Asymptotic Safety

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To appear in " Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter" ed. D. Oriti, Cambridge University Press.

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Abstract

Asymptotic safety is a set of conditions, based on the existence of a nontrivial fixed point for the renormalization group flow, which would make a quantum field theory consistent to arbitrarily high energies. After introducing the basic ideas of this approach, I review the present evidence in favor of an asymptotically safe quantum field theory of gravity.

1.1 Introduction

The problems of perturbative Quantum Field Theory (QFT) in relation to the UV behaviour of gravity have led to widespread pessimism about the possibility of constructing a fundamental QFT of gravity. Instead, we have become accustomed to thinking of General Relativity (GR) as an effective field theory, which only gives an accurate description of gravitational physics at low energies. The formalism of effective field theories provides a coherent framework in which quantum calculations can be performed even if the theory is not renormalizable. For example, quantum corrections to the gravitational potential have been discussed by several authors; see Bjerrum-Bohr *et al.* (2003) and references therein. This continuum QFT description is widely expected to break down at very short distances and to be replaced by something dramatically different beyond the Planck scale. There is however no proof that continuum QFT will fail, and the current situation may just be the result of the lack of suitable technical tools. Weinberg (1979) described a generalized, nonperturbative notion of renormalizability called "asymptotic safety" and suggested that GR may satisfy this condition, making it a consistent QFT at all energies. The essential ingredient of this approach is the existence of a Fixed Point (FP) in the Renormalization Group (RG) flow of gravitational couplings. Several calculations were performed using the ϵ–expansion around $d = 2$ dimensions, supporting the view that gravity is asymptotically safe (Gastmans *et al.* (1978), Christensen & Duff (1978), Kawai & Ninomiya (1990)). However, the continuation to four dimensions ($\epsilon \to 2$) was questionable and this line of research slowed down for some time. It was revived by Reuter (1998) who calculated the gravitational beta functions directly in $d = 4$ dimensions, using a truncation of an Exact Renormalization Group Equation (ERGE). Matter couplings were considered by Dou & Percacci (1998); then Souma (1999) found that these beta functions admit a non–Gaußian FP. Further work by Lauscher & Reuter (2002a,b), Percacci (2006), Codello & Percacci (2006) strongly supports the view that this FP is not a mere artifact of the approximations made. An extensive review of this subject can be found in Niedermaier & Reuter (2006).

In section 1.2 I introduce the general idea of asymptotic safety; the reader is referred to Weinberg (1979) for a more detailed discussion. In section 1.3 I describe some peculiarities of the gravitational RG, which derive from the dual character of the metric as a dynamical field and as definition of lengths. Recent evidence for a FP, coming mainly from the ERGE, is reviewed in section 1.4. Some relations to other approaches to quantum gravity are briefly mentioned in section 1.5.

1.2 The general notion of asymptotic safety

The techniques of effective QFT have been recognized as being of great generality and are now quite pervasive in particle physics. An effective field theory is described by an effective action Γ_k which can be thought of as the result of having integrated out all fluctuations of the fields with momenta larger than k. We need not specify here the physical meaning of k: for each application of the theory one will have to identify the physically relevant variable acting as k (in particle physics it is usually some external momentum). One convenient definition of Γ_k that we shall use here is as follows. We start from a ("bare") action $S[\phi_A]$ for multiplets of quantum fields ϕ_A, describing physics at an energy scale k_0. We add to it a term $\Delta S_k[\phi_A]$, quadratic in the ϕ_A, which in Fourier space has the form: $\Delta S_k[\phi] = \int d^d q \phi_A R_k^{AB}(q^2)\phi_B$. The kernel $R_k^{AB}(q^2)$, henceforth called the cutoff function, is chosen in such a way that the propagation of field modes $\phi_A(q)$ with momenta $q < k$ is suppressed, while field modes with momenta $k < q < k_0$ are unaffected. We formally define a k–dependent generating functional of connected Green functions

$$W_k[J^A] = -\log \int (d\phi_A) \exp\left(-S[\phi_A] - \Delta S_k[\phi_A] - \int J^A \phi_A\right) \quad (1.2.1)$$

and a modified k–dependent Legendre transform

$$\Gamma_k[\phi_A] = W_k[J^A] - \int J^A \phi_A - \Delta S_k[\phi_A] \ , \quad (1.2.2)$$

where ΔS_k has been subtracted. The "classical fields" $\frac{\delta W_k}{\delta J^A}$ are denoted again ϕ_A for notational simplicity. This functional interpolates continuously between S, for $k = k_0$, and the usual effective action $\Gamma[\phi_A]$, the generating functional of one–particle irreducible Green functions, for $k = 0$. It is similar in spirit, but distinct from, the Wilsonian effective action. In the following we will always use this definition of Γ_k, but much of what will be said should be true also with other definitions.

In the case of gauge theories there are complications due to the fact that the cutoff interferes with gauge invariance. One can use a background gauge condition, which circumvents these problems by defining a functional of two fields, the background field and the classical field; the effective action Γ_k is then obtained by identifying these fields. See Pawlowski (2005), or Reuter (1998) for the case of gravity.

The effective action $\Gamma_k[\phi_A]$, used at tree level, gives an accurate description of processes occurring at momentum scales of order k. In general it will have the form $\Gamma_k(\phi_A, g_i) = \sum_i g_i(k)\mathcal{O}_i(\phi_A)$, where g_i are runnning coupling constants and $\mathcal{O}_i$ are all possible operators constructed with the fields ϕ_A and their derivatives, which are compatible with the symmetries of the theory. It can be thought of as a functional on $\mathcal{F} \times \mathcal{Q} \times R^+$, where $\mathcal{F}$ is the configuration space of the fields, $\mathcal{Q}$ is an infinite dimensional manifold parametrized by the coupling constants, and R^+ is the space parametrized by k. The dependence of Γ_k on k is given by $\partial_t \Gamma_k(\phi_A, g_i) = \sum_i \beta_i(k)\mathcal{O}_i(\phi_A)$ where $t = \log(k/k_0)$ and $\beta_i(g_j, k) = \partial_t g_i$ are the beta functions.

Dimensional analysis implies the scaling property

$$\Gamma_k(\phi_A, g_i) = \Gamma_{bk}(b^{d_A}\phi_A, b^{d_i} g_i) \ , \quad (1.2.3)$$

where d_A is the canonical dimension of ϕ_A, d_i is the canonical dimension of g_i, and $b \in R^+$ is a positive real scaling parameter †. One can rewrite

† We assume that the coordinates are dimensionless, as is natural in curved space, resulting in unconventional canonical dimensions. The metric is an area.

the theory in terms of dimensionless fields $\tilde\phi_A = \phi_A k^{-d_A}$ and dimensionless couplings $\tilde g_i = g_i k^{-d_i}$. A transformation (1.2.3) with parameter $b = k^{-1}$ can be used to define a functional $\tilde\Gamma$ on $(\mathcal{F} \times \mathcal{Q} \times R^+)/R^+$:

$$\tilde\Gamma(\tilde\phi_A, \tilde g_i) := \Gamma_1(\tilde\phi_A, \tilde g_i) = \Gamma_k(\phi_A, g_i) \ . \tag{1.2.4}$$

Similarly, $\beta_i(g_j, k) = k^{d_i} a_i(\tilde g_j)$ where $a_i(\tilde g_j) = \beta_i(\tilde g_j, 1)$. There follows that the beta functions of the dimensionless couplings,

$$\tilde\beta_i(\tilde g_j) \equiv \partial_t \tilde g_i = a_i(\tilde g_j) - d_i \tilde g_i \tag{1.2.5}$$

depend on k only implicitly via the $\tilde g_j(t)$.

The effective actions Γ_k and $\Gamma_{k-\delta k}$ differ essentially by a functional integral over field modes with momenta between k and $k-\delta k$. Such integration does not lead to divergences, so the beta functions are automatically finite. Once calculated at a certain scale k, they are automatically determined at any other scale by dimensional analysis. Thus, the scale k_0 and the "bare" action S act just as initial conditions: when the beta functions are known, one can start from an arbitrary initial point on $\mathcal{Q}$ and follow the RG trajectory in either direction. The effective action Γ_k at any scale k can be obtained integrating the flow. In particular, the UV behaviour can be studied by taking the limit $k \to \infty$.

It often happens that the flow cannot be integrated beyond a certain limiting scale Λ, defining the point at which some "new physics" has to make its appearance. In this case the theory only holds for $k < \Lambda$ and is called an "effective" or "cutoff" QFT. It may happen, however, that the limit $t \to \infty$ can be taken; we then have a self-consistent description of a certain set of physical phenomena which is valid for arbitrarily high energy scales and does not need to refer to anything else outside it. In this case the theory is said to be "fundamental".

The couplings appearing in the effective action can be related to physically measurable quantities such as cross sections and decay rates. Dimensional analysis implies that aside from an overall power of k, such quantities only depend on dimensionless kinematical variables X, like scattering angles and ratios of energies, and on the dimensionless couplings $\tilde g_i$ (recall that usually k is identified with one of the momentum variables). For example, a cross section can be expressed as $\sigma = k^{-2}\tilde\sigma(X, \tilde g_i)$. If some of the couplings $\tilde g_i$ go to infinity when $t \to \infty$, also the function $\tilde\sigma$ can be expected to diverge. A sufficient condition to avoid this problem is to assume that in the limit $t \to \infty$ the RG trajectory tends to a FP of the RG, *i.e.* a point $\tilde g_*$ where $\tilde\beta_i(\tilde g_*) = 0$ for all i. The existence of such a FP is the first requirement for asymptotic safety. Before discussing the second requirement, we have to understand that one needs to impose this condition only on a subset of all couplings.

The fields ϕ_A are integration variables, and a redefinition of the fields does not change the physical content of the theory. This can be seen as invariance under a group $\mathcal{G}$ of coordinate transformations in $\mathcal{F}$. There is a similar arbitrariness in the choice of coordinates on $\mathcal{Q}$, due to the freedom of redefining the couplings g_i. Since, for given k, Γ_k is assumed to be the "most general" functional on $\mathcal{F} \times \mathcal{Q}$ (in some proper sense), given a field redefinition $\phi' = \phi'(\phi)$ one can find new couplings g'_i such that

$$\Gamma_k(\phi'_B(\phi_A), g_i) = \Gamma_k(\phi_A, g'_i) \ . \tag{1.2.6}$$

At least locally, this defines an action of $\mathcal{G}$ on $\mathcal{Q}$. We are then free to choose a coordinate system which is adapted to these trasformations, in the sense that a subset $\{g_{\hat{\imath}}\}$ of couplings transform nontrivially and can be used as coordinates in the orbits of $\mathcal{G}$, while a subset $\{g_{\bar{\imath}}\}$ are invariant under the action of $\mathcal{G}$ and define coordinates on $\mathcal{Q}/\mathcal{G}$. The couplings $g_{\hat{\imath}}$ are called redundant or inessential, while the couplings $g_{\bar{\imath}}$ are called essential. In an adapted parametrization there exists, at least locally, a field redefinition $\bar{\phi}(\phi)$ such that using (1.2.6) the couplings $g_{\hat{\imath}}$ can be given fixed values $(g_{\hat{\imath}})_0$. We can then define a new action $\bar{\Gamma}$ depending only on the essential couplings:

$$\bar{\Gamma}_k(\bar{\phi}_A, g_{\bar{\imath}}) := \Gamma_k(\bar{\phi}_A, g_{\bar{\imath}}, (g_{\hat{\imath}})_0) = \Gamma_k(\phi_A; g_{\bar{\imath}}, g_{\hat{\imath}}) \ . \tag{1.2.7}$$

Similarly, the values of the redundant couplings can be fixed also in the expressions for measurable quantities, so there is no need to constrain their RG flow in any way: they are not required to flow towards a FP.

For example, the action of a scalar field theory in a background $g_{\mu\nu}$,

$$\Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}) = \int d^4x \sqrt{g} \left[\frac{Z_\phi}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda_2 \phi^2 + \lambda_4 \phi^4 + \dots \right] \tag{1.2.8}$$

has the scaling invariance

$$\Gamma_k(c\phi, g_{\mu\nu}; c^{-2} Z_\phi, c^{-2i} \lambda_{2i}) = \Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}) \ , \tag{1.2.9}$$

which is a special case of (1.2.6). There exists an adapted coordinate system where Z is inessential and $\bar{\lambda}_{2i} = \lambda_{2i} Z_\phi^{-i}$ are the essential coordinates. A transformation with $c = \sqrt{Z_\phi}$ then leads to $Z_\phi = 1$, leaving the essential couplings unaffected.

A comparison of (1.2.4) and (1.2.7) shows that k behaves like a redundant coupling. In ordinary QFT's, it is generally the case that for each multiplet of fields ϕ_A there is a scaling invariance like (1.2.9) commuting with (1.2.3). One can use these invariances to eliminate simultaneously k and one other redundant coupling per field multiplet; the conventional choice is to eliminate the wave function renormalization Z_A. No conditions have to be imposed on the RG flow of the Z_A's, and the anomalous dimensions $\eta_A = \partial_t \log Z_A$, at a FP, can be determined by a calculation. More generally, (1.2.3) and (1.2.6) can be used to eliminate simultaneously the dependence of Γ_k on k and on the inessential couplings, and to define an effective action $\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_{\bar{\imath}})$, depending only on the dimensionless essential couplings $\tilde{g}_{\bar{\imath}} = g_{\bar{\imath}} k^{-d_{\bar{\imath}}}$. It is only on these couplings that one has to impose the FP condition $\partial_t \tilde{g}_{\bar{\imath}} = 0$.

We can now state the second requirement for asymptotic safety. Denote $\tilde{\mathcal{Q}} = (\mathcal{Q} \times R^+)/(\mathcal{G} \times R^+)$ the space parametrized by the dimensionless essential couplings $\tilde{g}_{\bar{\imath}}$. The set $\mathcal{C}$ of all points in $\tilde{\mathcal{Q}}$ that flow towards the FP in the UV limit is called the UV critical surface. If one chooses an initial point lying on $\mathcal{C}$, the whole trajectory will remain on $\mathcal{C}$ and will ultimately flow towards the FP in the UV limit. Points that lie outside $\mathcal{C}$ will generally flow towards infinity (or other FP's). Thus, demanding that the theory lies on the UV critical surface ensures that it has a sensible UV limit. It also has the effect of reducing the arbitrariness in the choice of the coupling constants. In particular, if the UV critical surface is finite dimensional, the arbitariness is reduced to a finite number of parameters, which can be determined by a finite number of experiments. Thus, a theory with a FP and a finite dimensional UV critical surface has a controllable UV behaviour, and is predictive. Such a theory is called "asymptotically safe".

A perturbatively renormalizable, asymptotically free field theory such as QCD is a special case of an asymptotically safe theory. In this case the FP is the Gaußian FP, where all couplings vanish, and the critical surface is spanned, near the FP, by the couplings that are renormalizable in the perturbative sense (those with dimension $d_{\bar{\imath}} \geq 0$).

The requirement of renormalizability played an important role in the construction of the Standard Model (SM) of particle physics. Given that the SM is not a complete theory, and that some of its couplings are not asymptotically free, nowadays it is regarded an effective QFT, whose nonrenormalizable couplings are suppressed by some power of momentum over cutoff. On the other hand, any theory that includes both the SM and gravity should better be a fundamental theory. For such a theory, the requirement of asymptotic safety will have the same significance that renormalizability originally had for the SM.

1.3 The case of gravity

We shall use a derivative expansion of Γ_k:

$$\Gamma_k(g_{\mu\nu}; g_i^{(n)}) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) \mathcal{O}_i^{(n)}(g_{\mu\nu}) \ , \tag{1.3.1}$$

where $\mathcal{O}_i^{(n)} = \int d^dx \sqrt{g} \mathcal{M}_i^{(n)}$ and $\mathcal{M}_i^{(n)}$ are polynomials in the curvature tensor and its derivatives containing $2n$ derivatives of the metric; i is an index that labels different operators with the same number of derivatives. The dimension of $g_i^{(n)}$ is $d_n = d - 2n$. The first two polynomials are just $\mathcal{M}^{(0)} = 1$, $\mathcal{M}^{(1)} = R$. The corresponding couplings are $g^{(1)} = -Z_g = -\frac{1}{16\pi G}$, $g^{(0)} = 2Z_g\Lambda$, Λ being the cosmological constant. Newton's constant G appears in Z_g, which in linearized Einstein theory is the wave function renormalization of the graviton. Neglecting total derivatives, one can choose as terms with four derivatives of the metric $\mathcal{M}_1^{(2)} = C^2$ (the square of the Weyl tensor) and $\mathcal{M}_2^{(2)} = R^2$. We also note that the coupling constants of higher derivative gravity are not the coefficients $g_i^{(2)}$ but rather their inverses $2\lambda = (g_1^{(2)})^{-1}$ and $\xi = (g_2^{(2)})^{-1}$. Thus,

$$\Gamma_k^{(n\leq 2)} = \int d^dx \sqrt{g} \left[2Z_g\Lambda - Z_g R + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 \right] \ . \tag{1.3.2}$$

As in any other QFT, Z_g can be eliminated from the action by a rescaling of the field. Under constant rescalings of $g_{\mu\nu}$, in d dimensions,

$$\Gamma_k(g_{\mu\nu}; g_i^{(n)}) = \Gamma_{bk}(b^{-2} g_{\mu\nu}; b^{d-2n} g_i^{(n)}) \ . \tag{1.3.3}$$

This relation is the analog of (1.2.9) for the metric, but also coincides with (1.2.3), the invariance at the basis of dimensional analysis; fixing it amounts to a choice of unit of mass. This is where gravity differs from any other field theory (Percacci & Perini (2004), Percacci (2007)). In usual QFT's such as (1.2.8), one can exploit the two invariances (1.2.3) and (1.2.9) to eliminate simultaneously k and Z from the action. In the case of pure gravity there is only one such invariance and one has to make a choice.

If we choose k as unit of mass, we can define the effective action,

$$\tilde{\Gamma}(\tilde{g}_{\mu\nu}; \tilde{Z}_g, \tilde{\Lambda}, \ldots) = \Gamma_1(\tilde{g}_{\mu\nu}; \tilde{Z}_g, \tilde{\Lambda}, \ldots) = \Gamma_k(g_{\mu\nu}; Z_g, \Lambda, \ldots) \ , \tag{1.3.4}$$

where $\tilde{g}_{\mu\nu} = k^2 g_{\mu\nu}$, $\tilde{Z}_g = \frac{Z_g}{k^2} = \frac{1}{16\pi\tilde{G}}$, $\tilde{\Lambda} = \frac{\Lambda}{k^2}$, etc.. There is then no freedom left to eliminate Z_g. Physically measurable quantities will depend explicitly on $\tilde{Z}_g$, so by the arguments of section 1.2, we have to impose that $\partial_t \tilde{Z}_g = 0$, or equivalently $\partial_t \tilde{G} = 0$, at a FP.

Alternatively, one can use (1.3.3) to set $Z_g = 1$: this amounts to working in Planck units. Then we can define a new action †:

$$\Gamma'_{k'}(g'_{\mu\nu};\Lambda',\ldots) = \Gamma_{k'}(g'_{\mu\nu};\Lambda',1,\ldots) = \Gamma_k(g_{\mu\nu};\Lambda,Z_g,\ldots)\ , \qquad (1.3.5)$$

where $g'_{\mu\nu} = 16\pi Z_g g_{\mu\nu}$, $\Lambda' = \frac{1}{16\pi Z_g}\Lambda$, $k' = \sqrt{\frac{1}{16\pi Z_g}}k$ etc. are the metric, cosmological constant and cutoff measured in Planck units. In this case, the dependence on G disappears; however, the beta functions and measurable quantities will depend explicitly on k'.

In theories of gravity coupled to matter, the number of these scaling invariances is equal to the number of field multiplets, so the situation is the same as for pure gravity. (Without gravity, it is equal to the number of field multiplets plus one, due to dimensional analysis.) The situation can be summarized by saying that when the metric is dynamical, k should be treated as one of the couplings, and that there exist parametrizations where k is redundant or G is redundant, but not both.

Scale invariance is usually thought to imply that a theory contains only dimensionless parameters, and the presence at a FP of nonvanishing dimensionful couplings may seem to be at odds with the notion that the FP theory is scale–invariant. This is the case if only the fields are scaled, and not the couplings. In an asymptotically safe QFT, scale invariance is realized in another way: all dimensionful couplings scale with k as required by their canonical dimension. In geometrical terms, the RG trajectories in $\mathcal{Q}$ lie asymptotically in an orbit of the transformations (1.2.3) and (1.2.6). This also has another consequence. At low momentum scales $p \ll \sqrt{Z_g}$ the couplings are not expected to run and the terms in the action (1.3.2) with four derivatives are suppressed relative to the term with two derivatives by a factor p^2/Z_g. On the other hand in the FP regime, if we evaluate the couplings at $k = p$, the running of Z_g exactly compensates the effect of the derivatives: both terms are of order p^4. From this point of view, *a priori* all terms in (1.3.1) could be equally important.

From the existence of a FP for Newton's constant there would immediately follow two striking consequences. First, the cutoff measured in Planck units would be bounded. This is because the cutoff in Planck units, $k' = k\sqrt{G}$, is equal to the square root of Newton's constant in cutoff units, $\sqrt{\tilde{G}}$. Since we have argued that the latter must have a finite limit at a FP, then also the former must do so. This seems to

† Note that to completely eliminate Z_g from the action one has to scale the whole metric, and not just the fluctuation, as is customary in perturbation theory.

contradict the notion that the UV limit is defined by $k \to \infty$. The point is that only statements about dimensionless quantities are physically meaningful, and the statement "$k \to \infty$" is meaningless until we specify the units. In a fundamental theory one cannot refer to any external "absolute" quantity as a unit, and any internal quantity which is chosen as a unit will be subject to the RG flow. If we start from low energy ($k' \ll 1$) and we increase k, k' will initially increase at the same rate, because in this regime $\partial_t G \approx 0$; however, when $k' \approx 1$ we reach the FP regime where $G(k) \approx \tilde{G}_*/k^2$ and therefore k' stops growing.

The second consequence concerns the graviton anomalous dimension, which in d dimensions is $\eta_g = \partial_t \log Z_g = \partial_t \log \tilde{Z}_g + d - 2$. Since we have argued that $\partial_t \tilde{Z}_g = 0$ at a gravitational FP, if $\tilde{Z}_{g*} \neq 0$ we must have $\eta_{g*} = d - 2$. The propagator of a field with anomalous dimension η behaves like $p^{-2-\eta}$, so one concludes that at a nontrivial gravitational FP the graviton propagator behaves like p^{-d} rather than p^{-2}, as would follow from a naive classical interpretation of the Einstein-Hilbert action. Similar behaviour is known also in other gauge theories away from the critical dimension, see *e.g.* Kazakov (2003).

1.4 The Gravitational Fixed Point

I will now describe some of the evidence that has accumulated in favor of a nontrivial gravitational FP. Early attempts were made in the context of the ϵ–expansion around two dimensions ($\epsilon = d - 2$), which yields

$$\beta_{\tilde{G}} = \epsilon \tilde{G} - q \tilde{G}^2 \ . \tag{1.4.1}$$

Thus there is a UV–attractive FP at $\tilde{G}_* = \epsilon/q$. The constant $q = \frac{38}{3}$ for pure gravity (Weinberg (1979), Kawai & Ninomiya (1990), see Aida & Kitazawa (1997) for two–loop results). Unfortunately, for a while it was not clear whether one could trust the continuation of this result to four dimensions ($\epsilon = 2$).

Most of the recent progress in this approach has come from the application to gravity of the ERGE. It was shown by Wetterich (1993) that the effective action Γ_k defined in (1.2.2) satisfies the equation

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi_A \delta\phi_B} + R_k^{AB} \right)^{-1} \partial_t R_k^{BA} \ , \tag{1.4.2}$$

where STr is a trace over momenta as well as over particle species and any spacetime or internal indices, including a sign -1 for fermionic fields and a factor 2 for complex fields. In the case of gauge theories, the ghost fields have to be included among the ϕ_A.

Comparing the r.h.s. of the ERGE with the t-derivative of (1.3.1) one can extract the beta functions. Note that in general the cutoff function R_k may depend on the couplings and therefore the term $\partial_t R_k$ in the r.h.s. of (1.4.2) contains the beta functions. Thus, extracting the beta functions from the ERGE implies solving an equation where the beta functions appear on both sides. At one loop, the effective action Γ_k is $\mathrm{Tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\phi\delta\phi}$; it satisfies an equation which is formally identical to (1.4.2) except that in the r.h.s. the running couplings $g_i(k)$ are replaced everywhere by the "bare" couplings $g_i(k_0)$, appearing in S. We will call "one–loop beta functions" those extracted from the ERGE ignoring the derivatives of the couplings that may appear in the r.h.s. of (1.4.2).

It is usually impossible to get the beta functions for all couplings, so a common procedure is to consider a truncation of the theory where the effective action Γ_k contains only a (finite or infinite) subset of all possible terms. In these calculations there is no small parameter to tell us what terms can be safely neglected, so the choice of truncation has to be motivated by physical insight. On the other hand, in this way one can obtain genuine nonperturbative information. This and other similar ERGEs have been applied to a variety of problems. One can reproduce the universal one loop beta functions of familiar theories, and in more advanced approximations the results are quantitatively comparable to those obtainable by other methods. See Bagnuls & Bervilliers (2001), Berges *et al.* (2002), Pawlowski (2005) for reviews.

The simplest way to arrive at a gravitational FP in four dimensions, avoiding the technical complications of graviton propagators, is through the contributions of matter loops to the beta functions of the gravitational couplings. Thus, consider gravity coupled to n_S scalar fields, n_D Dirac fields, n_M gauge (Maxwell) fields, all massless and minimally coupled. A priori, nothing is assumed about the gravitational action. For each type of field ϕ_A we choose the cutoff function in such a way that $P_k(\Delta^{(A)}) = \Delta^{(A)} + R_k(\Delta^{(A)})$, where $\Delta^{(S)} = -\nabla^2$ on scalars, $\Delta^{(D)} = -\nabla^2 + \frac{R}{4}$ on Dirac fields and $\Delta^{(M)} = -\nabla^2\delta^\mu_\nu + R^\mu{}_\nu$ on Maxwell fields in the gauge $\alpha = 1$. Then, the ERGE is simply

$$\partial_t\Gamma_k = \sum_{A=S,D,M}\frac{n_A}{2}\mathrm{STr}_{(A)}\left(\frac{\partial_t P_k}{P_k}\right) - n_M \mathrm{Tr}_{(S)}\left(\frac{\partial_t P_k}{P_k}\right) , \qquad (1.4.3)$$

where $\mathrm{STr} = \pm\mathrm{Tr}$ depending on the statistics, and the last term comes from the ghosts. Using integral transforms and the heat kernel expansion, the trace of a function f of Δ can be expanded as

$$\mathrm{Tr} f(\Delta) = \sum_{n=0}^{\infty} Q_{2-n}(f) B_{2n}(\Delta) \tag{1.4.4}$$

where the heat kernel coefficients $B_{2n}(\Delta)$ are linear combinations of the $\mathcal{O}_i^{(n)}$, $Q_n(f) = (-1)^n f^{(n)}(0)$ for $n \leq 0$ and $Q_n(f)$ are given by Mellin transforms of f for $n > 0$ †. In this way one can write out explicitly the r.h.s. of (1.4.3) in terms of the $\mathcal{O}_i^{(n)}$ and read off the beta functions.

When $N \to \infty$, this is the dominant contribution to the gravitational beta functions, and graviton loops can be neglected (Tomboulis (1977), Smolin (1982), Percacci (2005)). The functions $a_i^{(n)}$ defined in (1.2.5) become numbers; with the so–called optimized cutoff function $R_k(z) = (k^2 - z)\theta(k^2 - z)$, discussed in Litim (2001, 2004), they are

$$a^{(0)} = \frac{n_S - 4n_D + 2n_M}{32\pi^2} , \qquad a^{(1)} = \frac{n_S - 2n_D - 4n_M}{96\pi^2} ,$$
$$a_1^{(2)} = \frac{6n_S + 36n_D + 72n_M}{11520\pi^2} , \qquad a_2^{(2)} = \frac{10n_S}{11520\pi^2} ,$$

while $a_i^{(n)} = 0$ for $n \geq 3$. The beta functions (1.2.5) are then

$$\partial_t \tilde{g}_i^{(n)} = (2n - 4)\tilde{g}_i^{(n)} + a_i^{(n)} . \tag{1.4.5}$$

For $n \neq 2$ this leads to a FP

$$\tilde{g}_{i*}^{(n)} = \frac{a_i^{(n)}}{4 - 2n} , \tag{1.4.6}$$

in particular we get

$$\tilde{\Lambda}_* = -\frac{3}{4}\frac{n_S - 4n_D + 2n_M}{n_S - 2n_D - 4n_M} , \quad \tilde{G}_* = \frac{12\pi}{-n_S + 2n_D + 4n_M} . \tag{1.4.7}$$

For $n = 2$, one gets instead $g_i^{(2)}(k) = g_i^{(2)}(k_0) + a_i^{(2)} \ln(k/k_0)$, implying asymptotic freedom for the couplings λ and ξ of (1.3.2). Remarkably, with this cutoff all the higher terms are zero at the FP. The critical exponents are equal to the canonical dimensions of the $g^{(n)}$'s, so Λ and G are UV–relevant (attractive), λ and ξ are marginal and all the higher terms are UV–irrelevant. Note that in perturbation theory G would be UV–irrelevant (nonrenormalizable). At the nontrivial FP the quantum corrections conspire with the classical dimensions of Λ and G to reconstruct the dimensions of $g^{(0)}$ and $g^{(1)}$. This does not happen at the Gaußian FP, where the transformation between $\tilde{G}$ and $\tilde{g}^{(1)}$ is singular.

† This technique is used also in some noncommutative geometry models, see Chamseddine & Connes (1996).

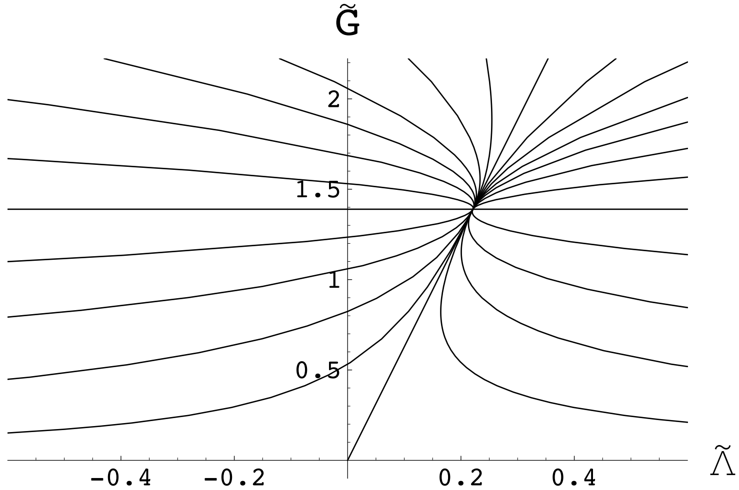

Fig. 1.1. The flow in the upper $\tilde{\Lambda}$–$\tilde{G}$ plane for pure gravity with higher derivative terms at one loop, eq.(1.4.8). All other couplings are set to zero. The nontrivial FP at (0.221,1.389) is UV–attractive with eigenvalues $(-4,-2)$, the one in the origin is UV–attractive along the $\tilde{\Lambda}$ axis with eigenvalue -2 and repulsive in the direction of the vector $(1/2\pi, 1)$ with eigenvalue 2.

Using the same techniques, the one loop beta functions for gravity with the action (1.3.2) have been calculated by Codello & Percacci (2006). The beta functions for λ and ξ agree with those derived in the earlier literature on higher derivative gravity (Fradkin & Tseytlin (1982), Avramidy & Barvinsky (1985), de Berredo–Peixoto & Shapiro (2005)). These couplings tend logarithmically to zero with a fixed ratio $\omega = -3\lambda/\xi \to \omega_* = -0.023$. The beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ differ from the ones that were given in the earlier literature essentially by the first two terms of the expansion (1.4.4). In a conventional calculation of the effective action these terms would correspond to quartic and quadratic divergences, which are normally neglected in dimensional regularization, but are crucial in generating a nontrivial FP. Setting the dimensionless couplings to their FP–values, one obtains:

$$\beta_{\tilde{\Lambda}} = 2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_*\tilde{G}\tilde{\Lambda} \ , \qquad \beta_{\tilde{G}} = 2\tilde{G} - q_*\tilde{G}^2 \ . \tag{1.4.8}$$

where $q_* \approx 1.440$. This flow is qualitatively identical to the flow in the $N \to \infty$ limit, and is shown in fig.1.

In order to appreciate the full nonperturbative content of the ERGE,

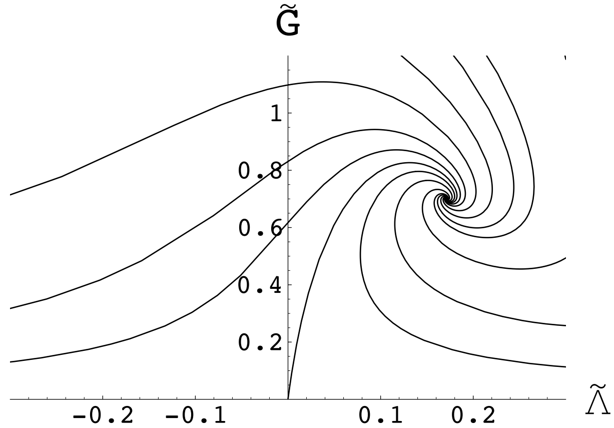

Fig. 1.2. The flow in the Einstein–Hilbert truncation, see Eq.(1.4.9-10). The nontrivial FP at $\tilde{\Lambda} = 0.171$, $\tilde{G} = 0.701$ is UV–attractive with eigenvalues $-1.69 \pm 2.49i$. The Gaußian FP is attractive along the $\tilde{\Lambda}$–axis with eigenvalue -2 and repulsive in the direction $(0.04, 1.00)$ with eigenvalue 2.

let us consider pure gravity in the Einstein–Hilbert truncation, *i.e.* neglecting terms with $n \geq 2$. In a suitable gauge the operator $\frac{\delta^2\Gamma_k}{\delta g_{\mu\nu}\delta g_{\rho\sigma}}$ is a function of $-\nabla^2$ only. Then, rather than taking as Δ the whole linearized wave operator, as we did before, we use (1.4.4) with $\Delta = -\nabla^2$. In this way we retain explicitly the dependence on Λ and R. Using the optimized cutoff, with gauge parameter $1/\alpha = Z$, the ERGE gives

$$\beta_{\tilde{\Lambda}} = \frac{-2(1-2\tilde{\Lambda})^2\tilde{\Lambda} + \frac{36-41\tilde{\Lambda}+42\tilde{\Lambda}^2-600\tilde{\Lambda}^3}{72\pi}\tilde{G} + \frac{467-572\tilde{\Lambda}}{288\pi^2}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}} \tag{1.4.9}$$

$$\beta_{\tilde{G}} = \frac{2(1-2\tilde{\Lambda})^2\tilde{G} - \frac{373-654\tilde{\Lambda}+600\tilde{\Lambda}^2}{72\pi}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}} \tag{1.4.10}$$

This flow is shown in Figure 2.

Lauscher & Reuter (2002a), Reuter & Saueressig (2002) have studied the gauge– and cutoff–dependence of the FP in the Einstein–Hilbert truncation. The dimensionless quantity $\Lambda' = \Lambda G$ (the cosmological constant in Planck units) and the critical exponents have a reassuringly weak dependence on these parameters. This has been taken as a sign that the FP is not an artifact of the truncation. Lauscher & Reuter (2002b) have also studied the ERGE including a term R^2 in the truncation. They find that in the subspace of $\tilde{\mathcal{Q}}$ spanned by $\tilde{\Lambda}, \tilde{G}, 1/\xi$, the non–Gaußian FP is very close to the one of the Einstein–Hilbert truncation, and is UV–attractive in all three directions. More recently, the FP has been shown to exist if the Lagrangian density is a polynomial in R of order up to six (Codello, Percacci and Rahmede (2007)). In this truncation the UV critical surface is three dimensional.

There have been also other generalizations. Niedermaier (2003) considered the RG flow for dimensionally reduced $d = 4$ gravity, under the hypothesis of the existence of two Killing vectors. This subsector of the theory is parametrized by infinitely many couplings, and has been proved to be asymptotically safe.

Matter couplings have been considered by Percacci & Perini (2003a,b). Consider the general action

$$\Gamma_k(g_{\mu\nu}, \phi) = \int d^4x \sqrt{g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi^2) + F(\phi^2) R \right) , \tag{1.4.11}$$

where V and F are arbitrary functions of ϕ^2, analytic at $\phi^2 = 0$. This action has a so-called Gaußian–Matter FP, meaning that only the coefficients of the ϕ-independent terms in (1.4.11) (namely $g^{(0)}$ and $g^{(1)}$) are nonzero. The critical surface has dimension four and there are no marginal operators. In the presence of other, minimally coupled matter fields, the dimension of the critical surface can be larger, and it is easy to find theories where a polynomial potential in ϕ is renormalizable and asymptotically free. Thus, gravity seems to provide a solution to the so–called triviality problem of scalar field theory.

It is tempting to speculate with Fradkin & Tseytlin (1982) that in the presence of gravity all matter interactions are asymptotically free. One loop calculations reported in Buchbinder *et al.* (1992), Robinson & Wilczek (2005) indicate that this may be the case also for gauge and Yukawa interactions. Then, in studying the FP, it would be consistent to neglect matter interactions, as we did in the $1/N$ expansion. If this is the case, it may become possible to show asymptotic safety for realistic unified theories including gravity and the SM.

For the time being, the gravitational FP has been found with a number of different approximations: the $2 + \epsilon$ expansion, the $1/N$ expansion, polynomial truncations with a variety of cutoffs and gauges, the two Killing vector reduction and the most general four–derivative gravity theory at one loop. The fact that all these methods yield broadly consistent results should leave little doubt about the existence of a nontrivial FP with the desired properties.

1.5 Other approaches and applications

In this final section we briefly comment on the relation of asymptotic safety to other approaches and results in quantum gravity.

Gravity with the Einstein–Hilbert action has been shown by Goroff & Sagnotti (1986) and van de Ven (1992) to be perturbatively nonrenormalizable at two loops. Stelle (1977) proved that the theory with action (1.3.2) and $\Lambda = 0$ is perturbatively renormalizable: all divergences can be absorbed into redefinitions of the couplings. In general, asymptotic safety does not imply that in the UV limit only a finite number of terms in (1.3.1) survive: there could be infinitely many terms, but there would be relations between their coefficients in such a way that only a finite number of parameters would be left free. At one loop or in the large–N limit, the ERGE predicts that the UV critical surface can be parametrized by the four couplings $\tilde{\Lambda}$, $\tilde{G}$, λ and ξ, the first two being nonzero at the FP and UV–relevant, the latter two being asymptotically free and marginal. Thus, at least in some approximations, asymptotic safety implies that near the FP quantum corrections to the action (1.3.2) will not generate new terms when one takes the UV limit. This is very similar to the result of Stelle. The main difference lies therein, that the perturbative proof holds at the Gaußian FP while the statement of asymptotic safety holds near the non–Gaußian one. According to the ERGE, the Gaußian FP is unstable, and moving by an infinitesimal amount towards positive $\tilde{G}$ (even with $\tilde{\Lambda} = 0$) would cause the system to be dragged in the direction of the repulsive eigenvector towards the non-Gaußian FP (see fig.1). It is unclear whether in a more accurate description it will still be possible to describe the UV limit of the theory by an action containing finitely many terms.

We now come to other nonperturbative approaches to quantum gravity. Monte Carlo simulations of quantum gravity have found evidence of a phase transition which can be related to the existence of a gravitational FP. Hamber & Williams (2004) review various results and arguments, mainly from quantum Regge calculus, supporting the claim that the mass critical exponent ν is equal to 1/3. In a theory with a single coupling constant $\tilde{G}$ we have $-1/\nu = \beta'_{\tilde{G}}(\tilde{G}_*)$, so for a rough comparison we can solve (1.4.10) with $\tilde{\Lambda} = 0$, finding a FP at $\tilde{G}_* = 1.21$ with $\beta'(\tilde{G}_*) \approx -2.37$. The agreement is numerically not very good for a universal quantity, but it might perhaps be improved by taking into account the flow of the cosmological constant.

In the so–called causal dynamical triangulation approach, recent numerical simulations have produced quantum worlds that exhibit several features of macroscopic four–dimensional spacetimes (see Ambjørn, Jurkiewicz and Loll's contribution to this volume). In particular they have also studied diffusion processes in such quantum spacetimes and found that the spectral dimension characterizing them is close to two for short diffusion times and to four for long diffusion times. This agrees with the expectation from asymptotic safety and can be seen as further independent evidence for a gravitational FP, as we shall mention below.

The physical implications of a gravitational FP and, more generally, of the running of gravitational couplings, are not yet well understood. First and foremost, one would expect asymptotic safety to lead to new insight into the local, short–distance structure of a region of spacetime. The boundedness of the cutoff in Planck units, derived in section 1.3, would be in accord with the widely held expectation of some kind of discrete spacetime structure at a fundamental level. In particular, it may help understand the connection to theories such as loop quantum gravity, which predict that areas and volumes have quantized values. However, the discussion in section 1.3 should make it clear that the issue of a minimal length in quantum gravity may have limited physical relevance, since the answer depends on the choice of units.

Another point that seems to emerge is that the spacetime geometry cannot be understood in terms of a single metric: rather, there will be a different effective metric at each momentum scale. This had been suggested by Floreanini & Percacci (1995a,b), who calculated the scale dependence of the metric using an effective potential for the conformal factor. Such a potential will be present in the effective action Γ_k before the background metric is identified with the classical metric (as mentioned in section 1.2). A scale dependence of the metric has also been postulated by Magueijo & Smolin (2004) in view of possible phenomenological effects. Lauscher & Reuter (2005) have suggested the following picture of a fractal spacetime. Dimensional analysis implies that in the FP regime $\langle g_{\mu\nu}\rangle_k = k^{-2}(\tilde{g}_0)_{\mu\nu}$, where $\tilde{g}_0$, defined as in (1.3.4), is a fiducial dimensionless metric that solves the equations of motion of Γ_{k_0}. For example, in the Einstein–Hilbert truncation, the effective metric $\langle g_{\mu\nu}\rangle_k$ is a solution of the equation $R_{\mu\nu} = \Lambda_k g_{\mu\nu}$, so

$$\langle g_{\mu\nu}\rangle_k = \frac{\Lambda_{k_0}}{\Lambda_k}\langle g_{\mu\nu}\rangle_{k_0} \approx \left(\frac{k_0}{k}\right)^2 \langle g_{\mu\nu}\rangle_{k_0} = k^{-2}(\tilde{g}_0)_{\mu\nu} \ , \tag{1.5.1}$$

where $\approx$ means "in the FP regime". The fractal spacetime is described by the collection of all these metrics.

A phenomenon characterized by an energy scale k will "see" the effective metric $\langle g_{\mu\nu}\rangle_k$. For a (generally off-shell) free particle with four–momentum p_μ it is natural to use $k \propto p$, where $p = \sqrt{(\tilde{g}_0)^{\mu\nu} p_\mu p_\nu}$. Its inverse propagator is then $\langle g^{\mu\nu}\rangle_p p_\mu p_\nu$. At low energy $\langle g_{\mu\nu}\rangle_k$ does not depend on k and the propagator has the usual p^{-2} behaviour; in the FP regime, (1.5.1) implies instead that it is proportional to p^{-4}. Its Fourier transform has a short–distance logarithmic behaviour which is characteristic of two dimensions, and agrees with the aforementioned numerical results on the spectral dimension in causal dynamical triangulations. This agreement is encouraging, because it suggests that the two approaches are really describing the same physics. When applied to gravitons in four dimensions (and only in four dimensions!) it also agrees with the general prediction, derived in the end of section 1.3, that $\eta_g = 2$ at a nontrivial gravitational FP.

The presence of higher derivative terms in the FP action raises the old issue of unitarity: as is well–known, the action (1.3.2) describes, besides a massless graviton, also particles with Planck mass and negative residue (ghosts). From a Wilsonian perspective, this is clearly not very significant: to establish the presence of a propagator pole at the mass m_P one should consider the effective action Γ_k for $k \approx m_P$, which may be quite different from the FP action. Something of this sort is known to happen in the theory of strong interactions: at high energy they are described by a renormalizable and asymptotically free theory (QCD), whose action near the UV (Gaußian) FP describes quarks and gluons. Still, none of these particles appears in the physical spectrum.

As in QCD, matching the UV description to low energy phenomena may turn out to be a highly nontrivial issue. A change of degrees of freedom could be involved. From this point of view one should not assume *a priori* that the metric appearing in the FP action is "the same" metric that appears in the low energy description of GR. Aside from a field rescaling, as discussed in section 1.2, a more complicated functional field redefinition may be necessary, perhaps involving the matter fields, as exemplified in Tomboulis (1996). Unless at some scale the theory was purely topological, it will always involve a metric and from general covariance arguments it will almost unavoidably contain an Einstein–Hilbert term. This explains why the Einstein–Hilbert action, which describes GR at macroscopic distances, may play an important role also in the UV limit, as the results of section 1.4 indicate. With this in mind, one can explore the consequences of a RG running of gravitational couplings also in other regimes.

Motivated in part by possible applications to the hierarchy problem, Percacci (2007) considered a theory with an action of the form (1.4.11), in the intermediate regime between the scalar mass and the Planck mass. Working in cutoff units (1.3.4), it was shown that the warped geometry of the Randall–Sundrum model can be seen as a geometrical manifestation of the quadratic running of the mass.

For applications to black hole physics, Bonanno & Reuter (2000) have included quantum gravity effects by substituting G with $G(k)$ in the Schwarzschild metric, where $k = 1/r$ and r is the proper distance from the origin. This is a gravitational analog of the Ühling approximation of QED. There is a softening of the singularity at $r = 0$, and it is predicted that the Hawking temperature goes to zero for Planck mass black holes, so that the evaporation stops at that point.

In a cosmological context, it would be natural to identify the scale k with a function of the cosmic time. Then, in order to take into account the RG evolution of the couplings, Newton's constant and the cosmological constant can be replaced in Friedman's equations by the effective Newton's constant and the effective cosmological constant calculated from the RG flow. With the identification $k = 1/t$, where t is cosmic time, Bonanno & Reuter (2002) have applied this idea to the Planck era, finding significant modifications to the cosmological evolution; a more complete picture extending over all of cosmic history has been given in Reuter & Saueressig (2005). It has also been suggested that an RG running of gravitational couplings may be responsible for several astrophysical or cosmological effects. There is clearly scope for various interesting speculations, which may even become testable against new cosmological data.

Returning to the UV limit, it can be said that asymptotic safety has so far received relatively little attention, when compared to other approaches to quantum gravity. Establishing this property is obviously only the first step: deriving testable consequences is equally important and may prove an even greater challenge. Ultimately, one may hope that asymptotic safety will play a similar role in the development of a QFT of gravity as asymptotic freedom played in the development of QCD.

1.6 Acknowledgements

I wish to thank R. Floreanini, D. Dou, D. Perini, A. Codello and C. Rahmede for past and present collaborations, and M. Reuter for many discussions.

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Questions and answers

Q: Could an asymptotically safe theory be regarded as an approximation to another more fundamental theory, or does it have to be regarded as a self-contained fundamental theory?

A: The asymptotic safety programme is very closely related to the formalism of effective field theories and both possibilities can be envisaged. If a fixed point with the desired properties did exist, then mathematically it would be possible to take the limit $k \to \infty$ and one could call this a fundamental theory. It would do for gravity what the Weinberg–Salam model originally did for electroweak interactions. However, experience shows that today's fundamental theory may become tomorrow' effective theory. The renormalizability of the Weinberg–Salam model was important in establishing it as a viable theory but nowadays this model is widely regarded as an effective theory whose nonrenormalizable couplings are suppressed by powers of momentum over some cutoff. In a distant future, the same could happen to an asymptotically safe theory of gravity.

To understand this point better, notice that in order to hit the fixed point as $k \to \infty$, one would have to place the initial point of the flow in the critical surface with "infinite precision". In the case of the standard model, where the use of perturbative methods is justified, this corresponds to setting all couplings with negative mass dimension *exactly* equal to zero. Even assuming that the property of asymptotic safety could be firmly established theoretically, because measurements are always imprecise, it is hard to see how one could ever establish experimentally that the world is described by such a theory. One could say at most that experiments are compatible with the theory being fundamental.

On the other hand suppose that the theory requires drastic modification at an energy scale of, say, a billion Planck masses, perhaps because of the existence of some presently unknown interaction. Then at the Planck scale one would expect the dimensionless couplings of the theory ($\tilde{g}_i$) to lie off the critical surface by an amount of the order of some power of one in a billion. Suppose we follow the flow in the direction of decreasing energies starting from a scale which is much larger than one, and much less than a billion Planck masses. Since the fixed point is IR–attractive in all directions except the ones in the critical surface, starting from a generic point in the space of coupling constants, the theory will be drawn quickly towards the critical surface. Going towards the infrared, the flow at sub–Planckian scales will then look as if it had originated from the fixed point, up to small deviations from the critical surface which may be hard or impossible to measure.

Thus, the formalism can accommodate both effective and fundamental theories of gravity. The most important point is that asymptotic safety would allow us to push QFT beyond the Planck scale, up to the next frontier, wherever that may be.

Q: What is your take on the issue of continuum versus discrete picture of spacetime, coming from a renormalization group perspective? If gravity is asymptotically safe, would it imply that a continuum description of spacetime is applicable at all scales, or one can envisage a role of discrete spacetime structures even in this case? How would a breakdown of the continuum description show up in the ERG approach?

A: First of all it should be said that the renormalization group can be realized both in continuum and discrete formulations and is likely to play a role in quantum gravity in either case. It should describe the transition from physics at the "lattice" or UV cutoff scale down to low energies.

Then, one has to bear in mind that when one formulates a quantum field theory in the continuum but with a cutoff Λ, it is impossible to resolve points closer that $1/\Lambda$, so the continuum should be regarded as a convenient kinematical framework that is devoid of physical reality. If the asymptotic safety program could be carried through literally as described, it would provide a consistent description of physics down to arbitrarily short length scales, and in this sense the continuum would become, at least theoretically, a reality.

Of course, it would be impossible to establish experimentally the continuity of spacetime in the mathematical sense, so this is not a well–posed physical question. What is in principle a meaningful physical question, and may become answerable sometimes in the future, is whether spacetime is continuous down to, say, one tenth of the Planck length. But even then, the answer may require further qualification. Recall that in order to define a distance one has to specify a unit of lengths. Units can ultimately be traced to some combination of the couplings appearing in the action. For example, in Planck units one takes the square root of Newton's constant as a unit of length. Because the couplings run, when the cutoff is sent to infinity the distance between two given points could go to zero, to a finite limit or to infinity depending on the asymptotic behaviour of the unit. In principle it seems possible that spacetime looks discrete in certain units and continuous in others. Then, even if asymptotic safety was correct, it need not be in conflict with models where spacetime is discrete.

Q: What differences, in formalism and results, can one expect in the ERG approach, if one adopts a 1st order (e.g. Palatini) or BF-like (e.g. Plebanski) description of gravity?

A: Writing the connection as the sum of the Levi–Civita connection and a three-index tensor Φ, one can always decompose an action for independent connection and metric into the same action written for the Levi-Civita connection, plus terms involving Φ. The effects due to Φ will be similar to those of a matter field. In the case when the action is linear in curvature, and possibly quadratic in torsion and nonmetricity, up to a surface term the action for Φ is just a mass term, implying that Φ vanishes on shell. In this case one expects the flow to be essentially equivalent to that obtained in the Einstein-Hilbert truncation plus some matter fields, although this has not been explicitly checked yet. The presence of a mass for Φ of the order of the Planck mass suggests that a decoupling theorem is at work and that Φ (or equivalently the connection) will become propagating degrees of freedom at the Planck scale. This is indeed the case when the action involves terms quadratic in curvature (which can be neglected at low energies). Then the field Φ propagates, and has quartic self-interactions. There will be new couplings, that may influence the running of Newton's constant, for example. But again, this should be equivalent to fourth-order gravity plus matter.

Q: You mention that the results of the ERG seem to point out that spacetime structure cannot be described in terms of a single metric for any momentum scale. How would one notice, in the RG approach, that it cannot be described by a metric field at all, but that a description in terms of connections or even a non-local one would be more appropriate, say, at the Planck scale?

A: I do in fact expect that an independent connection will manifest itself at the Planck scale, as I have indicated in my answer to another question, though I don't think that this will be forced upon us by the ERG.

The scale-dependence of the metric could manifest itself as violations of the equivalence principle, or perhaps as Lorentz-invariance violations or deformations of the Lorentz group. There is much work to be done to understand this type of phenomenology. Even more radically, it is possible that gravity is just the "low energy" manifestation of some completely different physics, as suggested in the article by Dreyer. This would probably imply a failure of the asymptotic safety programme, for example a failure to find a fixed point when certain couplings are considered.

Q: Can you please comment on possibility of extending the ERG approach to the Lorentzian signature or to the case of dynamical space topology?

A: So far the ERG has been applied to gravity in conjunction with the background field method. Calculations are often performed in a convenient background, such as (Euclidean) de Sitter space, but the beta functions obtained in this way are then completely general and independent of the background metric and spacetime topology. The choice of a background is merely a calculational trick. It is assumed that the beta functions are also independent of the signature of the background metric, although this point may require further justification. One should also stress in this connection that the use of the background field method and of the background field gauge does not make this a "background-dependent" approach. On the contrary, when properly implemented it guarantees that the results are background-independent.