..., $\mathcal{Q}^1$, $\mathcal{Q}$ and $\mathcal{A}$. One has to remember also that the form of the polynomial $\mathcal{Q}^k$ is well known [4, 5, 7, 10, 13, 14] and is uniquely specified by invariant ghost monomials of the form $\mathrm{Tr}\, c^k$ (k odd). This completes the general strategy.

We emphasize that this scheme represents an alternative way of solving the descent equations which is completely different from the usual homotopy setup given by the "$Russian-formula$" [3, 4, 5, 6]. However we will see that the two schemes identify the same class of solutions, i.e. they give expressions which differ only by a trivial cocycle.

It is interesting to note that the decomposition (1.5) naturally appears in the context of the topological field theories [15, 16]. In this case the operator δ is the generator of the topological vector supersymmetry and allows for a complete classification of anomalies, counterterms and non-trivial observables [17].

The paper is organized as follows. In Section 2 we briefly recall some basic properties concerning the cohomologies of d and s. Section 3 is devoted to the study of the algebraic structure implied by the decomposition (1.5). In Section 4, after giving some explicit examples, we present the solution of the descent equations in the general case.

We hope that this work will be of some help in understanding the algebraic structure which underlies the topological nature of the anomalies.

2 General results on the d and s cohomologies

2.1 Notations and functional idendities

Let $\mathcal{V}(A, dA, c, dc)$ be the space of polynomials of differential forms. The BRS transformations of the one-form gauge connection $A^a = A^a_\mu dx^\mu$ and of the zero-form ghost field c^a are:

$$
\begin{aligned}
sA^a &= dc^a + f^{abc}c^bA^c \ , \\
sc^a &= \tfrac{1}{2}f^{abc}c^bc^c \ ,
\end{aligned}
\tag{2.1}
$$

where f^{abc} are the structure constants of the gauge group G and d is the exterior derivative defined by

$$
d\omega_p = dx^\mu \partial_\mu \omega_p \ , \tag{2.2}
$$

for any p-form

$$
\omega_p = \frac{1}{p!}\omega_{i_1....i_p}dx^{i_1}....dx^{i_p} \ , \tag{2.3}
$$

where a wedge product has to be understood.

As usual, the adopted grading is given by the sum of the form degree and of the ghost number. The fields A and c are both of degree one, their ghost number being respectively zero and one. A p-form with ghost number q will be denoted by ω^q_p; its grading being $(p+q)$. The two-form field strength F^a reads:

$$
F^a = \frac{1}{2}F^a_{\mu\nu}dx^\mu dx^\nu \ = \ dA^a + \frac{1}{2}f^{abc}A^bA^c \ , \tag{2.4}
$$

and

$$
dF^a = f^{abc}F^bA^c \ , \tag{2.5}
$$

is its Bianchi identity.

To characterize the cohomology of s and d on the space of polynomials of differential forms it is convenient to switch from (A, dA, c, dc) to another set of more natural variables. Following [5], we choose as independent variables the set $(A, F, c, \xi = dc)$; i.e. we replace everiwhere dA with F by using (2.4) and we introduce the variable $\xi = dc$ to emphasize the local character of the descent equations (1.4). Indeed, since integration by parts is not allowed, the variable $\xi = dc$ is really an independent quantity. On the local space $\mathcal{V}(A,F,c,\xi)$ the BRS operator s and the exterior derivative d act as ordinary differential operators given by

$$s \;=\; (\xi^a + f^{abc}c^bA^c)\frac{\partial}{\partial A^a} \;+\; f^{abc}\frac{c^bc^c}{2}\frac{\partial}{\partial c^a} \;+\; f^{abc}c^b\xi^c\frac{\partial}{\partial \xi^a} \;+\; f^{abc}c^bF^c\frac{\partial}{\partial F^a} \;, \tag{2.6}$$

$$d \;=\; (F^a - f^{abc}\frac{A^bA^c}{2})\frac{\partial}{\partial A^a} \;+\; \xi^a\frac{\partial}{\partial c^a} \;+\; f^{abc}F^bA^c\frac{\partial}{\partial F^a} \;. \tag{2.7}$$

One easily checks that s and d are of degree one and satisfy

$$s^2 \;=\; d^2 \;=\; sd \;+\; ds \;=0 \;. \tag{2.8}$$

2.2 The d cohomology

Even if the vanishing of the cohomology of the exterior derivative is a well established result [5, 7, 10] we give here a simple proof which may be useful for the reader.

Proposition I

The exterior derivative d has vanishing cohomology on $\mathcal{V}(A,F,c,\xi)$.

Proof

The proof is easily done by introducing the filtering operator $\mathcal{N}$ [13, 14]

$$\mathcal{N} = \xi^a\frac{\partial}{\partial \xi^a} \;+\; c^a\frac{\partial}{\partial c^a} \;+\; A^a\frac{\partial}{\partial A^a} \;+\; F^a\frac{\partial}{\partial F^a} \;, \tag{2.9}$$

according to which the exterior derivative (2.7) decomposes as

$$d \;=\; d^{(0)} \;+\; d^{(1)} \;, \tag{2.10}$$

with

$$d^{(0)} \;=\; \xi^a\frac{\partial}{\partial c^a} \;+\; F^a\frac{\partial}{\partial A^a} \;, \tag{2.11}$$

and

$$d^{(0)}d^{(0)} \;=\; 0 \;. \tag{2.12}$$

It is apparent from (2.11) that $d^{(0)}$ has vanishing cohomology. It then follows that also d has vanishing cohomology, due to the fact that the cohomology of d is isomorphic to a subspace of the cohomology of $d^{(0)}$ [13, 14].

2.3 The s cohomology and the descent equations

The triviality of the cohomology of d allows for a simple algebraic derivation of the descent equations. To do this, let us begin by recalling the main result on the cohomolgy of s.

s-cohomology [5, 7, 10, 13, 14]

The cohomology of s on $\mathcal{V}(A,F,c,\xi)$ is spanned by polynomials in the variables (c,F) generated by elements of the form

$$\left(\mathrm{Tr}\,\frac{c^{2m+1}}{(2m+1)!}\right)\mathcal{P}_{2n+2}(F) \qquad m,n=1,2,....\ , \tag{2.13}$$

with $\mathcal{P}_{2n+2}(F)$ the invariant monomial of degree $(2n+2)$; i.e.

$$\mathcal{P}_{2n+2}(F) \ = \ \mathrm{Tr}\,F^{n+1}\ , \tag{2.14}$$

where in matrix notation

$$F=T^aF^a \qquad , \qquad c=T^ac^a\ , \tag{2.15}$$

$$[\,T^a\ ,\ T^b\,]\ =\ if^{abc}T^c \qquad , \qquad \mathrm{Tr}\,T^aT^b\ =\ \delta^{ab}\ , \tag{2.16}$$

T^a being the generators of a finite unitary representation of G.

Due to the Bianchi indentity (2.5), the invariant monomial $\mathcal{P}_{2n+2}(F)$ has also the remarkable property of being d-closed:

$$d\mathcal{P}_{2n+2}(F)\ =0\ . \tag{2.17}$$

The triviality of the cohomology of d implies then

$$\mathcal{P}_{2n+2}(F)\ =\ d\omega^0_{2n+1}\ , \tag{2.18}$$

which, due to (2.8), is easily seen to generate a tower of descent equations:

$$\begin{aligned}
&s\omega^0_{2n+1}\ +\ d\omega^1_{2n}=0\\
&s\omega^1_{2n}\ \ +\ d\omega^2_{2n-1}=0\\
&\quad\ldots\ldots\\
&\quad\ldots\ldots\\
&s\omega^{2n}_1\ \ +\ d\omega^{2n+1}_0=0\\
&s\omega^{2n+1}_0=0\ .
\end{aligned} \tag{2.19}$$

In particular, eq. (2.13) implies that the non-trivial solution of the last equation in (2.19) corresponding to $\mathcal{P}_{2n+2}(F)$ is given by the ghost monomial of degree $(2n+1)$:

$$\omega_0^{2n+1} \;=\; \mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!}\;. \tag{2.20}$$

One has to note that the descent equations (2.19) are still valid in the more general case of an invariant polynomial $\mathcal{Q}_{2n+2}(F)$ which is the product of several elements of the basis (2.14):

$$\mathcal{Q}_{2n+2}(F) \;=\; \prod_{i=1}^{I}\mathcal{P}_{m_i}(F)\;, \qquad\qquad \sum_i m_i = 2n+2\;. \tag{2.21}$$

In this case the corresponding non-trivial solution for the ghost cocycle ω_0^{2n+1} is given by the more complex expression

$$\omega_0^{2n+1} \;=\; \prod_{j=1}^{J}\left(\mathrm{Tr}\,\frac{c^{p_j}}{p_j!}\right)\;, \qquad\qquad \sum_j p_j = 2n+1\;. \tag{2.22}$$

However, as one can easily understand, the knowledge of the solution of the equations (2.19) for the basic monomials $\mathcal{P}_{2n+2}(F)$ allows to characterize also the more general case of eq. (2.21). We can assume then, without loss of generality, that the descent equations (2.19) refer always to the monomials of the basis (2.14)

3 Decomposition of the exterior derivative

The purpose of this section is to analyse in details the algebraic relations implied by a decomposition of the exterior derivative d as the one proposed in (1.5).

3.1 Algebraic relations

Let us introduce the operators δ and $\mathcal{G}$ defined by

$$\delta \;=\; -\,A^a\frac{\partial}{\partial c^a} \;+\; (F^a + f^{abc}\frac{A^bA^c}{2})\frac{\partial}{\partial \xi^a}\;, \tag{3.1}$$

and

$$\mathcal{G} \;=\; -\,F^a\frac{\partial}{\partial c^a} \;+\; f^{abc}F^bA^c\frac{\partial}{\partial \xi^a} \; . \tag{3.2}$$

It is easily verified that δ and $\mathcal{G}$ are respectively of degree zero and one and that the following algebraic relations hold:

$$d \;=\; -[\, s \,,\, \delta \,] \,, \tag{3.3}$$

$$[\, d \,,\, \delta \,] \;=\; 2\mathcal{G} \,, \tag{3.4}$$

$$\{\, d \,,\, \mathcal{G} \,\} \;=\; 0 \qquad , \qquad \mathcal{G}\mathcal{G} \;=\; 0 \,, \tag{3.5}$$

$$\{\, s \,,\, \mathcal{G} \,\} \;=\; 0 \qquad , \qquad [\, \mathcal{G} \,,\, \delta \,] = \; 0 \; . \tag{3.6}$$

One sees from (3.3) that, as announced, the operator δ allows to decompose the exterior derivative d as a BRS commutator. This property, as it will be shown in the next chapters, will give a systematic procedure for solving the descent equations (2.19). Notice that the closure of the algebra between d, s and δ requires the introduction of the operator $\mathcal{G}$. Remarkably, this operator, already present in the work of Brandt et al. [7], here appears in a natural way.

3.2 Properties of $\mathcal{G}$

We will establish here some algebraic properties concerning the operator $\mathcal{G}$ which will be useful in the following. Let us begin by showing that the action of $\mathcal{G}$ on the ghost-monomial (2.20) gives a trivial BRS cocycle.

Proposition II

$$\mathcal{G}\left(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!}\right) \;=\; s\Omega_2^{2n-1} \;, \tag{3.7}$$

for some two-form Ω_2^{2n-1} with ghost number $(2n-1)$.

Proof

The proof is easily done by noticing that the general result (2.13) on the cohomology of s and the relations (3.6) imply that

$$\mathcal{G}s\left(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!}\right) \;=\; -\,s\mathcal{G}\left(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!}\right) \;=\; 0 \;, \tag{3.8}$$

which shows that $\mathcal{G}(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!})$ is s-invariant. Moreover from (2.13) it follows that $\mathcal{G}(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!})$ cannot belong to the cohomolgy of s; hence it is trivial.

The two-form Ω_2^{2n-1} in (3.7) is easily computed by using the expression (3.2) for the operator $\mathcal{G}$ and its general form reads

$$\Omega_2^{2n-1} \;=\; -\frac{i}{(2n-3)}\frac{1}{(2n)!}\mathrm{Tr}\,Fc^{2n-1}\;. \tag{3.9}$$

This completes the proof.

Repeating now the same argument of eq. (3.8) one can prove that, if $\mathcal{G}\Omega_2^{2n-1} \neq 0$, one has

$$\mathcal{G}\Omega_2^{2n-1} \;+\; s\Omega_4^{2n-3} \;=\; 0\;. \tag{3.10}$$

Acting again with $\mathcal{G}$ on eq. (3.10) and using properties (3.5), (3.6) one gets

$$s\mathcal{G}\Omega_4^{2n-3} \;=\; 0\;, \tag{3.11}$$

from which it follows that, if $\mathcal{G}\Omega_4^{2n-3} \neq 0$,

$$\mathcal{G}\Omega_4^{2n-3} \;+\; s\Omega_6^{2n-5} \;=\; 0\;. \tag{3.12}$$

As one can easily understand, this process gives a tower of descent equations between the operators s and $\mathcal{G}$. They read

$$\begin{array}{l} \mathcal{G}\left(\mathrm{Tr}\,\frac{c^{2n+1}}{(2n+1)!}\right) \;=\; s\Omega_2^{2n-1} \\ \mathcal{G}\Omega_2^{2n-1} \;+\; s\Omega_4^{2n-3} \;=\; 0 \\ \ldots\ldots \\ \ldots\ldots \\ \mathcal{G}\Omega_{2n-2}^{3} \;+\; s\Omega_{2n}^{1} \;=\; 0\;. \end{array} \tag{3.13}$$

To close the tower, let us apply once more the operator $\mathcal{G}$ on the last of the equations (3.13). Using (3.5) and (3.6) one has:

$$s\mathcal{G}\Omega_{2n}^{1} \;=\; 0\;, \tag{3.14}$$

from which it follows that $\mathcal{G}\Omega_{2n}^1$ is a BRS invariant $(2n+2)$-form with zero ghost number. Expressions (2.13), (2.14) show then that $\mathcal{G}\Omega_{2n}^1$ is nothingh but the invariant monomial of degree $(2n+2)$:

$$\mathcal{G}\Omega_{2n}^{1} \;=\; (const.)\;\mathcal{P}_{2n+2}(F)\;, \tag{3.15}$$

where the *constant* can be computed by means of the general formula (3.9). Equation (3.15) closes the tower of descent equations (3.13) generated by $\mathcal{G}$ and s.

To conclude this section let us compute, for a better understanding of equations (3.13) - (3.15), the Ω-cocycles for the BRS invariant ghost monomials of degree three and five:

$$f^{abc}\frac{c^a c^b c^c}{3!} \qquad , \qquad d^{abc} f^{bmn} f^{cpq} \frac{c^a c^m c^n c^p c^q}{5!} \ , \tag{3.16}$$

where d^{abc} is the invariant totally symmetric tensor of rank three

$$d^{abc} \ = \ \frac{1}{2} \mathrm{Tr}\, T^a \{ \ T^b \ , \ T^c \ \} \ . \tag{3.17}$$

In the case of $\frac{1}{3!} f^{abc} c^a c^b c^c$ the tower (3.13) - (3.15) reads

$$\mathcal{G} f^{abc} \frac{c^a c^b c^c}{3!} \ = \ s\Omega_2^1 \ , \tag{3.18}$$

$$\mathcal{G}\Omega_2^1 = -\mathcal{P}_4(F) \ , \tag{3.19}$$

with Ω_2^1 and $\mathcal{P}_4(F)$ given by

$$\Omega_2^1 \ = \ F^a c^a \qquad , \qquad \mathcal{P}_4(F) \ = \ F^a F^a \ . \tag{3.20}$$

For the ghost monomial of degree five the descent equations (3.13) - (3.15) are a little more extended

$$\mathcal{G} \ \left(d^{abc} f^{bmn} f^{cpq} \frac{c^a c^m c^n c^p c^q}{5!} \right) \ = \ s\Omega_2^3 \ , \tag{3.21}$$

$$\mathcal{G}\Omega_2^3 \ + \ s\Omega_4^1 \ = \ 0 \ , \tag{3.22}$$

$$\mathcal{G}\Omega_4^1 \ = \ \frac{1}{3}\mathcal{P}_6(F) \ , \tag{3.23}$$

where Ω_2^3, Ω_4^1 and $\mathcal{P}_6(F)$ are computed to be

$$\begin{aligned} \Omega_2^3 \ &= \ \tfrac{1}{12} d^{abc} F^a c^b f^{cmn} c^m c^n \\ \Omega_4^1 \ &= \ -\ \tfrac{1}{3} d^{abc} F^a F^b c^c \\ \mathcal{P}_6(F) \ &= \ d^{abc} F^a F^b F^c \ . \end{aligned} \tag{3.24}$$

4 Solution of the descent equations

In this chapter we will apply the results established in the previous sections to obtain in a closed form a class of solutions of the descent equations (2.19). This will be done by using the decomposition of the exterior derivative (3.3) as well as the descent equations (3.13) - (3.15) involving the operators $\mathcal{G}$ and s.

For the sake of clarity and to make contact with the solutions given by the "*Russian − formula*" [3, 4, 5, 6] let us proceed by discussing some explicit examples.

4.1 The case n=1

In this case, relevant for the two-dimensional gauge anomaly and for the three-dimensional Chern-Simons term, the descent equations (2.19) read:

$$\begin{array}{l} s\omega_3^0 \ + \ d\omega_2^1 = 0 \\ s\omega_2^1 \ + \ d\omega_1^2 = 0 \\ s\omega_1^2 \ + \ d\omega_0^3 = 0 \\ s\omega_0^3 = 0 \ , \end{array} \tag{4.1}$$

where, from eq. (2.20), ω_0^3 is given by

$$\omega_0^3 \ = \ f^{abc}\frac{c^a c^b c^c}{3!} \ . \tag{4.2}$$

Acting with the operator δ of eq. (3.1) on the last of the equations (4.1) one gets

$$[\ \delta\ ,\ s\ ]\omega_0^3 \ + \ s\delta\omega_0^3 \ = \ 0\ , \tag{4.3}$$

which, using the decomposition (3.3), becomes

$$s\delta\omega_0^3 \ + \ d\omega_0^3 = \ 0\ . \tag{4.4}$$

This equation shows that $\delta\omega_0^3$ gives a solution for the cocycle ω_1^2 in eqs. (4.1). Acting again with δ on the equation (4.4) and using the algebraic relations (3.3), (3.4) one has

$$s\frac{\delta\delta}{2}\omega_0^3 \ - \mathcal{G}\omega_0^3 \ + \ d\delta\omega_0^3 \ = 0\ . \tag{4.5}$$

Moreover, from the previous results (3.7) and (3.18) - (3.20), we get

$$
\begin{aligned}
\mathcal{G}\omega^3_0 \; &= \; s\Omega^1_2 \\
\Omega^1_2 &= F^a c^a \ ,
\end{aligned}
\tag{4.6}
$$

so that eq. (4.5) can be rewritten as:

$$
s(\frac{\delta\delta}{2}\omega^3_0 - \Omega^1_2) \; + \; d\delta\omega^3_0 \; = 0 \ . \tag{4.7}
$$

One sees that $(\frac{1}{2}\delta\delta\omega^3_0 - \Omega^1_2)$ gives a solution for ω^1_2. To solve completely the tower (4.1) let us apply once more the operator δ on the equation (4.7). After a little algebra we get:

$$
s(\frac{\delta\delta\delta}{3!}\omega^3_0 - \delta\Omega^1_2) \; + \; d(\frac{\delta\delta}{2}\omega^3_0 - \Omega^1_2) \; = 0 \ . \tag{4.8}
$$

which shows that the cocycle ω^0_3 can be identified with $(\frac{1}{3!}\delta\delta\delta\omega^3_0 - \delta\Omega^1_2)$. It is apparent then how repeated applications of the operator δ on the zero-form cocycle ω^3_0 and the use of the tower (3.13) - (3.15) give in a simple way a solution of the descent equations.

Summarizing, the solution of the descent equations (4.1) is given by

$$
\omega^0_3 \; = \; \frac{1}{3!}\delta\delta\delta\omega^3_0 \; - \; \delta\Omega^1_2 \ , \tag{4.9}
$$

$$
\omega^1_2 \; = \; \frac{1}{2}\delta\delta\omega^3_0 \; - \; \Omega^1_2 \ , \tag{4.10}
$$

$$
\omega^2_1 \; = \; \delta\omega^3_0 \ , \tag{4.11}
$$

where, using expressions (4.2), (4.6), ω^0_3, ω^1_2 and ω^2_1 read:

$$
\omega^2_1 \; = \; -\frac{1}{2}f^{abc}A^a c^b c^c \; = \; \xi^a c^a \; - s(A^a c^a) \ , \tag{4.12}
$$

$$
\omega^1_2 \; = \; \frac{1}{2}f^{abc}A^a A^b c^c - F^a c^a \; = \; - dA^a c^a \ , \tag{4.13}
$$

$$
\omega^0_3 \; = \; F^a A^a - \frac{1}{6}f^{abc}A^a A^b A^c \; = \; \mathrm{Tr}\,(FA + \frac{i}{3}A^3) \ . \tag{4.14}
$$

One easily recognizes that the expressions (4.12) - (4.14) coincide, modulo an s or a d-coboundary, with the solution given, for instance, in the work of Zumino, Wu and Zee [3]. In particular, (4.13) and (4.14) give respectively the two-dimensional gauge anomaly (modulo a d-coboundary) and the three-dimensional Chern-Simons term.

4.2 The case n=2

In this example, relevant for the gauge anomaly in four dimensions, the tower (2.19) takes the form:

$$\begin{array}{l} s\omega_5^0 \ + \ d\omega_4^1 = 0 \\ s\omega_4^1 \ + \ d\omega_3^2 = 0 \\ s\omega_3^2 \ + \ d\omega_2^3 = 0 \\ s\omega_2^3 \ + \ d\omega_1^4 = 0 \\ s\omega_1^4 \ + \ d\omega_0^5 = 0 \\ s\omega_0^5 = 0 \ , \end{array} \tag{4.15}$$

where, according to equation (2.20):

$$\omega_0^5 \ = \ d^{abc} f^{bmn} f^{cpq} \frac{c^a c^m c^n c^p c^q}{5!} \ . \tag{4.16}$$

As in the previous case, a solution of eqs. (4.15) is easily obtained by applying the operator δ on the cocycle (4.16):

$$\omega_5^0 \ = \ \frac{1}{5!}\delta\delta\delta\delta\delta\omega_0^5 \ - \ \frac{1}{3!}\delta\delta\delta\Omega_2^3 \ - \ \delta\Omega_4^1 \ , \tag{4.17}$$

$$\omega_4^1 \ = \ \frac{1}{4!}\delta\delta\delta\delta\omega_0^5 \ - \ \frac{1}{2}\delta\delta\Omega_2^3 \ - \ \Omega_4^1 \ , \tag{4.18}$$

$$\omega_3^2 \ = \ \frac{1}{3!}\delta\delta\delta\omega_0^5 \ - \ \delta\Omega_2^3 \ , \tag{4.19}$$

$$\omega_2^3 \ = \ \frac{1}{2}\delta\delta\omega_0^5 \ - \ \Omega_2^3 \ , \tag{4.20}$$

$$\omega_1^4 \ = \ \delta\omega_0^5 \ , \tag{4.21}$$

where Ω_2^3 and Ω_4^1 belong to the tower (3.21) - (3.23) and are given by (see also eq. (3.24)):

$$\Omega_2^3 \ = \ \frac{1}{12} d^{abc} F^a c^b f^{cmn} c^m c^n \ , \tag{4.22}$$

$$\Omega_4^1 \ = \ -\frac{1}{3} d^{abc} F^a F^b c^c \ . \tag{4.23}$$

In particular, ω_5^0 and ω_4^1 are computed to be:

$$\begin{aligned} \omega_5^0 \ &= -\tfrac{1}{3} d^{abc} F^a F^b A^c \ - \ \tfrac{1}{120} d^{abc} f^{bmn} f^{cpq} A^a A^m A^n A^p A^q \\ &\quad + \tfrac{1}{12} d^{abc} F^a A^b f^{cmn} A^m A^n \\ &= - \tfrac{1}{3} \mathrm{Tr} \left(F^2 A + \tfrac{i}{2} F A^3 - \tfrac{1}{10} A^5 \right) \ . \end{aligned} \tag{4.24}$$

$$
\begin{aligned}
\omega_4^1 \;=\; & \tfrac{1}{3} d^{abc} F^a F^b c^c \;-\; \tfrac{1}{12} d^{abc} F^a c^b f^{cmn} A^m A^n \\
& - \tfrac{1}{6} d^{abc} F^a A^b f^{cmn} A^m c^n \;+\; \tfrac{1}{24} d^{abc} f^{bmn} f^{cpq} A^a A^m A^n A^p c^q \\
=\; & \tfrac{1}{3} c^a \; d\left(d^{abc} A^b dA^c \;+\; \tfrac{1}{4} d^{abc} A^b f^{cmn} A^m A^n \right) \; .
\end{aligned}
\tag{4.25}
$$

and give respectively the generalized five-dimensional Chern-Simons term and the four-dimensional gauge anomaly. Again, expressions (4.24) and (4.25) coincide, modulo a d-coboundary, with that of ref. [3].

4.3 The general case

It is strightforward now to iterate the previous construction to obtain the solution of the descent equations (2.19) in the general case of an arbitrary n $(n \geq 1)$.

The solution of the ladder

$$
\begin{aligned}
& s\omega^0_{2n+1} \;+\; d\omega^1_{2n} = 0 \\
& s\omega^1_{2n} \;\;\;\; + \; d\omega^2_{2n-1} = 0 \\
& \quad \ldots\ldots \\
& \quad \ldots\ldots \\
& s\omega^{2n}_1 \;\;\;\; + \; d\omega^{2n+1}_0 = 0 \\
& s\omega^{2n+1}_0 = 0 \; ,
\end{aligned}
\tag{4.26}
$$

is given by

$$
\omega^{2n+1}_0 \;=\; \mathrm{Tr}\, \frac{c^{2n+1}}{(2n+1)!} \; , \tag{4.27}
$$

$$
\omega^{2n+1-2p}_{2p} \;=\; \frac{\delta^{2p}}{(2p)!}\omega^{2n+1}_0 \;-\; \sum_{j=0}^{p-1} \frac{\delta^{2j}}{(2j)!} \Omega^{2n+1-2p+2j}_{2p-2j} \; , \tag{4.28}
$$

for the even space-time form sector and

$$
\omega^{2n}_1 \;=\; \delta\omega^{2n+1}_0 \; , \tag{4.29}
$$

$$
\omega^{2n-2p}_{2p+1} \;=\; \frac{\delta^{2p+1}}{(2p+1)!}\omega^{2n+1}_0 \;-\; \sum_{j=0}^{p-1} \frac{\delta^{2j+1}}{(2j+1)!} \Omega^{2n+1-2p+2j}_{2p-2j} \; , \tag{4.30}
$$

for the odd sector and $p = 1, 2, ...n$.

The Ω-cocycles in (4.28), (4.30) belong to the tower (3.13) - (3.15) and are computed by using the general formula (3.9).

Equations (4.27) - (4.30) generalize the results of the previous examples and show how the use of the operator δ gives a simple way of generating explicit solutions. It is easy to check indeed that, for $n > 2$, the expressions (4.27) - (4.30) coincide, modulo an s or a d-coboundary, with the results obtained in [3, 4, 5, 6].

5 Conclusion

We have presented a new way of solving the descent equations associated with the Wess-Zumino consistency conditions. The main ingredient has been the introduction of the operator δ which decomposes the exterior derivative d as a BRS commutator and which allows to introduce in a natural way the operator $\mathcal{G}$. This operator, already used by Brandt et al. [7], generates together with the BRS operator s a new tower of descent equations which are easily disentangled using the general results on the cohomology of s. Moreover the algebraic properties of δ and $\mathcal{G}$ allow for a complete characterization of the solutions of Wess-Zumino consistency conditions. These solutions turn out to coincide, modulo trivial cocycles, with the ones already obtained by using the homotopy "$Russian - formula$" [3, 4, 5, 6].

Applications to gravity and topological theories are under investigation [18].

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