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Super coset space geometry

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Abstract

Super coset spaces play an important role in the formulation of supersymmetric theories. The aim of this paper is to review and discuss the geometry of super coset spaces with particular focus on the way the geometrical structures of the super coset space G/H are inherited from the super Lie group G. The isometries of the super coset space are discussed and a definition of Killing supervectors – the supervectors associated with infinitesimal isometries – is given that can be easily extended to spaces other than coset spaces.

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1 Introduction

Coset spaces are widely used in a variety of contexts, for example as target spaces within string theory or more generally within non-linear sigma models. For supersymmetric theories of this kind, a thorough understanding of the geometry of super coset spaces is therefore essential. On the other hand, supersymmetric theories with coset spaces as the base naturally have a superspace formulation in terms of super coset spaces, the most prominent examples being supersymmetric theories in flat space. These can be formulated in terms of flat superspace, which is the quotient of the super Poincaré group by its Lorentz subgroup. We aim to provide in this paper a firm mathematical basis for the geometry of super coset spaces, collating and making rigorous results often only sketched in the literature and extending results only discussed for ordinary coset spaces to the supersymmetric case.

The geometry of a (super) coset space G/H, i.e. its frame and connection, can be determined in terms of the geometry of G. In fact, it is well known that by pulling back the Maurer-Cartan one-form on the group G to the base G/H one can obtain the frame and connection on the base, see e.g. [1, 2]. While this is often stated in the literature a geometric explanation of this is usually omitted. Treating the super Lie group G as a principal bundle over G/H we give

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a self-contained account of the geometry of coset spaces focussing in particular on the relation between the geometry in the bundle and the geometry in the base.

Coset spaces are characterized by high symmetry; most of their isometries can be derived from the left action of G on the coset space and as such the isometry group of G/H contains G. On the other hand, the isometries of the coset space can be determined directly from its geometry: In the case of ordinary space the infinitesimal isometries, i.e. the Killing vector fields, are defined as the directions along which the metric tensor is dragged into itself. In the case of superspace – where the tangent space group of the supermanifold under consideration must correspond to the even Grassmann extension of the tangent space group of the body of the supermanifold, i.e. to $SO(p, q)$ – such a definition is no longer feasible. This is as, in this case, there exists no physically natural superspace generalization of the concept of metric [3, 4]: If one were to introduce a supermetric the tangent space group of the supermanifold derived from this would be too large, and hence would not correspond to the even Grassmann extension of $SO(p, q)$, thus rendering the theory unphysical. While this problem is often noted in the literature its relevance to the definition of Killing supervectors is seldom elucidated. One aim of this paper is therefore to make rigorous and clarify the notion of Killing supervector fields in the context of superspace. As we shall see it is possible to define Killing supervector fields as those infinitesimal transformations that leave the frame and connection invariant up to a gauge transformation, see also [3]. This condition can be rephrased in terms of the commutator of some generalized Lie derivative and the covariant derivative, cf. [3]. In the case of super coset spaces we shall see that this definition reproduces the Killing supervectors derived from the left action of the group on the super coset space which justifies this definition also for more general spaces.

Derivations such as the covariant derivative and generalized Lie derivative are an integral part of the geometry of coset spaces. We give a geometric interpretation to derivations such as these on G/H by defining them in terms of maps on the bundle G and associated bundles.

The organization of this paper is as follows: First we briefly review the construction of G as a principal bundle over the coset space G/H. We then discuss the geometry of G focussing in particular on its invariant vector fields and the Maurer-Cartan form in the bundle. In Sections 4–6 we discuss in detail how the geometry in the base, i.e. connection and frame, torsion and curvature, can be obtained from the geometry in the bundle. In Section 7 we introduce associated bundles and briefly review some aspects of supertensor bundles that will be important when discussing derivations on associated bundles. We then introduce those (local) bundle maps that will allow us to define in Section 10 derivations on supertensor bundles, such as the covariant derivative, the Lie derivative and the so-called H-covariant Lie derivative. As we shall see in Section 11, Killing supervectors, i.e. the supervectors associated to infinitesimal isometries, can be defined in terms of a so-called generalized Lie derivative which combines an arbitrary transformation in the base with an arbitrary gauge transformation in the bundle, and which we require to commute with the covariant derivative.

Finally, in Section 12, we shall apply the concepts and methods introduced in the previous sections to flat superspace as an example.

2 G as a principal bundle over G/H

Consider a super Lie group G with super Lie subgroup H. A super Lie group shall be defined in the sense of DeWitt, see [5], as a group which is also a supermanifold and which has a differentiable group multiplication. We define the super coset space G/H via the equivalence of group elements in G under right multiplication by an element of H,

$$G/H = \{gH : g \in G\}. \tag{2.1}$$

The coset space G/H naturally inherits a supermanifold structure from the supermanifold G [5]. The geometry of the coset space G/H is also inherited from G. To study this we consider G as a principal bundle over G/H with fibre H. The construction in the non-supersymmetric case [6, 7] is directly transferred to the supersymmetric case. First we have the bundle projection map

$$\begin{aligned} \pi &: G \to G/H \\ &: g \mapsto gH. \end{aligned} \tag{2.2}$$

The inverse image $\pi^{-1}(p)$ of a point p in the base gives us the fibre above that point which is clearly isomorphic to H. To define the local trivializations of the bundle we consider charts $U_i \subset G/H$ on the base. Within a particular chart it is always possible to choose a local section. By a local section we mean a map $L_i : U_i \to \pi^{-1}(U_i)$ which satisfies $\pi \circ L_i = \mathrm{id}_{U_i}$. The local section L_i provides us with a coset representative $L_i(p)$ for any $p \in G/H$. Using this local section we define the canonical local trivialization for the bundle

$$\begin{aligned} \phi_i &: U_i \times H \to \pi^{-1}(U_i) \\ &: (p, h) \mapsto L_i(p)h. \end{aligned} \tag{2.3}$$

Note that the inverse is easily constructed as $\phi_i^{-1}(g) = (\pi(g), L_i(\pi(g))^{-1}g)$. The transition functions for $U_i \cap U_j \neq \varnothing$ are defined as

$$t_{ij}(p) = \phi_{i,p}^{-1}\phi_{j,p} : H \to H, \tag{2.4}$$

where $\phi_{i,p}(h) \equiv \phi_i(p, h)$. The map $t_{ij}(p)$ is clearly just left multiplication by an element of H, and we shall use the notation $t_{ij}(p)(h) = t_{ij}(p)h$. The trivializations are then related as

$$\phi_j(p, h) = \phi_i(p, t_{ij}(p)h). \tag{2.5}$$

The structure group is therefore H and acts on the fibre H by left multiplication. We thus have that G is a principal bundle over G/H with structure group H. We denote this as $G = P(G/H, H)$.

3 Group geometry

3.1 Invariant vector fields

Let us denote the super Lie algebra associated to the super Lie group G by $\mathfrak{g}$. The generators of $\mathfrak{g}$ will be denoted by T_p, $p = 1, \ldots, \dim \mathfrak{g}$; they have definite parity, either even or odd and we will set $(-1)^p = 1$ for T_p even and $(-1)^p = -1$ for T_p odd. The index in the exponent of (-1) is as such to be understood as taking the values 0 or 1 according to whether it is even or odd. A general element of the super Lie algebra is then expanded in the generators as $X = X^p T_p$, where the X^p are pure supernumbers chosen such that X is even. As such the super Lie algebra consists of even elements only, in fact $\mathfrak{g}$ can be viewed as the even part of a larger Berezin superalgebra [3]. The super Lie group G can then be obtained from its super Lie algebra via the exponential mapping.

For each element of the algebra $A \in \mathfrak{g}$ we can construct two different supervector fields, $A^\sharp$ and $A^\flat$, defined by their action on a function f on G as

$$A^\sharp\big|_g[f] \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(f\left(ge^{tA}\right)\right)\Big|_{t=0} \tag{3.1a}$$

$$A^\flat\big|_g[f] \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(f\left(e^{tA}g\right)\right)\Big|_{t=0}. \tag{3.1b}$$

We have thus obtained two maps from the algebra $\mathfrak{g}$ to the space of vector fields on G given by $\sharp : A \mapsto A^\sharp$ and $\flat : A \mapsto A^\flat$. Clearly these maps are linear.

The definition of the vector fields given above may be modified in the case that A is an odd element of the Berezin superalgebra simply by choosing the parameter t to be an odd supernumber. This ensures that e^{tA} is still a group element. This way it possible to define the supervectors ${T_p}^\sharp$ and ${T_p}^\flat$ for all p.

It is easy to see that $A^\sharp$ is a left-invariant vector field, whereas $A^\flat$ is right-invariant:

$$L_{g_1*}\left(A^\sharp\big|_g\right) = A^\sharp\big|_{g_1 g} \tag{3.2a}$$

$$R_{g_1*}\left(A^\flat\big|_g\right) = A^\flat\big|_{g g_1}. \tag{3.2b}$$

Here L_g and R_g are the group operations of left and right multiplication by $g \in G$, the lowered asterisk $(*)$ is used to denote the corresponding induced map on vector fields (the pushforward). The vector fields also satisfy

$$R_{g_1*}\left(A^\sharp\big|_g\right) = \left(\mathrm{Ad}_{g_1^{-1}} A\right)^\sharp\big|_{g g_1} \tag{3.3a}$$

$$L_{g_1*}\left(A^\flat\big|_g\right) = \left(\mathrm{Ad}_{g_1} A\right)^\flat\big|_{g_1 g}. \tag{3.3b}$$

Here Ad_g is the adjoint action of the group on its algebra which is induced from the adjoint action of the group on itself. For the latter we also use the notation Ad_g. We have $\mathrm{Ad}_g h = ghg^{-1}$ and thus, in a matrix representation, the adjoint action on the algebra is thus just $\mathrm{Ad}_g A = gAg^{-1}$.

Under the Lie bracket of supervector fields we find

$$[A^\sharp, B^\sharp] = [A, B]^\sharp \tag{3.4a}$$
$$[A^\flat, B^\flat] = -[A, B]^\flat \tag{3.4b}$$
$$[A^\sharp, B^\flat] = 0. \tag{3.4c}$$

The bracket occurring on the right is the super Lie algebra bracket.

3.2 The Maurer-Cartan one-form

The Maurer-Cartan one-form ζ is a super Lie algebra valued one-form defined on the super Lie group as

$$\zeta(A^\sharp) \stackrel{\text{def}}{=} A, \quad \forall A \in \mathfrak{g}. \tag{3.5}$$

Note that one should be careful when dealing with forms acting on vectors in the case of supersymmetry, our conventions are given in Appendix A.1.

The Maurer-Cartan one-form is clearly left-invariant, whereas under right translations it transforms as

$$R_g^* \zeta = \mathrm{Ad}_{g^{-1}} \zeta. \tag{3.6}$$

The Maurer-Cartan form can be shown to satisfy

$$\mathrm{d}\zeta + \zeta \wedge \zeta = 0, \tag{3.7}$$

which is the so-called *Maurer-Cartan structure equation*.

When working in a matrix representation of the group the Maurer-Cartan form may be represented as $\zeta = g^{-1}\mathrm{d}g$ [8]. Throughout the remainder of this paper we will occasionally work in a matrix representation where it is more convenient.

4 The connection on G/H

In this section we will construct a connection on the coset space G/H. We will see that this connection is naturally induced from the group G, in particular from the Maurer-Cartan form. A connection on a fibre bundle is defined quite abstractly in terms of a decomposition of the tangent space into so-called vertical and horizontal subspaces. While this definition may not be the definition of connection familiar from physics it will provide us with a deeper insight into the geometry. In Section 4.3 we will review how this abstract definition relates to the familiar notion of connection in physics.

4.1 Horizontal and vertical subspaces

Geometrically a connection on the bundle G can be thought of as a decomposition of the tangent space to the super Lie group, $T_gG = T_g^{\mathrm{v}}G \oplus T_g^{\mathrm{h}}G$, into vertical and horizontal subspaces, respectively. This decomposition must be supersmooth and the horizontal subspaces must satisfy

$$T^{\mathrm{h}}_{gh}G = R_{h*}T^{\mathrm{h}}_{g}G. \tag{4.1}$$

Such a decomposition of the tangent space can be achieved by utilizing the left-invariant vector fields and the natural decomposition of the algebra arising from the subgroup H.

Let us denote the super Lie algebra of the subgroup H by $\mathfrak{h}$. We then choose a subspace $\mathfrak{k}$ in $\mathfrak{g}$ complementary to $\mathfrak{h}$, i.e.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}. \tag{4.2}$$

The generators of the full algebra T_p, $p = 1, \dots, \dim \mathfrak{g}$, can then be split up into the generators of $\mathfrak{h}$, H_I, $I = 1, \dots \dim \mathfrak{h}$, and the remaining generators K_A, $A = 1, \dots \dim \mathfrak{k}$. The structure constants ${f_{pq}}^r$ of $\mathfrak{g}$ are then defined by

$$[H_I, H_J] = {f_{IJ}}^K H_K \tag{4.3a}$$
$$[H_I, K_A] = {f_{IA}}^J H_J + {f_{IA}}^B K_B \tag{4.3b}$$
$$[K_A, K_B] = {f_{AB}}^J H_J + {f_{AB}}^C K_C. \tag{4.3c}$$

The bracket here is the graded Lie bracket which satisfies the symmetry $[T_p, T_q] = -(-1)^{pq}[T_q, T_p]$.

If $\mathfrak{k}$ can be chosen such that the structure constants ${f_{IA}}^J$ vanish, i.e. $[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{k}$, then the group G is said to be *reductive*. As we shall see this is an important property and we shall assume that G is reductive throughout the remainder of this paper.

The decomposition of the algebra in Eq. (4.2) naturally gives a decomposition of the tangent space to the group into horizontal and vertical subspaces. We may use the generators of the algebra to define a basis of the vertical and horizontal subspaces. We take $\{{H_I}^\sharp\}$ to be a basis of $T^{\mathrm{v}}_g G$ and $\{{K_A}^\sharp\}$ to be a basis of $T^{\mathrm{h}}_g G$, i.e.

$$T^{\mathrm{v}}_g G \equiv \{X^I {H_I}^\sharp\big|_g : X^I \in \mathbb{R}_\infty\} \tag{4.4a}$$
$$T^{\mathrm{h}}_g G \equiv \{X^A {K_A}^\sharp\big|_g : X^A \in \mathbb{R}_\infty\}, \tag{4.4b}$$

where here $\mathbb{R}_\infty$ denotes the real supernumbers. This decomposition is clearly smooth and of the form $T_g G = T^{\mathrm{v}}_g G \oplus T^{\mathrm{h}}_g G$. From Eq. (3.3a) and the reductive property f the group it is easily seen that Eq. (4.1) is satisfied.

Given a notion of horizontal and vertical vectors, it is natural to define horizontal and vertical differential forms. A differential form on the bundle is called vertical (respectively horizontal) if it vanishes whenever one of the vectors on which it is evaluated is horizontal (respectively vertical). For example, expanding the Maurer-Cartan form, Eq. (3.5), in the algebra generators we have

$$\zeta = \zeta^A K_A + \zeta^I H_I. \tag{4.5}$$

It is then easy to see that

$$\begin{aligned} \zeta^A(K_B{}^\sharp) &= \delta_B{}^A \qquad & \zeta^A(H_J{}^\sharp) &= 0 \\ \zeta^I(K_B{}^\sharp) &= 0 \qquad & \zeta^I(H_J{}^\sharp) &= \delta_J{}^I. \end{aligned} \tag{4.6}$$

Hence ζ^A will vanish when acting on any vertical vector whereas ζ^I vanishes on any horizontal vector. ζ^A is therefore a horizontal form and ζ^I a vertical form. Eq. (4.5) can thus be thought of as a decomposition of the Maurer-Cartan form into its horizontal and vertical parts.

4.2 The connection one-form

It is usually more convenient to define a connection in terms of a *connection one-form*. This is a super Lie algebra valued one-form, Ω, on the bundle and it is required to satisfy

$$\Omega(H_I{}^\sharp) = H_I, \quad I = 1, \dots, \dim \mathfrak{h} \tag{4.7a}$$

$$R_h^* \Omega = \mathrm{Ad}_{h^{-1}} \Omega, \quad \forall h \in H. \tag{4.7b}$$

The horizontal subspace is then defined as the kernel of Ω, i.e.

$$T_g^{\mathrm{h}} G = \{X \in T_g G : \Omega(X) = 0\}. \tag{4.8}$$

We will now show that the vertical part of the Maurer-Cartan form, $\zeta^I H_I$, can be taken to be the connection one-form consistent with the definition of horizontal and vertical subspaces defined in Eqs. (4.7a, 4.7b). Firstly, as $\zeta^A K_A$ is horizontal, it follows immediately from Eq. (3.5) that

$$(\zeta^I H_I)(H_J{}^\sharp) = H_J, \quad J = 1, \dots, \dim \mathfrak{h}. \tag{4.9a}$$

We also find immediately from Eq. (3.6) that

$$R_h^*(\zeta^I H_I) = \mathrm{Ad}_{h^{-1}}(\zeta^I H_I), \quad \forall h \in H. \tag{4.9b}$$

These two equations can then be compared directly with Eqs. (4.7a, 4.7b). Also, as the form $\zeta^I H_I$ is vertical, its kernel is precisely the horizontal subspace. Thus we may choose the connection one-form as $\Omega = \zeta^I H_I$.

4.3 The local connection and parallel transport

The definition of a connection in terms of horizontal and vertical subspaces may seem a little abstract, and the connection one-form Ω is not the connection usually dealt with in physics. In this section we shall review the relation of this abstract definition of connection to the more familiar notion of connection in physics. To do this we shall consider parallel transport.

Given a curve $\gamma : [0,1] \to G/H$ in the base we define a *horizontal lift* of γ to be a curve $\tilde{\gamma} : [0,1] \to G$ in the bundle which satisfies $\pi \circ \tilde{\gamma} = \gamma$ and for which the tangent vector to $\tilde{\gamma}$ lies in $T^{\mathrm{h}}_{\tilde{\gamma}(t)} G$. For each point $g \in \pi^{-1}(\gamma(0))$ there is a unique horizontal lift of γ for which $\tilde{\gamma}(0) = g$. Further, for $h \in H$, the horizontal lift passing through gh is simply $\tilde{\gamma}(t)h$.

Consider a point $g_0 \in G$ above $\gamma(0)$, and construct the horizontal lift $\tilde{\gamma}$ of γ satisfying $\tilde{\gamma}(0) = g_0$. Then the value $g_1 = \tilde{\gamma}(1)$ is said to be the *parallel transport* of g_0 along γ.

Let us now analyze this construction as viewed from a local trivialization. Let us choose the local section[1] such that $L(p) = g_0$, we may then decompose the horizontal lift of γ as

$$\tilde{\gamma}(t) = L(\gamma(t))\tilde{h}(t), \tag{4.10}$$

where $\tilde{h}(0) = 1$. Then, if we let X be the tangent vector to γ and let $\tilde{X}$ be the tangent vector to $\tilde{\gamma}$, it is possible to show (see Appendix A.2) that

$$\tilde{X} = R_{\tilde{h}(t)_*}(L_* X) + (\tilde{h}(t)^{-1}\mathrm{d}\tilde{h}(X))^\sharp. \tag{4.11}$$

Note that this result is here written in a matrix representation. Also, d is the exterior derivative on the base G/H and should not be confused with the exterior derivative on the group G which we used in Section 3.2. Now, since $\tilde{X}$ is horizontal we have that $\Omega(\tilde{X}) = 0$, hence using Eqs. (4.7a, 4.7b) we find

$$\begin{aligned} 0 &= \mathrm{Ad}_{\tilde{h}(t)^{-1}}\Omega(L_* X) + \tilde{h}(t)^{-1}\mathrm{d}\tilde{h}(X) \\ &= \tilde{h}(t)^{-1}\left(L^*\Omega(X)\tilde{h}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}(t)\right). \end{aligned} \tag{4.12}$$

With this equation in mind we introduce the *local connection*, $\omega^{(L)}$, a super Lie algebra valued one-form in U, defined as

$$\omega^{(L)} \stackrel{\text{def}}{=} L^*\Omega. \tag{4.13}$$

Thus we see from Eq. (4.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}(t) = -\omega^{(L)}(X)\tilde{h}(t). \tag{4.14}$$

From this equation, or its formal solution in terms of the path ordered exponential, we see that the local connection is precisely the connection familiar to us from physics. This will become even more apparent in Section 10.1 when we work with associated bundles.

Note that $\omega^{(L)}$ clearly depends on the choice of local section. The choice of local section can be seen to be precisely what we would naturally call the choice of gauge. This view of gauge choice is one of the many nice aspects of working in the language of bundles. To see this in a little more detail let us consider the connection for two different choices of local section: $L(p)$ and $L'(p)$. The two local sections can clearly be related by a right multiplication, $L'(p) = L(p)h(p)$, with $h(p) \in H$ dependent on $p \in U$. It is possible to show (see Appendix A.2) that for a vector $X \in T_pU$ we have

$$L'_* X = R_{h*}(L_* X) + (h^{-1}\mathrm{d}h(X))^\sharp. \tag{4.15}$$

From this it follows directly that

$$\omega^{(L')}(X) = \omega^{(Lh)}(X) = h^{-1}\omega^{(L)}(X)h + h^{-1}\mathrm{d}h(X).$$

[1] We will drop subscript i's where we only need to consider one chart.

Since X is arbitrary we deduce

$$\omega^{(Lh)} = h^{-1}\omega^{(L)}h + h^{-1}\mathrm{d}h, \tag{4.16}$$

which is the usual transformation rule for a connection under the gauge transformation given by the local field $h(p) \in H$.

5 The frame and coframe on G/H

In the previous section we have discussed how the connection one-form on the bundle G can be taken to be the vertical part of the Maurer-Cartan one-form. When pulled back under a local section this gives us the usual connection on the base. It is then natural to ask what the horizontal part of the Maurer-Cartan form is associated with. We will show in this section that it is naturally associated with a frame.

Consider the pullback of the horizontal components of the Maurer-Cartan form. We define

$$E^A_{(L)} \stackrel{\mathrm{def}}{=} L^*\zeta^A. \tag{5.1}$$

Note that a coframe in U is defined as a set of $\dim(G/H)$ one-forms that are linearly independent at each $p \in U$. Since $A = 1, \ldots, \dim(G/H)$ there is clearly the right number of $E^A_{(L)}$; all that remains is to show pointwise linear independence. Consider a point $p \in U$. Suppose we introduce a set of supernumbers λ_A and impose that $\sum_A E^A_{(L)}|_p\lambda_A = 0$. Acting on an arbitrary vector $v \in T_pU$ and using the definition of pullback this gives

$$\sum_A (\zeta^A\lambda_A)(L_*v) = 0.$$

Now, since ζ_A is horizontal it vanishes when acting on any vertical vector $V_{\mathrm{vert}} \in T^{\mathrm{v}}_{L(p)}G$ and hence we have

$$\sum_A (\zeta^A\lambda_A)(L_*v + V_{\mathrm{vert}}) = 0.$$

The vector $L_*v + V_{\mathrm{vert}}$ can be seen to be a completely arbitrary supervector in $T_{L(p)}G$, this follows from the fact that an arbitrary curve $g(t) \in \pi^{-1}(U)$ can be decomposed in terms of the local trivialization as $g(t) = L(\gamma(t))h(t)$ where $\gamma(t)$ is a curve in U, and $h(t) \in H$. So let us expand $L_*v + V_{\mathrm{vert}}$ in the basis of the $K^\sharp_A$ as $L_*v + V_{\mathrm{vert}} = V^A K_A{}^\sharp$, where V^A are therefore arbitrary supernumbers of parity A since $L_*v + V_{\mathrm{vert}}$ is even. We then have

$$\sum_A V^A\lambda_A = 0.$$

Choosing V^A appropriately we see that $\lambda_A = 0$ for all A, hence the $E^A_{(L)}$ are linearly independent.

The dual frame of vectors, $E^{(L)}_A$, satisfying $E^B_{(L)}(E^{(L)}_A) = \delta_A{}^B$, is given by

$$E^{(L)}_A\big|_p = \pi_*\left(K_A{}^\sharp|_{L(p)}\right). \tag{5.2}$$

This is easily checked as

$$\begin{aligned} E^B_{(L)}(E^{(L)}_A) &= L^*\zeta^B(\pi_* K_A{}^\sharp) \\ &= \zeta^B(L_*\pi_* K_A{}^\sharp) \\ &= \zeta^B(K_A{}^\sharp + V_{\text{vert}}) \\ &= \delta_A{}^B. \end{aligned}$$

Here we have noted that the horizontal part of the supervector $L_*\pi_* K_A{}^\sharp$ is just $K_A{}^\sharp$, its vertical part we have called V_{vert} which vanishes when acted on by ζ^B. The final line then follows immediately from Eq. (4.6).

Let us consider the behavior of the frame under a gauge transformation, which, as remarked in Section 4.3, is merely a change of local section. We again introduce two local sections $L(p)$ and $L'(p) = L(p)h(p)$ and will consider how the frame $E^{(L')}_A$ is related to the frame $E^{(L)}_A$. First consider, for $A \in \mathfrak{k}$ and f some arbitrary function on G,

$$\begin{aligned} \pi_*\big(A^\sharp\big|_{L(p)h(p)}\big)[f] &= A^\sharp\big|_{L(p)h(p)}[f\circ\pi] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\circ\pi\left(L(p)h(p)e^{tA}\right)\right)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\circ\pi\left(L(p)e^{t\mathrm{Ad}_{h(p)}A}h(p)\right)\right)\Big|_{t=0} \\ &= \left(\mathrm{Ad}_{h(p)}A\right)^\sharp\big|_{L(p)}[f\circ\pi] \\ &= \pi_*\big((\mathrm{Ad}_{h(p)}A)^\sharp\big|_{L(p)}\big)[f]. \end{aligned} \tag{5.3}$$

Now expand A in the basis of $\mathfrak{k}$ as $A = X^A K_A$, where the X^A are constant supernumbers of parity A. Then, as the function f is arbitrary, and by appropriate choice of the X^A, we find

$$\pi_*\big(K_A{}^\sharp\big|_{L(p)h(p)}\big) = \pi_*\big((\mathrm{Ad}_{h(p)}K_A)^\sharp)\big|_{L(p)}\big). \tag{5.4}$$

Let us introduce the coadjoint representation, $g \mapsto \Lambda_p{}^q(g)$, of the group G defined by

$$\mathrm{Ad}_{g^{-1}}T_p \overset{\text{def}}{=} \Lambda_p{}^q(g)T_q. \tag{5.5}$$

We will now again see the importance of the assumption that the group G is reductive. In this case we have $[\mathfrak{h},\mathfrak{k}] \subseteq \mathfrak{k}$ and thus the coadjoint representation furnishes us with a representation, $h \mapsto \Lambda_A{}^B(h)$, of the subgroup H on the space $\mathfrak{k}$

$$\mathrm{Ad}_{h^{-1}}K_A = \Lambda_A{}^B(h)K_B. \tag{5.6}$$

Thus, from Eq. (5.4), we finally have

$$E^{(Lh)}_A = \Lambda_A{}^B(h^{-1})E^{(L)}_B. \tag{5.7}$$

It follows immediately from Eq. (5.7) that the coframe transforms under a gauge transformation as

$$E^A_{(Lh)} = E^B_{(L)}\Lambda_B{}^A(h). \tag{5.8}$$

This can also be derived directly from the definition Eq. (5.1) using Eq. (4.15) in a similar manner to that used in the case of the connection.

We are primarily, though not solely, interested in the case when G/H is a superspace, see Section 1. Recall that one crucial property of superspace is that its tangent space group must coincide with the even Grassmann shell [3] of the tangent space group of the body of the superspace under consideration. This will be the case when H is the even Grassmann shell of $SO(p,q)$. Further to this we require for a superspace that the representation $\Lambda_A{}^B(h)$ is completely reducible, acting as $SO(p,q)$ rotations on the even coordinates and as $\mathrm{Spin}(p,q)$ rotations on the odd coordinates. In this case we will sometimes refer to the frame as being orthonormal although this is only true for the even part of the frame.

6 Torsion and curvature

In the previous sections we have prescribed a supergeometry on the coset space G/H by determining a local frame and connection. Eqs. (4.13, 5.1) may be combined into one equation

$$L^*\zeta = E^A_{(L)} K_A + \omega^I_{(L)} H_I, \tag{6.1}$$

where we have expanded the connection in the generators of the algebra $\mathfrak{h}$. This equation is often stated in the literature [2, 9] in a matrix representation so that $L^*\zeta = L(p)^{-1}\mathrm{d}L(p)$, see Section 3.2. Given a supergeometry it is natural to next calculate the torsion and curvature. We will see that the torsion and curvature can be straightforwardly determined from the Maurer-Cartan structure equation, Eq. (3.7).

6.1 Local torsion and curvature

First we define the torsion and curvature on the base. In this section we will not explicitly display the dependence of the frame and connection on the local section L, all quantities derived from them do, however, obviously retain this gauge dependence. From the local connection ω we define the curvature on the base as

$$R \stackrel{\text{def}}{=} \mathrm{d}\omega + \omega \wedge \omega, \tag{6.2}$$

which can be expanded in components as $R = R^I H_I$. This gives

$$R^I = \mathrm{d}\omega^I - \frac{1}{2}\omega^J \wedge \omega^K f_{KJ}{}^I. \tag{6.3}$$

The torsion is naturally defined in terms of the exterior covariant derivative of the frame[2]

$$T^A \stackrel{\text{def}}{=} -\mathrm{D}E^A = -\mathrm{d}E^A - E^B \wedge \omega_B{}^A. \tag{6.4}$$

[2]Note that the definition we use has the opposite sign to much of the literature.

Here $\omega_B{}^A$ is the connection ω written in terms of the coadjoint representation. Using the definition of the coadjoint representation, Eq. (5.5), we see that to first order

$$\Lambda_A{}^B(1+\epsilon^I H_I) = \delta_A{}^B - \epsilon^I f_{IA}{}^B, \tag{6.5}$$

hence

$$\omega_A{}^B = -\omega^I f_{IA}{}^B. \tag{6.6}$$

We may also consider the component form of the definitions of curvature and torsion. We expand $\omega = E^A\omega_A$, and similarly $\omega_B{}^C = E^A\omega_{AB}{}^C$, and also introduce the *anholonomy supercoefficients*, $\mathcal{C}_{AB}{}^C$, via

$$[E_A, E_B] = \mathcal{C}_{AB}{}^C E_C. \tag{6.7}$$

It is then possible to show that

$$T_{AB}{}^C = \mathcal{C}_{AB}{}^C + \omega_{AB}{}^C - (-1)^{AB}\omega_{BA}{}^C \tag{6.8a}$$

$$R_{AB} = E_A[\omega_B] - (-1)^{AB}E_B[\omega_A] - \mathcal{C}_{AB}{}^C\omega_C + [\omega_A, \omega_B]. \tag{6.8b}$$

Using the conventions of Appendix A.1 we can also show that

$$R(X,Y) = (-1)^{XY}\big(X[\omega(Y)] - (-1)^{XY}Y[\omega(X)] - \omega([X,Y]) + [\omega(X), \omega(Y)]\big) \tag{6.9}$$

for supervectors X and Y. These expressions will be useful later.

With these definitions in mind let us consider the pullback of the Maurer-Cartan structure equation, Eq. (3.7), under the local section. We have

$$\mathrm{d}L^*\zeta + L^*\zeta \wedge L^*\zeta = 0.$$

Substituting in Eq. (6.1) and separating out the $\mathfrak{k}$ and $\mathfrak{h}$ parts we find

$$\mathrm{d}E^A - \frac{1}{2}E^B \wedge E^C f_{CB}{}^A - E^B \wedge \omega^I f_{IB}{}^A = 0$$
$$\mathrm{d}\omega^I - \frac{1}{2}E^B \wedge E^C f_{CB}{}^I - \frac{1}{2}\omega^J \wedge \omega^K f_{KJ}{}^I = 0.$$

Comparing these two equations with the definitions of the curvature and torsion two-forms we see that

$$T^A = -\frac{1}{2}E^B \wedge E^C f_{CB}{}^A \tag{6.10a}$$

$$R^I = \frac{1}{2}E^B \wedge E^C f_{CB}{}^I. \tag{6.10b}$$

Comparing to the component expansion of a p-form as given in Eq. (A.11) we see that

$$T_{AB}{}^C = f_{AB}{}^C, \qquad R_{AB}{}^I = -f_{AB}{}^I. \tag{6.11}$$

6.2 Torsion and curvature from the bundle

Whilst it is sufficient for most applications to consider the torsion and curvature as defined in the previous section, it is nice to see how they are obtained in terms of the geometry of the bundle. The first concept we need to define is the exterior covariant derivative, D, on the bundle, not to be confused with that defined on the base used in the previous section. It is defined by its action on an n-form ϕ on the bundle as

$$\mathrm{D}\phi(X_1, \ldots, X_{n+1}) \stackrel{\text{def}}{=} \mathrm{d}\phi(X_1^{\mathrm{h}}, \ldots, X_{n+1}^{\mathrm{h}}). \tag{6.12}$$

Here the supervector X^{h} is the horizontal part of the supervector X.

Now recall that the vertical part of the Maurer-Cartan form gave us the connection on the bundle $\zeta^I H_I = \Omega$. Denoting the horizontal part by $\zeta^A K_A = \Theta$, we have

$$\zeta = \Theta + \Omega. \tag{6.13}$$

The form Θ is essentially the *solder form* on the bundle [8] which can be seen as follows. The solder form is a form defined on a frame bundle; we will see in Section 7.2 how the principal bundle may be thought of as the frame bundle. In particular, a frame $E_A^{(L)}|_p$ corresponds to the point $L(p)$ in the principal bundle. Given a vector $V|_{L(p)}$ tangent to the bundle at this point the components of the solder form, Θ^A, are defined to satisfy

$$\pi_* \left(V|_{L(p)}\right) = \Theta^A(V|_{L(p)}) E_A^{(L)}|_p. \tag{6.14}$$

By expanding V in the basis of the $T_p^\sharp$ it is straightforward to check that $\Theta^A = \zeta^A$ solves this equation. $\Theta = \zeta^A K_A$ can therefore be considered as the collection of the components of the solder form into a single algebra valued form, thus in the following we will simply refer to Θ as the solder form.

We may then define the quantities $\mathcal{T}$ and $\mathcal{R}$ which we call the torsion and curvature on the bundle. These are defined in terms of the exterior covariant derivatives of the solder form and connection form, respectively, and are calculated to be

$$-\mathcal{T} \stackrel{\text{def}}{=} \mathrm{D}\Theta = \mathrm{d}\Theta + \Theta \wedge \Omega + \Omega \wedge \Theta \tag{6.15a}$$

$$\mathcal{R} \stackrel{\text{def}}{=} \mathrm{D}\Omega = \mathrm{d}\Omega + \Omega \wedge \Omega. \tag{6.15b}$$

Note that $\mathcal{T}$ is a $\mathfrak{k}$-valued two-form, whereas $\mathcal{R}$ is $\mathfrak{h}$-valued. A straightforward calculation shows that when pulled back under a local section these quantities give the definitions of the local torsion and curvature two-forms in Eqs. (6.2, 6.4)

$$L^*\mathcal{T} = T^A K_A, \qquad L^*\mathcal{R} = R. \tag{6.16}$$

Let us now consider how the torsion and curvature on the bundle may be expressed in terms of the structure constants of the algebra. Using Eq. (6.13) and Eqs. (6.15a, 6.15b) we see that

$$\begin{aligned}\mathrm{D}\zeta = -\mathcal{T} + \mathcal{R} &= \mathrm{d}\Theta + \Theta \wedge \Omega + \Omega \wedge \Theta + \mathrm{d}\Omega + \Omega \wedge \Omega \\ &= \mathrm{d}\zeta + \zeta \wedge \zeta - \Theta \wedge \Theta \\ &= -\Theta \wedge \Theta,\end{aligned} \tag{6.17}$$

where in the last line we have used the Maurer-Cartan structure equation Eq. (3.7). As $\Theta = \zeta^A K_A$ we find

$$\begin{aligned}
-\mathcal{T} + \mathcal{R} &= \frac{1}{2}\zeta^A \wedge \zeta^B [K_B, K_A] \\
&= \frac{1}{2}\zeta^A \wedge \zeta^B f_{BA}{}^C K_C + \frac{1}{2}\zeta^A \wedge \zeta^B f_{BA}{}^I H_I.
\end{aligned}$$

If we then split this up into its $\mathfrak{k}$ and $\mathfrak{h}$ parts we find

$$\mathcal{T} = -\frac{1}{2}\zeta^A \wedge \zeta^B f_{BA}{}^C K_C \tag{6.18a}$$

$$\mathcal{R} = \frac{1}{2}\zeta^A \wedge \zeta^B f_{BA}{}^I H_I. \tag{6.18b}$$

If we pullback these equations to the base as in Eq. (6.16) we obtain the expressions for the local torsion and curvature we derived earlier in Eqs. (6.10a, 6.10b).

7 Associated bundles and supertensor bundles

7.1 Associated bundles

Given a principal bundle $P(G/H, H) = G$ we construct an associated fibre bundle as follows. Consider a manifold F on which the structure group H acts on the left. Then we define an equivalence relation on $G \times F$ by

$$(g, f) \sim (gh, h^{-1} f), \tag{7.1}$$

where $(g, f) \in G \times F$. In the following we shall denote the equivalence class of the point (g, f) as $[(g, f)]$. From this equivalence relation we define the associated fibre bundle as the coset space $(G \times F)/H$, see [6]. In the following we shall consider associated bundles only where the fibre F is a supervector space[3]. In these cases H acts on the fibre via some representation ρ and the equivalence relation, Eq. (7.1), reads

$$(g, \xi) \sim (gh, \rho(h)^{-1}\xi), \quad \xi \in F. \tag{7.2}$$

Defining the local bundle map

$$R^{(L)}_{h_1} : L(p)h \;\mapsto\; L(p)h_1 h \tag{7.3}$$

we can define the action of H on a local section of a general associated bundle $s(p) = [(L(p), \xi(p))]$ as

$$\tilde{R}^{(L)}_h : [(L(p), \xi(p))] \;\mapsto\; [(R^{(L)}_h\big(L(p)\big), \xi(p))] \tag{7.4}$$

which, using the equivalence relation, can be rewritten as

$$\tilde{R}^{(L)}_h s(p) = [(L(p), \rho(h)\xi(p))]. \tag{7.5}$$

[3]Note that local sections of associated bundles where the fibre F is a supervector space can be linearly combined to give new local sections in a way that will become apparent in Section 7.3.

Note that $R_h^{(L)}$ is a group homomorphism, i.e. we have

$$R_{h_1}^{(L)} \circ R_{h_2}^{(L)} = R_{h_1 h_2}^{(L)}. \tag{7.6}$$

Based on the definition of $\tilde{R}_h^{(L)}$ we can define the action of $\mathfrak{h}$ on a local section $s(p) = [(L(p), \xi(p))]$ of a general associated bundle. For $h_\epsilon = 1 + \epsilon H + \mathcal{O}(\epsilon^2)$ we define

$$\tilde{R}_H^{(L)} s(p) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{R}_{h_t}^{(L)} s(p)\Big)\Big|_{t=0}. \tag{7.7}$$

We hence have

$$\tilde{R}_H^{(L)} s(p) = \Big[\Big(L(p), \frac{\mathrm{d}}{\mathrm{d}t}(\rho(h_t)\xi(p))\Big|_{t=0}\Big)\Big]. \tag{7.8}$$

7.2 Equivalence of the principal bundle with the frame bundle

In this section we shall show that the orthonormal[4] frame bundle $F(G/H)$ is equivalent to the principal bundle. Note that the fibre of the frame bundle above the point p is given by $\{E_A^{(L_i h)}\big|_p : h \in H\} \cong H$. We define the local trivialization of the frame bundle as

$$\begin{aligned} \psi_i &: U_i \times H \to \pi_F^{-1}(U_i) \\ &: (p, h) \mapsto \{\Lambda_A{}^B(h^{-1}) E_B^{(L_i)}\big|_p\} = \{E_A^{(L_i h)}\big|_p\}, \end{aligned} \tag{7.9}$$

where we have defined the frame bundle projection map π_F as

$$\begin{aligned} \pi_F &: F(G/H) \to G/H \\ &: \pi_*\big(K_A{}^\sharp\big|_g\big) \mapsto gH. \end{aligned} \tag{7.10}$$

The transition functions $\tilde{t}_{ij}$ of the frame bundle are then given by

$$\tilde{t}_{ij}(p) = \psi_{i,p}^{-1}\psi_{j,p} : H \to H, \tag{7.11}$$

where $\psi_i(p, h) = \psi_{i,p}(h)$. We thus already see that the structure group of the orthonormal frame bundle is equal to the structure group of the principal bundle, see Section 2. In order to fully establish the equivalence between the principal bundle and the frame bundle we shall now show that the transition functions of the principal bundle are equal to those of the frame bundle. In order to do this consider the action of $\tilde{t}_{ij}(p)$ on h

$$\tilde{t}_{ij}(p)h = \psi_{i,p}^{-1}\left(\Lambda_A{}^B(h^{-1}) E_B^{(L_j)}\big|_p\right). \tag{7.12}$$

Now we have using the definition of the transition functions of the principal bundle $P(G/H, H)$, see Eq. (2.3),

$$L_i(p) = \phi_i(p, e) = \phi_j(p, t_{ji}(p)e) = L_j(p)t_{ji}(p) \tag{7.13}$$

[4]Note that here we use the term orthonormal in the sense discussed in the last paragraph of Section 5.

and hence we find

$$\begin{aligned} E_B^{(L_j)}\big|_p &= \pi_*\big(K_B{}^\sharp\big|_{L_j(p)}\big) \\ &= \pi_*\big(K_B{}^\sharp\big|_{L_i(p)t_{ij}}\big) \\ &= \Lambda_B{}^C(t_{ij}(p)^{-1})E_C^{(L_i)}\big|_p. \end{aligned} \tag{7.14}$$

We thus have

$$\begin{aligned} \tilde{t}_{ij}(p)h &= \psi_{i,p}^{-1}\left(\Lambda_A{}^B(h^{-1})\Lambda_B{}^C(t_{ij}(p)^{-1})E_C^{(L_i)}\big|_p\right) \\ &= \psi_{i,p}^{-1}\left(\Lambda_A{}^B\left((t_{ij}(p)h)^{-1}\right)E_B^{(L_i)}\big|_p\right) \\ &= t_{ij}(p)h. \end{aligned} \tag{7.15}$$

This proves that the transition functions of the frame bundle equal those of the principal bundle. We thus see that the principal bundle and the frame bundle are equivalent bundles. In the following we shall, for convenience, use the principal bundle rather than the frame bundle in order to formulate associated bundles.

7.3 Supertensor bundles

In this section we shall briefly discuss the equivalence relations Eq. (7.1) in the case of general tensor bundles. Although it is more natural to consider tensor bundles associated to the frame bundle we will in this section use the equivalent principal bundle for the ease of notation.

Let us consider a pure, i.e. even or odd, section $s(p)$ of a general tensor bundle with both contravariant and covariant indices. We write

$$s(p) = [(L(p), \{s^{A_1\cdots A_i}{}_{\tilde{A}_{i+1}\cdots\tilde{A}_{i+j}}{}^{A_{i+j+1}\cdots}\cdots\})]. \tag{7.16}$$

Here we define the equivalence by

$$\begin{aligned} &(L(p), \{s^{A_1\cdots A_i}{}_{\tilde{A}_{i+1}\cdots\tilde{A}_{i+j}}{}^{A_{i+j+1}\cdots}\cdots\}) \\ &\qquad\sim (L(p)h, \rho(h^{-1})\{s^{A_1\cdots A_i}{}_{\tilde{A}_{i+1}\cdots\tilde{A}_{i+j}}{}^{A_{i+j+1}\cdots}\cdots\}) \end{aligned} \tag{7.17}$$

where we set

$$\begin{aligned} &\rho(h)\{s^{A_1\cdots A_i}{}_{\tilde{A}_{i+1}\cdots\tilde{A}_{i+j}}{}^{A_{i+j+1}\cdots}\cdots\} \\ &\quad= \{(-1)^{\Delta_n(A+B,B+\tilde{B})+\Delta_n(A+\tilde{B},\tilde{A}+\tilde{B})}\Lambda_{\tilde{A}_{i+1}}{}^{\tilde{B}_{i+1}}(h)\cdots\Lambda_{\tilde{A}_{i+j}}{}^{\tilde{B}_{i+j}}(h)\cdots \\ &\qquad\qquad s^{B_1\cdots B_i}{}_{\tilde{B}_{i+1}\cdots\tilde{B}_{i+j}}{}^{B_{i+j+1}\cdots}\cdots \\ &\qquad\qquad \Lambda_{B_1}^{A_1}(h^{-1})\cdots\Lambda_{B_i}^{A_i}(h^{-1})\Lambda_{B_{i+j+1}}{}^{A_{i+j+1}}(h^{-1})\cdots\}. \end{aligned} \tag{7.18}$$

Here Δ_n is the parity function as defined in Eq. (A.12), n denotes the total number of indices and we have set $\Delta_n(A, B+\tilde{B}) \equiv \Delta_n(A,B) + \Delta_n(A,\tilde{B})$.

We define the local section of a general supertensor bundle to satisfy the following linearity property under left multiplication by an arbitrary pure supernumber λ

$$\lambda s(p) \stackrel{\text{def}}{=} [(L(p), \{(-1)^{\lambda(\sum \tilde{A})} \lambda s^{A_1 \cdots A_i}{}_{\tilde{A}_{i+1} \cdots \tilde{A}_{i+j}}{}^{A_{i+j+1} \cdots} \cdots \})], \tag{7.19}$$

where $\sum \tilde{A} \equiv (\tilde{A}_{i+1} + \ldots + \tilde{A}_{i+j} + \ldots)$ is the sum over the parities of the lower indices. From Eq. (7.19) and from the fact that $\lambda s(p) = (-1)^{\lambda s} s(p) \lambda$ we can infer the linearity property under right multiplication

$$s(p)\lambda = [(L(p), \{(-1)^{\lambda(\sum A)} s^{A_1 \cdots A_i}{}_{\tilde{A}_{i+1} \cdots \tilde{A}_{i+j}}{}^{A_{i+j+1} \cdots} \cdots \lambda \})]. \tag{7.20}$$

From these last two equations we see that contravariant tensor bundles are left-linear, i.e. they satisfy

$$\lambda s(p) = [(L(p), \{\lambda s^{A_1 \cdots A_n}\})], \tag{7.21}$$

whereas covariant tensor bundles are right-linear, i.e.

$$s(p)\lambda = [(L(p), \{s_{A_1 \cdots A_n} \lambda\})]. \tag{7.22}$$

Defining the basis of a general supertensor bundle by[5]

$$\begin{aligned} E^{(L)}_{A_1} \otimes \cdots \otimes E^{(L)}_{A_i} \otimes E^{\tilde{A}_{i+1}}_{(L)} \otimes \cdots \otimes E^{\tilde{A}_{i+j}}_{(L)} \otimes E^{(L)}_{A_{i+j+1}} \otimes \cdots \stackrel{\text{def}}{=} [(L(p), \\ \{(-1)^{\Delta_n(A,A) + \Delta_n(\tilde{A},\tilde{A})} \delta_{A_1}{}^{B_1} \cdots \delta_{A_i}{}^{B_i} \delta_{\tilde{B}_{i+1}}{}^{\tilde{A}_{i+1}} \cdots \delta_{\tilde{B}_{i+j}}{}^{\tilde{A}_{i+j}} \delta_{A_{i+j+1}}{}^{B_{i+j+1}} \cdots \})] \end{aligned} \tag{7.23}$$

we can rewrite the local section $s(p)$ of Eq. (7.16) in terms of this basis as

$$\begin{aligned} s(p) &= [(L(p), \{s^{A_1 \cdots A_i}{}_{\tilde{A}_{i+1} \cdots \tilde{A}_{i+j}}{}^{A_{i+j+1} \cdots} \cdots \})] \\ &= (-1)^{\Delta_n(A,A) + \Delta_n(\tilde{A},\tilde{A})} (-1)^{(s + \sum A + \sum \tilde{A})(\sum \tilde{A})} s^{A_1 \cdots A_i}{}_{\tilde{A}_{i+1} \cdots \tilde{A}_{i+j}}{}^{A_{i+j+1} \cdots} \cdots \\ &\quad E^{(L)}_{A_1} \otimes \cdots \otimes E^{(L)}_{A_i} \otimes E^{\tilde{A}_{i+1}}_{(L)} \otimes \cdots \otimes E^{\tilde{A}_{i+j}}_{(L)} \otimes E^{(L)}_{A_{i+j+1}} \otimes \cdots \\ &= (-1)^{\Delta_n(A+\tilde{A}, A+\tilde{A})} (-1)^{(s + \sum \tilde{A})(\sum \tilde{A})} s^{A_1 \cdots A_i}{}_{\tilde{A}_{i+1} \cdots \tilde{A}_{i+j}}{}^{A_{i+j+1} \cdots} \cdots \\ &\quad E^{(L)}_{A_1} \otimes \cdots \otimes E^{(L)}_{A_i} \otimes E^{\tilde{A}_{i+1}}_{(L)} \otimes \cdots \otimes E^{\tilde{A}_{i+j}}_{(L)} \otimes E^{(L)}_{A_{i+j+1}} \otimes \cdots , \end{aligned}$$

where we have used the linearity property Eq. (7.19). Note that in the case of contravariant tensor bundles this formula simplifies to

$$s(p) = (-1)^{\Delta_n(A,A)} s^{A_1 \cdots A_n} E^{(L)}_{A_1} \otimes \cdots \otimes E^{(L)}_{A_n} \tag{7.24}$$

and in the case of covariant tensor bundles we have

$$s(p) = (-1)^{\Delta_n(A,A)} E^{A_1}_{(L)} \otimes \cdots \otimes E^{A_n}_{(L)} s_{A_1 \cdots A_n}. \tag{7.25}$$

[5]One should note that the collection of δ's that occurs in the definition of the basis is to be understood as an ordered collection, i.e. the δ's ought not be swapped.

Using the definition of the action of H on a local section $s(p)$ as given in Eq. (7.5) we find after a slightly lengthy calculation for the transformation of the basis under H

$$\begin{aligned}\tilde{R}_h^{(L)}\left(E_{A_1}^{(L)}\otimes\cdots\otimes E_{A_i}^{(L)}\otimes E_{(L)}^{\tilde{A}_{i+1}}\otimes\cdots\otimes E_{(L)}^{\tilde{A}_{i+j}}\otimes E_{A_{i+j+1}}^{(L)}\otimes\cdots\right)\\ = E_{A_1}^{(Lh)}\otimes\cdots\otimes E_{A_i}^{(Lh)}\otimes E_{(Lh)}^{\tilde{A}_{i+1}}\otimes\cdots\otimes E_{(Lh)}^{\tilde{A}_{i+j}}\otimes E_{A_{i+j+1}}^{(Lh)}\otimes\cdots\end{aligned} \tag{7.26}$$

as expected.

From the definition of the basis Eq. (7.23) and the linearity property Eq. (7.19) one can easily deduce the following properties of the tensor product

$$\lambda(s_1\otimes s_2)=(\lambda s_1)\otimes s_2\equiv\lambda s_1\otimes s_2 \tag{7.27a}$$
$$(s_1\otimes s_2)\lambda=s_1\otimes(s_2\lambda)\equiv s_1\otimes s_2\lambda \tag{7.27b}$$
$$(s_1\lambda)\otimes s_2=s_1\otimes(\lambda s_2)\equiv s_1\lambda\otimes s_2 \tag{7.27c}$$
$$(s_1\otimes s_2)\otimes s_3=s_1\otimes(s_2\otimes s_3)\equiv s_1\otimes s_2\otimes s_3, \tag{7.27d}$$

where s_1, s_2 and s_3 are local sections of general tensor bundles and λ is a supernumber.

8 Bundle maps

Let us define the following (local) bundle maps

$$L_g : L(p)h \;\mapsto\; gL(p)h \tag{8.1a}$$
$$L_g^{(L)} : L(p)h \;\mapsto\; gL(p)\tilde{h}_L^{(L)}(p,g)^{-1}h \tag{8.1b}$$
$$R_g^{(L)} : L(p)h \;\mapsto\; L(p)gh. \tag{8.1c}$$

While L_g, see Section 3.1, is a global bundle map from $G\to G$ – here written locally in the patch $\pi^{-1}(U)$ – the map $L_g^{(L)}$ depends on the local section $L(p)$ and, for it to be well-defined, both its domain and range must be restricted to $\pi^{-1}(U)$. In the following we shall however, for convenience, also restrict both the domain and range of the map L_g to $\pi^{-1}(U)$. Then the *left* H-compensator $\tilde{h}_L^{(L)}(p,g)$ is defined by the equation, see [9, 10, 11],

$$gL(p)=L(q)\tilde{h}_L^{(L)}(p,g)\quad\text{with}\, p,q\in U, \tag{8.2a}$$

i.e., we have

$$L_g^{(L)}\left(L(p)h\right)=L(q)h \tag{8.2b}$$

and

$$L_g\left(L(p)h\right)=L(q)\tilde{h}_L^{(L)}(p,g)h \tag{8.2c}$$

and hence we see that the map $L_g^{(L)}$ preserves the local section $L(p)$, while this is not true for the map L_g.

For the map $R_g^{(L)}$ – which also depends on the local section $L(p)$ – only the domain need be restricted to $\pi^{-1}(U)$; in the following, however, we shall for convenience also require its range to be restricted to $\pi^{-1}(U)$. We can then, in accordance with Eq. (8.2a), also define the *right* H-compensator $\tilde{h}_R^{(L)}(p,g)$ by

$$L(p)g = L(q)\tilde{h}_R^{(L)}(p,g), \quad \text{with}\, p,q \in U, \tag{8.3a}$$

i.e.,

$$R_g^{(L)}\left(L(p)h\right) = L(q)\tilde{h}_R^{(L)}(p,g)h. \tag{8.3b}$$

Note that the map $R_h^{(L)}$ introduced in Section 7.1, see Eq. (7.3), is just a specific case of the map $R_g^{(L)}$ introduced here.

The maps $L_g^{(L)}$ and $R_g^{(L)}$ induce maps on the base that we shall denote by l_g and $r_g^{(L)}$, respectively. We have

$$\pi \circ L_g^{(L)} = \pi \circ L_g = l_g \circ \pi \tag{8.4a}$$

$$\pi \circ R_g^{(L)} = r_g^{(L)} \circ \pi, \tag{8.4b}$$

where one should note that the map $r_g^{(L)}$ on the base depends on the local section $L(p)$. We can then rewrite Eqs. (8.2b, 8.2c) and Eq. (8.3b), respectively, as

$$L_g^{(L)}\left(L(p)h\right) = L(l_g(p))h \tag{8.5a}$$

$$L_g\left(L(p)h\right) = L(l_g(p))\tilde{h}_L^{(L)}(p,g)h \tag{8.5b}$$

$$R_g^{(L)}\left(L(p)h\right) = L(r_g^{(L)}(p))\tilde{h}_R^{(L)}(p,g)h. \tag{8.5c}$$

Using Eqs. (8.4a, 8.4b) we can derive relations for the vectors $\pi_*(A^\sharp|_{L(p)})$ and $\pi_*(A^\flat|_{L(p)})$ similar to those in Eqs. (3.1a, 3.1b). We have

$$\begin{aligned}
\pi_*\left(A^\sharp\big|_{L(p)}\right)[f] &= A^\sharp\big|_{L(p)}[f\circ\pi] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\circ\pi\left(L(p)e^{tA}\right)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\circ\pi\circ R_{e^{tA}}^{(L)}L(p)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\circ r_{e^{tA}}^{(L)}\circ\pi\circ L(p)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}f(r_{e^{tA}}^{(L)}(p))\Big|_{t=0}
\end{aligned} \tag{8.6}$$

and similarly we find

$$\pi_*\left(A^\flat\big|_{L(p)}\right)[f] = \frac{\mathrm{d}}{\mathrm{d}t}f(l_{e^{tA}}(p))\Big|_{t=0}. \tag{8.7}$$

Note that the right hand side of this last equation is independent of the local section $L(p)$, and hence we have that

$$\pi_*\left(A^\flat\big|_{L(p)}\right) = \pi_*\left(A^\flat\big|_{L(p)h(p)}\right). \tag{8.8}$$

Thus we may unambiguously write $\pi_* A^\flat$ as a well defined vector field on G/H without the need to specify which point in the fibre the supervector $A^\flat$ was based. Note that this not true for $A^\sharp$.

The maps L_g, $L_g^{(L)}$ and $R_g^{(L)}$ defined in Eqs. (8.1a–8.1c), respectively, can be extended to maps on associated bundles as follows. Consider a local section $s(p) = [(L(p), \xi(p))]$ of some general associated bundle. We then define the maps $\tilde{L}_g$, $\tilde{L}_g^{(L)}$ and $\tilde{R}_g^{(L)}$ by

$$\tilde{L}_g : [(L(p), \xi(p))] \mapsto [(L_g\big(L(p)\big), \xi(p))] \tag{8.9a}$$

$$\tilde{L}_g^{(L)} : [(L(p), \xi(p))] \mapsto [(L_g^{(L)}\big(L(p)\big), \xi(p))] \tag{8.9b}$$

$$\tilde{R}_g^{(L)} : [(L(p), \xi(p))] \mapsto [(R_g^{(L)}\big(L(p)\big), \xi(p))]. \tag{8.9c}$$

8.1 Properties of the H-compensators

In this section we will discuss properties of the left and right H-compensators.

8.1.1 Properties of the left H-compensator

Recall that the left H-compensator is defined by the equation

$$L(l_g(p)) = gL(p)\tilde{h}_L^{(L)}(p, g)^{-1}. \tag{8.10}$$

Now, setting $l_g(p) = q$ we can rearrange this formula to give

$$L(p) = g^{-1}L(q)\tilde{h}_L^{(L)}(l_{g^{-1}}(q), g),$$

but we also have

$$L(p) = L(l_{g^{-1}}(q)) = g^{-1}L(q)\tilde{h}_L^{(L)}(q, g^{-1})^{-1}$$

from which we can deduce the relation

$$\tilde{h}_L^{(L)}(l_{g^{-1}}(q), g) = \tilde{h}_L^{(L)}(q, g^{-1})^{-1}. \tag{8.11}$$

This relation will be important when considering derivations on local sections of associated bundles.

Next we shall derive the composition rule for the left H-compensators. We have

$$\begin{aligned} L_{g_1}^{(L)} \circ L_{g_2}^{(L)}\,(L(p)h) &= L_{g_1}^{(L)}(L(l_{g_2}(p))h) = L(l_{g_1} \circ l_{g_2}(p))h && (8.12a) \\ &= g_1 L(l_{g_2}(p))\tilde{h}_L^{(L)}(l_{g_2}(p), g_1)^{-1}h \\ &= g_1 g_2 L(p)\tilde{h}_L^{(L)}(p, g_2)^{-1}\tilde{h}_L^{(L)}(l_{g_2}(p), g_1)^{-1}h. && (8.12b) \end{aligned}$$

On the other hand we have

$$\begin{aligned} L_{g_1 g_2}^{(L)}\,(L(p)h) &= L(l_{g_1 g_2}(p))h && (8.13a) \\ &= g_1 g_2 L(p)\tilde{h}_L^{(L)}(p, g_1 g_2)^{-1}h. && (8.13b) \end{aligned}$$

Now, from Eqs. (8.12b, 8.13b), we see that

$$\pi \circ L_{g_1 g_2}^{(L)}\,(L(p)h) = \pi \circ L_{g_1}^{(L)} \circ L_{g_2}^{(L)}\,(L(p)h) \tag{8.14}$$

and hence we can deduce from Eqs. (8.12a, 8.13a) and Eqs. (8.12b, 8.13b), respectively,

$$l_{g_1 g_2} = l_{g_1} \circ l_{g_2} \tag{8.15a}$$
$$\tilde{h}_L^{(L)}(p, g_1 g_2)^{-1} = \tilde{h}_L^{(L)}(p, g_2)^{-1} \tilde{h}_L^{(L)}(l_{g_2}(p), g_1)^{-1}. \tag{8.15b}$$

Now we shall also consider the transformation of $\tilde{h}_L^{(L)}$ under a change of local section $L \to L' = Lh$, i.e. under a gauge transformation. We have

$$\begin{aligned} gL'(p)\tilde{h}_L^{(L')}(p,g)^{-1} &= L'(l_g(p)) &(8.16a)\\ &= L(l_g(p))h(l_g(p)) \\ &= gL(p)h(p)h(p)^{-1}\tilde{h}_L^{(L)}(p,g)^{-1}h(l_g(p)). &(8.16b)\end{aligned}$$

From Eqs. (8.16a, 8.16b) we thus find

$$\tilde{h}_L^{(Lh)}(p, g)^{-1} = h(p)^{-1}\tilde{h}_L^{(L)}(p, g)^{-1}h(l_g(p)). \tag{8.17}$$

In the following we shall consider infinitesimal versions of Eqs. (8.15a, 8.15b) and Eq. (8.17). Defining

$$W_L^{(L)}(p, A) \overset{\text{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_L^{(L)}(p, e^{tA})\Big|_{t=0}, \tag{8.18}$$

we can write the expansion of $\tilde{h}_L^{(L)}(p, g)$ for $g = e^{\epsilon A}$ with $\epsilon \ll 1$ and $A \in \mathfrak{g}$ to first order as

$$\tilde{h}_L^{(L)}(p, e^{\epsilon A}) = 1 - \epsilon W_L^{(L)}(p, A) + \mathcal{O}(\epsilon^2). \tag{8.19}$$

Note that using Eq. (8.15b) we easily see that $W_L^{(L)}(p, A)$ is left linear in A. In the following we shall for the ease of notation suppress the p dependence in $W_L^{(L)}$.

Now using the composition rule Eq. (8.15a) and the fact that $(\mathrm{Ad}_{g_1} g_2)g_1 = g_1 g_2$ we find for the infinitesimal version of Eq. (8.15a) with $g_1 = e^{\epsilon_1 A}$ and $g_2 = e^{\epsilon_2 B}$ for $A, B \in \mathfrak{g}$

$$[\pi_* A^\flat, \pi_* B^\flat] = -\pi_*[A, B]^\flat. \tag{8.20}$$

This result can also be seen as a consequence of the well definedness of the supervector field $\pi_* A^\flat$, Eq. (8.8), which allows us to take the action of the pushforward under the projection, π_*, inside the Lie bracket of Eq. (3.4b).

Now consider Eq. (8.15b). We have for the expansion of $\tilde{h}_L^{(L)}(l_{g_2}(p), g_1)^{-1}$

$$\tilde{h}_L^{(L)}(l_{e^{\epsilon_2 B}}(p), e^{\epsilon_1 A})^{-1} = 1 + \epsilon_1 W_L^{(L)}(A) + \epsilon_1\epsilon_2 \pi_* B^\flat[W_L^{(L)}(A)] + \ldots.$$

The infinitesimal version of Eq. (8.15b) is thus given by

$$W_L^{(L)}([A, B]) = \pi_* B^\flat[W_L^{(L)}(A)] - \pi_* A^\flat[W_L^{(L)}(B)] + [W_L^{(L)}(B), W_L^{(L)}(A)]. \tag{8.21}$$

In the literature this last equation is normally referred to as the integrability condition for the left H-compensator [9].

Expanding Eq. (8.17) we find for the transformation of $W_L^{(L)}$ under a gauge transformation $L \to Lh$

$$W_L^{(Lh)}(A) = h(p)^{-1} W_L^{(L)}(A) h(p) + h(p)^{-1} \pi_* A^\flat h(p). \tag{8.22}$$

Therefore we see that $W_L^{(L)}$ transforms like a connection under a change of local section.

8.1.2 Properties of the right H-compensator

In a similar fashion as for the left H-compensator we can derive a composition rule for the right H-compensator, Eq. (8.3a), $\tilde{h}_R^{(L)}(p,g)$. We find

$$r_{g_1}^{(L)} \circ r_{g_2}^{(L)}(p) = r^{(L)}_{g_2 \mathrm{Ad}_{\tilde{h}_R^{(L)}(p,g_2)^{-1}} g_1}(p) \tag{8.23a}$$

$$\tilde{h}_R^{(L)}(r_{g_2}^{(L)}(p), g_1) \tilde{h}_R^{(L)}(p, g_2) = \tilde{h}_R^{(L)}(p, g_2 \mathrm{Ad}_{\tilde{h}_R^{(L)}(p,g_2)^{-1}} g_1). \tag{8.23b}$$

Again proceeding in a similar fashion as in the case of the left H-compensator we can derive an expression for the transformation properties of the right H-compensator under a change of local section $L \to Lh$. Note, however, that in this case also the map $r_g^{(L)}$ will transform under a change of local section. We find

$$r^{(Lh)}_{\mathrm{Ad}_{h(p)^{-1}} g}(p) = r_g^{(L)}(p) \tag{8.24a}$$

$$\tilde{h}_R^{(Lh)}(p, \mathrm{Ad}_{h(p)^{-1}} g)^{-1} = h(p)^{-1} \tilde{h}_R^{(L)}(p,g)^{-1} h(r_g^{(L)}(p)). \tag{8.24b}$$

In the following we shall consider the infinitesimal versions of Eqs. (8.23a–8.24b). Defining

$$W_R^{(L)}(p, A) \overset{\mathrm{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}t} \tilde{h}_R^{(L)}(p, e^{tA}) \Big|_{t=0}, \tag{8.25}$$

we can write the expansion of $\tilde{h}_R^{(L)}(p,g)$ for $g = e^{\epsilon A}$ with $\epsilon \ll 1$ and $A \in \mathfrak{g}$ to first order as

$$\tilde{h}_R^{(L)}(p, e^{\epsilon A}) = 1 - \epsilon W_R^{(L)}(p, A) + \mathcal{O}(\epsilon^2). \tag{8.26}$$

Note that using Eq. (8.23b) one can show that $W_R^{(L)}(p,A)$ is left linear in A. In the following we shall for the ease of notation suppress the p dependence in $W_R^{(L)}$.

We shall now show that $W_R^{(L)}(A)$ for $A \in \mathfrak{k}$ is related to the spin connection. In order to do this consider first

$$R^{(L)}_{e^{tK}}(L(p)) = L(p) e^{tK} = L(r^{(L)}_{e^{tK}}(p)) \tilde{h}_R^{(L)}(p, e^{tK}),$$

for $K \in \mathfrak{k}$. Now, setting $\tilde{\gamma}(t) \equiv L(p) e^{tK}$ and $\gamma(t) \equiv r^{(L)}_{e^{tK}}(p)$, we have $\pi \circ \tilde{\gamma} = \gamma$. Clearly the tangent vector to the curve $\tilde{\gamma}(t)$ is given by $K^\sharp \in T^{\mathrm{h}}_{\tilde{\gamma}(t)} G$ and hence $\tilde{\gamma}(t)$ is the horizontal lift of the curve $\gamma(t)$. Also note that $\tilde{\gamma}(0) = L(p)$. As such we can identify the right H-compensator $\tilde{h}_R^{(L)}(p, e^{tK})$ with $\tilde{h}(\gamma(t))$ of Eq. (4.14). We can thus write

$$\omega^{(L)}(\pi_*(K^\sharp\big|_{L(p)})) = -\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_R^{(L)}(p, e^{tK})\Big|_{t=0} \tag{8.27}$$

and hence

$$W_R^{(L)}(K) = \omega^{(L)}(\pi_*(K^\sharp\big|_{L(p)})) \quad \text{for} \quad K \in \mathfrak{k}. \tag{8.28}$$

On the other hand consider $W_R^{(L)}(A)$ for $A \in \mathfrak{h}$. We have

$$R_{h_1}^{(L)}(L(p)h) = L(p)h_1h = L(r_{h_1}^{(L)}(p))\tilde{h}_R^{(L)}(p, h_1)h$$

and using the fact that $r_h^{(L)}(p) = p$ we thus find

$$\tilde{h}_R^{(L)}(p, e^{tH}) = e^{tH} \tag{8.29a}$$

and hence

$$W_R^{(L)}(H) = -\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_R^{(L)}(p, e^{tH})\Big|_{t=0} = -H. \tag{8.29b}$$

For the ease of notation we shall for the rest of this section drop the explicit $L(p)$ dependence on $\pi_*A^\sharp$.

Now consider the infinitesimal versions of Eqs. (8.23a, 8.23b). We find with $g_1 = e^{\epsilon_1 A}$ and $g_2 = e^{\epsilon_2 B}$

$$[\pi_*A^\sharp, \pi_*B^\sharp] = \pi_*[A, B]^\sharp + \pi_*[W_R^{(L)}(A), B]^\sharp - \pi_*[W_R^{(L)}(B), A]^\sharp \tag{8.30a}$$

$$\begin{aligned} W_R^{(L)}([A, B]) = {} & [W_R^{(L)}(A), W_R^{(L)}(B)] + \pi_*A^\sharp W_R^{(L)}(B) - \pi_*B^\sharp W_R^{(L)}(A) \\ & + W_R^{(L)}\left([W_R^{(L)}(B), A]\right) - W_R^{(L)}\left([W_R^{(L)}(A), B]\right). \end{aligned} \tag{8.30b}$$

Now, in the case where $A, B \in \mathfrak{k}$ we find that Eq. (8.30a) corresponds to the expression for the torsion as given in Eq. (6.8a), whereas Eq. (8.30b) corresponds to the expression for the curvature as given in Eq. (6.8b).

The infinitesimal versions of Eqs. (8.24a, 8.24b) read

$$E_A^{(Lh)} = \Lambda_A{}^B(h^{-1})E_B^{(L)} \tag{8.31a}$$

$$W_R^{(Lh)}(\mathrm{Ad}_{h(p)^{-1}}A) = h(p)^{-1}W_R^{(L)}(A)h(p) + h(p)^{-1}\pi_*A^\sharp h(p). \tag{8.31b}$$

In the case where $A \in \mathfrak{k}$ we find using $\pi_*((\mathrm{Ad}_{h(p)^{-1}}A)^\sharp\big|_{L(p)h(p)}) = \pi_*(A^\sharp\big|_{L(p)})$, see Eq. (5.3), that Eq. (8.31b) gives the transformation of the spin connection under gauge transformations, cf. Eq. (4.16).

8.1.3 Composition rules of mixed left and right actions

Finally let us consider the successive action of $L_g^{(L)}$ and $R_g^{(L)}$ on $L(p)$. Proceeding in the same way as when deriving the composition rules for the left H-compensators, Eqs. (8.15a, 8.15b), we find

$$l_{g_1} \circ r_{g_2}^{(L)}(p) = r^{(L)}_{\mathrm{Ad}_{\tilde{h}_L^{(L)}(p,g_1)}g_2} \circ l_{g_1}(p) \tag{8.32a}$$

$$\tilde{h}_R^{(L)}(l_{g_1}(p), \mathrm{Ad}_{\tilde{h}_L^{(L)}(p,g_1)}g_2) = \tilde{h}_L^{(L)}(r_{g_2}^{(L)}(p), g_1)\tilde{h}_R^{(L)}(p, g_2)\tilde{h}_L^{(L)}(p, g_1)^{-1}. \tag{8.32b}$$

The infinitesimal versions of Eq. (8.32a) and Eq. (8.32b) with $g_1 = e^{\epsilon_1 A}$ and $g_2 = e^{\epsilon_2 B}$ read

$$[\pi_* A^\flat, \pi_* B^\sharp] = \pi_*[W_L^{(L)}(A), B]^\sharp \tag{8.33a}$$

$$[W_L^{(L)}(A), W_R^{(L)}(B)] - \pi_* B^\sharp [W_L^{(L)}(A)] = W_R^{(L)}([W_L^{(L)}(A), B]) - \pi_* A^\flat [W_R^{(L)}(B)]. \tag{8.33b}$$

9 Isometries

When studying the geometry of superspace we do not, in general, have a definition of a metric on the superspace [3, 4]. Thus the notion of isometry from ordinary geometry, i.e. transformations which leave the metric invariant, cannot be carried over to supergeometry. Instead we must work with a definition of isometries based on the geometrical objects at hand, that is the frame and connection. Imposing that the frame and connection remain invariant under an isometry turns out to be too restrictive and must be relaxed by demanding them to be invariant only up to a gauge transformation, see e.g. [12]. This is required to be a single gauge transformation transforming frame and connection together, not independent transformations for each quantity. For instance, the map $f : G/H \to G/H$ will be an isometry if

$$f^*\left(E^A_{(L)}\big|_{f(p)}\right) = E^A_{(Lh)}\big|_p = E^B_{(L)}\big|_p \Lambda_B{}^A(h) \tag{9.1a}$$

$$f^*\left(\omega^{(L)}\big|_{f(p)}\right) = \omega^{(Lh)}\big|_p = h^{-1}\omega^{(L)}\big|_p h + h^{-1}\mathrm{d}h, \tag{9.1b}$$

for some $h \in H$ which, as a gauge transformation, need not be constant. It can be shown that if we impose such a condition on the map f for one choice of local section L then it will automatically be satisfied for other choices of L with a different value for the gauge transformation h.

As stated earlier the left action L_g on G induces a map l_g on the coset space G/H. Such maps can be thought of as isometries of the coset space, as can be demonstrated by considering how the frame and connection transform under the action of l_g. Recall from Eq. (6.1) that the pullback of the Maurer-Cartan form under a local section provides us with both the local coframe and connection. Thus we will consider how $L^*\zeta$ behaves under a pullback by l_g.

Consider a curve in the base $\gamma : [0,1] \to G/H$ with tangent vector X and $\gamma(0) = p$. Using a matrix representation we have

$$L^*\big(\zeta|_{L(p)}\big)\big(X|_p\big) = L(p)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}L(\gamma(t))\Big|_{t=0}. \tag{9.2}$$

Now the curve $l_g \circ \gamma$ has tangent vector $l_{g_*}X$ and thus

$$\begin{aligned} L^*\big(\zeta|_{L(l_g(p))}\big)\big(l_{g_*}(X|_p)\big) &= L(l_g(p))^{-1}\frac{\mathrm{d}}{\mathrm{d}t}L(l_g(\gamma(t)))\Big|_{t=0} \\ &= L(l_g(p))^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(gL(\gamma(t))\tilde{h}_L^{(L)}(\gamma(t),g)^{-1}\right)\Big|_{t=0} \\ &= \tilde{h}_L^{(L)}(p,g)L(p)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(L(\gamma(t))\tilde{h}_L^{(L)}(\gamma(t),g)^{-1}\right)\Big|_{t=0} \end{aligned}$$

$$= \tilde{h}_L^{(L)}(p,g)L^*\big(\zeta|_{L(p)}\big)\big(X|_p\big)\tilde{h}_L^{(L)}(p,g)^{-1} + \tilde{h}_L^{(L)}(p,g)\mathrm{d}\tilde{h}_L^{(L)}(p,g)^{-1}\big(X|_p\big).$$

Here we have used Eq. (8.10) to obtain the second and third lines, the final line follows from evaluating the derivative and using Eq. (9.2). From this we thus see that

$$l_g^* L^*\big(\zeta|_{L(l_g(p))}\big) = \mathrm{Ad}_{\tilde{h}_L^{(L)}(p,g)} L^*\big(\zeta|_{L(p)}\big) + \tilde{h}_L^{(L)}(p,g)\mathrm{d}\tilde{h}_L^{(L)}(p,g)^{-1}. \tag{9.3}$$

If we now decompose this equation into its $\mathfrak{k}$ and $\mathfrak{h}$ parts, we find

$$l_g^*\big(E_{(L)}^A|_{l_g(p)}\big) = E_{(L)}^B|_p \Lambda_B{}^A\big(\tilde{h}_L^{(L)}(p,g)^{-1}\big), \tag{9.4a}$$

$$l_g^*\big(\omega^{(L)}|_{l_g(p)}\big) = \mathrm{Ad}_{\tilde{h}_L^{(L)}(p,g)}\omega^{(L)}|_p + \tilde{h}_L^{(L)}(p,g)\mathrm{d}\tilde{h}_L^{(L)}(p,g)^{-1}. \tag{9.4b}$$

Comparing with Eqs. (9.1a, 9.1b) we see that the pulled back coframe and connection from $l_g(p)$ to p are simply a gauge transform of the coframe and connection already at p; the parameter of the gauge transformation is given by the H-compensator $\tilde{h}_L^{(L)}(p,g)$.

From how the coframe transforms under the pullback by l_g we may deduce how the frame transforms under the pushforward by l_g. We find

$$l_{g*}\big(E_A^{(L)}|_p\big) = \Lambda_A{}^B\big(\tilde{h}_L^{(L)}(p,g)^{-1}\big)E_B^{(L)}|_{l_g(p)}. \tag{9.5}$$

The maps l_g therefore are isometries of G/H. They may be composed as in Eq. (8.15a) and thus form a group of isometries isomorphic to G. The maps l_g may, however, not be all the isometries. For instance, in the case when the normalizer $N(H) = \{g \in G : gHg^{-1} = H\}$ is non-trivial then right multiplication in G by an element $g \in N(H)$ results in a well defined map on the coset space which, if non-trivial, i.e. $g \notin H$, also satisfies the conditions for it to be an isometry. For a more detailed discussion of this see [9, 13].

For an infinitesimal isometry given by l_g with $g = e^{\epsilon A}$ we have the associated supervector field $\pi_* A^\flat$, c.f. Eq. (8.7). The supervectors $\pi_* A^\flat$ are therefore Killing supervectors for the coset space, i.e. supervectors which give infinitesimal isometries. We see from Eq. (8.20) that the Killing supervectors satisfy an algebra. The set of $\pi_* T_p^\flat$, $p = 1, \ldots, \dim \mathfrak{g}$, form a set of independent Killing supervectors, and from Eq. (8.20) satisfy

$$[\pi_* T_p^\flat, \pi_* T_q^\flat] = -f_{pq}{}^r \pi_* T_r^\flat. \tag{9.6}$$

The definition of Killing supervectors will be discussed in more detail in Section 11.

10 Derivations on associated bundles

In this section we shall consider derivations, – such as the covariant derivative, the Lie derivative and the so-called H-covariant Lie derivative – on associated bundles. A derivation, or more precisely a graded derivation, is here defined as a linear map on an abstract algebra satisfying the graded Leibniz rule.

10.1 Covariant derivative

In the following we shall define the covariant derivative $\nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)}$ on a local section $s(p) = [(L(p), \xi(p))]$ of a general associated bundle in the direction of the push-forward of an even horizontal vector $K^\sharp$ in terms of the map $R_g^{(L)}$. Before we do this let us, however, first review the standard definition of the covariant derivative on associated bundles, see [6].

Consider a curve $\gamma(t)$ in the base with $t \in [0,1]$ and $\gamma(0) = p$. We can then write, using Eq. (4.10),

$$\begin{aligned} s(\gamma(t)) &= \Big[\Big(L(\gamma(t)), \xi(\gamma(t))\Big)\Big] \\ &= \Big[\Big(\tilde{\gamma}(t)\tilde{h}(t)^{-1}, \xi(\gamma(t))\Big)\Big] \\ &= \Big[\Big(\tilde{\gamma}(t), \rho(\tilde{h}(t)^{-1})\xi(\gamma(t))\Big)\Big]. \end{aligned} \tag{10.1}$$

Now, setting $\eta(\gamma(t)) \equiv \rho(\tilde{h}(t)^{-1})\xi(\gamma(t))$, we have for the standard definition of the covariant derivative

$$\nabla_X s(p) \stackrel{\text{def}}{=} \left[\left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t}\eta(\gamma(t))\Big|_{t=0}\right)\right], \tag{10.2}$$

where X is the tangent vector to $\gamma(t)$ at p. From this we see that a local section $s(\gamma(t))$ is parallel transported along $\gamma(t)$ if η is constant along $\gamma(t)$. It is easy to see that the covariant derivative does not depend on the specific choice of horizontal lift $\tilde{\gamma}(t)$.

On the other hand we shall now see that one can also define the covariant derivative of a local section $s(p)$ in terms of the map $R_g^{(L)}$. We will define

$$\nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)} s(p) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\left(\tilde{R}^{(L)}_{e^{\epsilon K}}\right)^{-1} s(r^{(L)}_{e^{\epsilon K}}(p)) - s(p)\right). \tag{10.3}$$

Using the definition of the map $\tilde{R}_g^{(L)}$ on $s(p)$, see Eq. (8.9c), this can be rewritten as

$$\begin{aligned} \nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)} s(p) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,(R^{(L)}_{e^{\epsilon K}})^{-1}L(r^{(L)}_{e^{\epsilon K}}(p))\,,\,\xi(r^{(L)}_{e^{\epsilon K}}(p))\,\Big)\Big] \\ &\qquad - \Big[\Big(L(p), \xi(p)\Big)\Big]\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,R^{(L)}_{\mathrm{Ad}_{\tilde{h}_R^{(L)}(p,e^{\epsilon K})}e^{-\epsilon K}}L(r^{(L)}_{e^{\epsilon K}}(p))\,,\,\xi(r^{(L)}_{e^{\epsilon K}}(p))\,\Big)\Big] \\ &\qquad - \Big[\Big(L(p), \xi(p)\Big)\Big]\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,L(p)\tilde{h}_R^{(L)}(p,e^{\epsilon K})^{-1}\,,\,\xi(r^{(L)}_{e^{\epsilon K}}(p))\,\Big)\Big] \\ &\qquad - \Big[\Big(L(p), \xi(p)\Big)\Big]\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\Big[\Big(\,L(p)\,,\,\rho(\tilde{h}_R^{(L)}(p,e^{\epsilon K})^{-1})\xi(r^{(L)}_{e^{\epsilon K}}(p)) - \xi(p)\,\Big)\Big]\right) \\ &= \left[\left(L(p), \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(\tilde{h}_R^{(L)}(p,e^{tK})^{-1})\xi(r^{(L)}_{e^{tK}}(p))\right)\Big|_{t=0}\right)\right]. \end{aligned} \tag{10.4}$$

Note that in deriving this we have used

$$(R_g^{(L)})^{-1}(L(p)h) = R^{(L)}_{\mathrm{Ad}_{\tilde{h}_R^{(L)}(p,g)}g^{-1}}(L(p)h)$$

as well as the equivalence relation Eq. (7.2). Noting that $\tilde{\gamma}(0) = L(p)$ and $\tilde{h} = \tilde{h}_R^{(L)}$ we easily see from Eq. (10.4) that our definition of the covariant derivative, Eq. (10.3), is – in the specific case of the vector $\pi_*\left(K^\sharp|_{L(p)}\right)$ – equivalent to the standard definition given by Eq. (10.2). Note that although the range of the map $R^{(L)}_{e^{\epsilon K}}$ is considered to be restricted to $\pi^{-1}(U)$ this does not pose a problem for our definition of the covariant derivative as ϵ can always be chosen sufficiently small such that $R^{(L)}_{e^{\epsilon K}}L(p) \in \pi^{-1}(U)$.

Now, from Eq. (10.4) we easily find

$$\begin{aligned}\nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)}s(p) &= \Big[\Big(\, L(p)\,,\, \frac{\mathrm{d}}{\mathrm{d}t}\xi(r^{(L)}_{e^{tK}}(p))\Big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\rho(\tilde{h}_R^{(L)}(p,e^{tK})^{-1})\Big|_{t=0}\xi(p)\,\Big)\Big] \\ &= \Big[\Big(\, L(p)\,,\, \pi_*(K^\sharp|_{L(p)})[\xi(p)] + \rho\big(\omega^{(L)}(\pi_*(K^\sharp|_{L(p)}))\big)\xi(p)\,\Big)\Big].\end{aligned} \tag{10.5}$$

Here we consider $\omega^{(L)}$ in the representation appropriate for acting on $\xi(p)$. It is easy to see that the covariant derivative is invariant under the equivalence transformations on associated bundles, see Eq. (7.1). We have

$$\begin{aligned}&\nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)}[(L(p)h, \rho(h^{-1})\xi(p))] \\ &\qquad = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,(R^{(L)}_{e^{\epsilon K}})^{-1}\big(L(r^{(L)}_{e^{\epsilon K}}(p))h\big)\,,\, \rho(h^{-1})\xi(r^{(L)}_{e^{\epsilon K}}(p))\,\Big)\Big] \\ &\qquad\qquad - \Big[\Big(L(p)h, \rho(h^{-1})\xi(p)\Big)\Big]\Big) \\ &\qquad = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\, L(p)\tilde{h}_R^{(L)}(p,e^{\epsilon K})^{-1}h\,,\, \rho(h^{-1})\xi(r^{(L)}_{e^{\epsilon K}}(p))\,\Big)\Big] \\ &\qquad\qquad - \Big[\Big(L(p)h, \rho(h^{-1})\xi(p)\Big)\Big]\Big) \\ &\qquad = \nabla_{\pi_*\left(K^\sharp|_{L(p)}\right)}[(L(p), \xi(p))].\end{aligned}$$

The covariant derivative of $s(p)$ is therefore, as a section, well defined.

Using $\pi_*\left(K^\sharp|_{L(p)}\right) = X^A E_A^{(L)}$ and $\nabla_{E_A^{(L)}} \equiv \nabla_A^{(L)}$, we have

$$\nabla_A^{(L)}s(p) = \Big[\Big(\, L(p)\,,\, E_A^{(L)}[\xi(p)] + \rho\big(\omega_A^{(L)}\big)\xi(p)\,\Big)\Big], \tag{10.6}$$

which will transform under a change of local section as

$$\nabla_A^{(L)}s(p) = \Lambda_A{}^B(h)\nabla_B^{(Lh)}s(p). \tag{10.7}$$

From Eq. (10.6) we see that our definition of the covariant derivative, Eq. (10.3), gives us an expression for the covariant derivative in the direction of the basis vectors $E_A^{(L)}$. As such it extends to a derivative in the direction of an arbitrary vector field $X = X^A E_A^{(L)}$, although our initial definition was only given for vector fields $X = X^A E_A^{(L)}$ with X^A constant. Note that for convenience of notation we shall for the rest of this section drop the explicit $L(p)$ dependence on the vectors $\pi_*(K^\sharp\big|_{L(p)})$ as well as on $E_A^{(L)}$.

As a practical example let us consider calculating the covariant derivative of a local section of a contravariant vector bundle $X(p) = [(L(p), \{X^A\})] = X^A E_A$. We have

$$\begin{aligned}
\nabla_{\pi_* K^\sharp} X(p) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,L(p)\tilde{h}_R^{(L)}(p, e^{\epsilon K})^{-1}\,,\,\Big\{X^A(r^{(L)}_{e^{\epsilon K}}(p))\Big\}\,\Big)\Big] \\
&\qquad - \Big[\Big(L(p), \{X^A(p)\}\Big)\Big]\Big) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(\,L(p)\,,\,\Big\{X^A(r^{(L)}_{e^{\epsilon K}}(p))\Lambda_A{}^B(\tilde{h}_R^{(L)}(p, e^{\epsilon K}))\Big\}\,\Big)\Big] \\
&\qquad - \Big[\Big(L(p), \{X^A(p)\}\Big)\Big]\Big),
\end{aligned} \tag{10.8}$$

where we have used the equivalence relation for tensor bundles, see Eqs. (7.17, 7.18). Using Eqs. (8.26, 6.5) we can expand $\Lambda_A{}^B(\tilde{h}_R^{(L)}(p, e^{\epsilon K}))$ as

$$\Lambda_A{}^B(\tilde{h}_R^{(L)}(p, e^{\epsilon K})) = \delta_A{}^B + \epsilon\omega^I(\pi_* K^\sharp) f_{IA}{}^B + \mathcal{O}(\epsilon^2),$$

where we have also dropped the $L(p)$ dependence on ω. Using Eq. (8.6) we can expand $X^A(r^{(L)}_{e^{\epsilon K}}(p))$ as

$$X^A(r^{(L)}_{e^{\epsilon K}}(p)) = X^A(p) + \epsilon\pi_* K^\sharp[X^A(p)] + \mathcal{O}(\epsilon^2),$$

which allows us to rewrite Eq. (10.8) as

$$\nabla_{\pi_* K^\sharp} X(p) = \Big[\Big(L(p), \Big\{\pi_* K^\sharp[X^A(p)] + X^B(p)\omega^I(\pi_* K^\sharp) f_{IB}{}^A\Big\}\Big)\Big]. \tag{10.9}$$

Finally let us consider the commutator of two covariant derivatives

$$\begin{aligned}
[\nabla_{\pi_* K_1^\sharp}, \nabla_{\pi_* K_2^\sharp}]s(p) &= \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\frac{1}{\epsilon_1\epsilon_2}\Big(\Big[\Big(L(p), \\
&\rho(\tilde{h}_R^{(L)}(p, e^{\epsilon_1 K_1})^{-1})\rho(\tilde{h}_R^{(L)}(r_{e^{\epsilon_1 K_1}}(p), e^{\epsilon_2 K_2})^{-1})\xi(r_{e^{\epsilon_2 K_2}}(r_{e^{\epsilon_1 K_1}}(p))) \\
&- \rho(\tilde{h}_R^{(L)}(p, e^{\epsilon_2 K_2})^{-1})\rho(\tilde{h}_R^{(L)}(r_{e^{\epsilon_2 K_2}}(p), e^{\epsilon_1 K_1})^{-1})\xi(r_{e^{\epsilon_1 K_1}}(r_{e^{\epsilon_2 K_2}}(p)))\Big)\Big]\Big).
\end{aligned}$$

Expanding and taking the limit we have

$$\begin{aligned}
[\nabla_{\pi_* K_1^\sharp}, \nabla_{\pi_* K_2^\sharp}]s(p) &= \Big[\Big(L(p),\ [\omega(\pi_* K_1^\sharp), \omega(\pi_* K_2^\sharp)]\xi(p) \\
&\quad + [\pi_* K_1^\sharp, \pi_* K_2^\sharp]\xi(p) + \pi_* K_1^\sharp[\omega(\pi_* K_2^\sharp)]\xi(p) - \pi_* K_2^\sharp[\omega(\pi_* K_1^\sharp)]\xi(p)\Big)\Big].
\end{aligned}$$

Note that for the ease of notation we have suppressed the representation ρ on $\omega(\pi_* K_2^\sharp)$. Now, using the expressions for the curvature as given in Eq. (6.9), we can rewrite this last equation as

$$\begin{aligned}
[\nabla_{\pi_* K_1^\sharp}, \nabla_{\pi_* K_2^\sharp}]s(p) &= \Big[\Big(L(p), \\
&R(\pi_* K_1^\sharp, \pi_* K_2^\sharp)\xi(p) + [\pi_* K_1^\sharp, \pi_* K_2^\sharp]\xi(p) + \omega([\pi_* K_1^\sharp, \pi_* K_2^\sharp])\xi(p)\Big)\Big]
\end{aligned}$$

and hence we finally have, using the expression for the covariant derivative as given in Eq. (10.5),

$$[\nabla_{\pi_* K_1^\sharp}, \nabla_{\pi_* K_2^\sharp}] s(p) = R(\pi_* K_1^\sharp, \pi_* K_2^\sharp) s(p) + \nabla_{[\pi_* K_1^\sharp, \pi_* K_2^\sharp]} s(p), \tag{10.10}$$

where $R(\pi_* K_1^\sharp, \pi_* K_2^\sharp) s(p)$ is an abbreviation for the action of $\tilde{R}_H^{(L)}$ on the section $s(p)$ with $H = R(\pi_* K_1^\sharp, \pi_* K_2^\sharp)$. Now, setting $\pi_* K_1^\sharp = X^A E_A$ and $\pi_* K_2^\sharp = Y^A E_A$ we can rewrite the left hand side of this equation as

$$\begin{aligned}
[\nabla_{X^A E_A}, \nabla_{Y^B E_B}] s(p) &= X^A (\nabla_A Y)^B \nabla_B s(p) - Y^B (\nabla_B X)^A \nabla_A s(p) \\
&\quad + Y^B X^A [\nabla_A, \nabla_B] s(p) \\
&= Y^B X^A [\nabla_A, \nabla_B] s(p) \\
&\quad - Y^B X^A \left(\omega_{AB}{}^C \nabla_C + (-1)^{AB} \omega_{BA}{}^C \nabla_C \right) s(p),
\end{aligned}$$

where we have used Eq. (10.9) for constant X. The right hand side of Eq. (10.10) can be rewritten as

$$R(X^A E_A, Y^B E_B) + \nabla_{[X^A E_A, Y^B E_B]} = Y^B X^A \left(R_{AB} + \mathcal{C}_{AB}{}^C \nabla_C \right)$$

and we hence find in total, using the component expression of the torsion, Eq. (6.8a),

$$[\nabla_A, \nabla_B] = R_{AB} + T_{AB}{}^C \nabla_C. \tag{10.11}$$

10.2 Lie derivative

In this section we will introduce the Lie derivative on a local section $s(p)$ in the direction of an isometry. We define

$$\mathcal{L}_{\pi_* A^\flat} s(p) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left(\tilde{L}_{e^{\epsilon A}} \right)^{-1} s(l_{e^{\epsilon A}}(p)) - s(p) \right). \tag{10.12}$$

We can rewrite the Lie derivative as

$$\begin{aligned}
\mathcal{L}_{\pi_* A^\flat} s(p) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left[\left(L_{e^{-\epsilon A}} \left(L(l_{e^{\epsilon A}}(p)) \right), \xi \left(l_{e^{\epsilon A}}(p) \right) \right) \right] - \left[\left(L(p), \xi(p) \right) \right] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left[\left(e^{-\epsilon A} L(l_{e^{\epsilon A}}(p)), \xi \left(l_{e^{\epsilon A}}(p) \right) \right) \right] - \left[\left(L(p), \xi(p) \right) \right] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left[\left(L(p) \tilde{h}_L^{(L)}(p, e^{-\epsilon A}), \xi \left(l_{e^{\epsilon A}}(p) \right) \right) \right] - \left[\left(L(p), \xi(p) \right) \right] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\left(L(p), \rho(\tilde{h}_L^{(L)}(p, e^{-\epsilon A})) \xi \left(l_{e^{\epsilon A}}(p) \right) - \xi(p) \right) \right] \\
&= \left[\left(L(p), \frac{\mathrm{d}}{\mathrm{d}t} \left(\rho(\tilde{h}_L^{(L)}(p, e^{-tA})) \xi(l_{e^{tA}}(p)) \right) \Big|_{t=0} \right) \right].
\end{aligned} \tag{10.13}$$

One should note the similarity to the expression of the covariant derivative given in Eq. (10.4). Note also that, although the range of the map $L_{e^{\epsilon A}}$ is considered to be restricted to $\pi^{-1}(U)$, this does not pose a problem for our definition of the Lie derivative as ϵ can always be chosen sufficiently small such that $L_{e^{\epsilon A}}L(p) \in \pi^{-1}(U)$. Now we easily find

$$\begin{aligned}\mathcal{L}_{\pi_* A^\flat} s(p) &= \Big[\Big(L(p)\,,\ \frac{\mathrm{d}}{\mathrm{d}t}\xi(l_{e^{tA}}(p))\Big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\rho(\tilde{h}_L^{(L)}(p, e^{-tA}))\Big|_{t=0}\xi(p)\Big)\Big] \\ &= \Big[\Big(L(p)\,,\ \pi_* A^\flat[\xi(p)] + \rho\big(W_L^{(L)}(A)\big)\xi(p)\Big)\Big], \end{aligned} \tag{10.14}$$

where we have used Eqs. (8.7, 8.18). Here we consider $W_L^{(L)}$ to be in the representation appropriate for acting on $\xi(p)$. It is easy to see that the Lie derivative is invariant under the equivalence transformations $[(L(p), \xi(p))] = [(L(p)h, \rho(h^{-1})\xi(p))]$. The Lie derivative of $s(p)$ is therefore, as a section, well defined. Again for the ease of notation we shall for the rest of this section drop the explicit $L(p)$ on the $E_A^{(L)}$.

As an example we shall now calculate the Lie derivative of a vector and a one-form, respectively. First consider a local section of a vector bundle $X(p) = [(L(p), \{X^A\})] = X^A E_A$. We have

$$\begin{aligned}\mathcal{L}_{\pi_* A^\flat} X(p) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L(p)\tilde{h}_L^{(L)}(p, e^{-\epsilon A})\,,\ \Big\{X^A(l_{e^{\epsilon A}}(p))\Big\}\Big)\Big] \\ &\qquad - \Big[\Big(L(p), \{X^A(p)\}\Big)\Big]\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L(p)\,,\ \Big\{X^A(l_{e^{\epsilon A}}(p))\Lambda_A{}^B(\tilde{h}_L^{(L)}(p, e^{-\epsilon A})^{-1})\Big\}\Big)\Big] \\ &\qquad - \Big[\Big(L(p), \{X^A(p)\}\Big)\Big]\Big), \end{aligned} \tag{10.15}$$

where we have used the equivalence relation for tensor bundles, see Eqs. (7.17, 7.18). Using Eqs. (8.19, 6.5) we can expand $\Lambda_A{}^B(\tilde{h}_L^{(L)}(p, e^{-\epsilon A})^{-1})$ as

$$\Lambda_A{}^B(\tilde{h}_L^{(L)}(p, e^{-\epsilon A})^{-1}) = \delta_A{}^B + \epsilon W_L^I f_{IA}{}^B + \mathcal{O}(\epsilon^2),$$

where we have now also dropped the $L(p)$ dependence on W_L. Using Eq. (8.7) we can expand $X^A(l_{e^{\epsilon A}}(p))$ as

$$X^A(l_{e^{\epsilon A}}(p)) = X^A(p) + \epsilon\pi_* A^\flat[X^A(p)] + \mathcal{O}(\epsilon^2),$$

which allows us to rewrite Eq. (10.15) as

$$\mathcal{L}_{\pi_* A^\flat} X(p) = \Big[\Big(L(p), \Big\{\pi_* A^\flat[X^A(p)] + X^B(p) W_L^I(A) f_{IB}{}^A\Big\}\Big)\Big]. \tag{10.16}$$

In the particular case of the Lie derivative of the basis vector E_A we find from this

$$\begin{aligned}\mathcal{L}_{\pi_* A^\flat} E_A &= \Big[\Big(L(p), \Big\{\delta_A{}^B W_L^I(A) f_{IB}{}^C\Big\}\Big)\Big] \\ &= \Big[\Big(L(p), \Big\{W_L^I(A) f_{IA}{}^B \delta_B{}^C\Big\}\Big)\Big] \\ &= W_L^I(A) f_{IA}{}^B E_B. \end{aligned} \tag{10.17}$$

By a similar calculation as in the case of the Lie derivative of a vector we find for the Lie derivative of a one-form $\phi(p) = [(L(p), \{\phi_A\})] = E^A\phi_A$

$$\mathcal{L}_{\pi_* A^\flat}\phi(p) = \Big[\Big(L(p), \Big\{\pi_* A^\flat[\phi_A(p)] - W_L^I(A) f_{IA}{}^B \phi_B(p)\Big\}\Big)\Big] \tag{10.18}$$

and hence in the particular case of the Lie derivative of the basis vector E^A

$$\begin{aligned}\mathcal{L}_{\pi_* A^\flat} E^A &= \Big[\Big(L(p), \Big\{ - W_L^I(A) f_{IB}{}^C \delta_C{}^A(p)\Big\}\Big)\Big] \\ &= \Big[\Big(L(p), \Big\{ - \delta_B{}^C W_L^I(A) f_{IC}{}^A\Big\}\Big)\Big] \\ &= -E^C W_L^I(A) f_{IC}{}^A.\end{aligned} \tag{10.19}$$

Now, in the case of the connection one-from ω we find from Eq. (10.18) together with the composition rule of the right and left H-compensators, see Eq. (8.33b),

$$\begin{aligned}\mathcal{L}_{\pi_* A^\flat}\omega(p) &= \Big[\Big(L(p), \Big\{\pi_* A^\flat[\omega_A(p)] - W_L^I(A) f_{IA}{}^B \omega_B(p)\Big\}\Big)\Big] \\ &= \Big[\Big(L(p), \Big\{E_A[\rho\big(W_L(A)\big)] - [\rho\big(W_L(A)\big), \omega_A]\Big\}\Big)\Big].\end{aligned} \tag{10.20}$$

Now consider the algebra of two Lie derivatives. We find, writing for simplicity $W_L(A)$ instead of $\rho\big(W_L(A)\big)$,

$$\begin{aligned}[\mathcal{L}_{\pi_* A^\flat}, \mathcal{L}_{\pi_* B^\flat}]s(p) = \Big[\Big(L(p)\ ,\ &[\pi_* A^\flat, \pi_* B^\flat]\xi(p) \\ &+ (\pi_* A^\flat[W_L(B)] - \pi_* B^\flat[W_L(A)] + [W_L(A), W_L(B)])\xi(p)\Big)\Big]\end{aligned}$$

and hence, using Eq. (8.21), we find

$$[\mathcal{L}_{\pi_* A^\flat}, \mathcal{L}_{\pi_* B^\flat}]s(p) = \Big[\Big(L(p)\ ,\ [\pi_* A^\flat, \pi_* B^\flat]\xi(p) + W_L([B, A])\xi(p)\Big)\Big]. \tag{10.21}$$

Now, noting that the algebra element corresponding to the vector $[\pi_* A^\flat, \pi_* B^\flat] = \pi_*[B, A]^\flat$ is given by $[B, A]$, cf. Eq. (8.20), we find

$$[\mathcal{L}_{\pi_* A^\flat}, \mathcal{L}_{\pi_* B^\flat}] = \mathcal{L}_{[\pi_* A^\flat, \pi_* B^\flat]}. \tag{10.22}$$

Finally let us consider the commutator of the Lie derivative with the covariant derivative on a local section $s(p) = [(L(p), \xi(p))]$. We find

$$\begin{aligned}[\mathcal{L}_{\pi_* A^\flat}, \nabla_{\pi_* B^\sharp}]s(p) &= \Big[\Big(L(p)\,,\ [\pi_* A^\flat, \pi_* B^\sharp]\xi(p) + \pi_* A^\flat[\omega(\pi_* B^\sharp)]\xi(p) \\ &\qquad - \pi_* B^\sharp[W_L(A)]\xi(p) + [W_L(A), \omega(\pi_* B^\sharp)]\xi(p)\Big)\Big] \\ &= \Big[\Big(L(p)\,,\ [\pi_* A^\flat, \pi_* B^\sharp]\xi(p) + \omega([\pi_* A^\flat, \pi_* B^\sharp])\xi(p)\Big)\Big] \\ &= \nabla_{[\pi_* A^\flat, \pi_* B^\sharp]} s(p),\end{aligned}$$

where we have used Eqs. (8.33a, 8.33b). We thus have

$$[\mathcal{L}_{\pi_* A^\flat}, \nabla_{\pi_* B^\sharp}]s(p) = \nabla_{(\mathcal{L}_{\pi_* A^\flat}\pi_* B^\sharp)} s(p). \tag{10.23}$$

Now, setting $\pi_* B^\sharp = X^A E_A$, X^A constant, we have, evaluating the left hand side of Eq. (10.23) directly

$$[\mathcal{L}_{\pi_* A^\flat}, \nabla_{\pi_* B^\sharp}]s(p) = (\mathcal{L}_{\pi_* A^\flat} \pi_* B^\sharp)^A \nabla_A s(p) + X^A [\mathcal{L}_{\pi_* A^\flat}, \nabla_A] s(p).$$

Combining this with Eq. (10.23) we finally find

$$[\mathcal{L}_{\pi_* A^\flat}, \nabla_A] = 0. \tag{10.24}$$

10.3 H-covariant Lie derivative

In this section we shall define the so-called H-covariant Lie derivative, the characteristic property of which is that when differentiating the frame it gives zero.

We define the H-covariant Lie derivative of a section $s(p)$ of a general tensor bundle in the direction of an isometry as

$$\mathbb{L}^{(L)}_{\pi_* A^\flat} s(p) \stackrel{\mathrm{def}}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left(\tilde{L}^{(L)}_{e^{\epsilon A}} \right)^{-1} s(l_{e^{\epsilon A}}(p)) - s(p) \right). \tag{10.25}$$

We can rewrite the H-covariant Lie derivative as

$$\begin{aligned}
\mathbb{L}^{(L)}_{\pi_* A^\flat} s(p) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\left[\left(L^{(L)}_{e^{-\epsilon A}} \left(L(l_{e^{\epsilon A}}(p)) \right), \xi \left(l_{e^{\epsilon A}}(p) \right) \right) \right] - \left[\left(L(p), \xi(p) \right) \right] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\Big[\Big(e^{-\epsilon A} L(l_{e^{\epsilon A}}(p)) \tilde{h}^{(L)}_L (l_{e^{\epsilon A}}, e^{-\epsilon A})^{-1}, \xi \left(l_{e^{\epsilon A}}(p) \right) \Big) \Big] \\
&\qquad - \Big[\Big(L(p), \xi(p) \Big) \Big] \Big) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\Big[\Big(e^{-\epsilon A} L(l_{e^{\epsilon A}}(p)) \tilde{h}^{(L)}_L (p, e^{\epsilon A}), \xi \left(l_{e^{\epsilon A}}(p) \right) \Big) \Big] \\
&\qquad - \Big[\Big(L(p), \xi(p) \Big) \Big] \Big) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\Big(L(p), \xi \left(l_{e^{\epsilon A}}(p) \right) - \xi(p) \Big) \Big] \\
&= \Big[\Big(L(p), \frac{\mathrm{d}}{\mathrm{d}t} \xi(l_{e^{tA}}(p)) \Big|_{t=0} \Big) \Big],
\end{aligned} \tag{10.26}$$

where we have used Eq. (8.11). Again note that, although the range of the map $L^{(L)}_{e^{\epsilon A}}$ is considered to be restricted to $\pi^{-1}(U)$, this does not pose a problem for our definition of the H-covariant Lie derivative as ϵ can always be chosen sufficiently small such that $L^{(L)}_{e^{\epsilon A}} L(p) \in \pi^{-1}(U)$. Now we immediately find from Eq. (10.26)

$$\mathbb{L}^{(L)}_{\pi_* A^\flat} s(p) = \Big[\Big(L(p)\, , \, \pi_* A^\flat [\xi(p)] \Big) \Big]. \tag{10.27}$$

It is easy to see that the H-covariant Lie derivative is invariant under the equivalence transformations on associated bundles, see Eq. (7.1). We have

$$
\begin{aligned}
&\mathbb{L}^{(L)}_{\pi_* A^\flat}[(L(p)h, \rho(h^{-1})\xi(p))] \\
&\qquad = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L^{(L)}_{e^{-\epsilon A}}\left(L(l_{e^{\epsilon A}}(p))h\right), \rho(h^{-1})\xi\left(l_{e^{\epsilon A}}(p)\right)\Big)\Big] \\
&\qquad\qquad - \Big[\Big(L(p)h, \rho(h^{-1})\xi(p)\Big)\Big]\Big) \\
&\qquad = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L(p)h, \rho(h^{-1})\xi\left(l_{e^{\epsilon A}}(p)\right)\Big)\Big] \\
&\qquad\qquad - \Big[\Big(L(p)h, \rho(h^{-1})\xi(p)\Big)\Big]\Big) \\
&\qquad = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\Big(L(p), \xi\left(l_{e^{\epsilon A}}(p)\right) - \xi(p)\Big)\Big] \\
&\qquad = \mathbb{L}^{(L)}_{\pi_* A^\flat}[(L(p), \xi(p))].
\end{aligned}
$$

On the other hand we find for the transformation of $\mathbb{L}^{(L)}_{\pi_* A^\flat}$ under gauge transformations

$$
\begin{aligned}
\mathbb{L}^{(Lh)}_{\pi_* A^\flat}s(p) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\Big[\Big(L^{(Lh)}_{e^{-\epsilon A}}\left(L(l_{e^{\epsilon A}}(p))\right), \xi\left(l_{e^{\epsilon A}}(p)\right)\Big)\Big] - \Big[\Big(L(p), \xi(p)\Big)\Big]\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L(p)h(p), \rho(h(l_{e^{\epsilon A}}(p))^{-1})\xi\left(l_{e^{\epsilon A}}(p)\right) - \rho(h(p)^{-1})\xi(p)\Big)\Big] \\
&= \tilde{R}^{(L)}_{h(p)}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Big[\Big(L(p), \rho(h(l_{e^{\epsilon A}}(p))^{-1})\xi\left(l_{e^{\epsilon A}}(p)\right) - \rho(h(p)^{-1})\xi(p)\Big)\Big] \\
&= \tilde{R}^{(L)}_{h(p)}\ \mathbb{L}^{(L)}_{\pi_* A^\flat}\left(\tilde{R}^{(L)}_{h(p)^{-1}}s(p)\right),
\end{aligned}
$$

i.e.

$$\mathbb{L}^{(Lh)}_{\pi_* A^\flat} = \tilde{R}^{(L)}_{h(p)}\ \mathbb{L}^{(L)}_{\pi_* A^\flat}\tilde{R}^{(L)}_{h(p)^{-1}}. \tag{10.28}$$

From Eq. (10.27) we easily find for the algebra of two H-covariant derivatives

$$[\mathbb{L}^{(L)}_{\pi_* A^\flat}, \mathbb{L}^{(L)}_{\pi_* B^\flat}] = \mathbb{L}^{(L)}_{[\pi_* A^\flat, \pi_* B^\flat]}. \tag{10.29}$$

Now, from Eq. (10.27) we immediately find that the H-covariant Lie derivative of the frame $E^{(L)}_A$ gives zero

$$\mathbb{L}^{(L)}_{\pi_* A^\flat}E^{(L)}_A = \Big[\Big(L(p), \left\{\pi_* A^\flat[\delta_A{}^B]\right\}\Big)\Big] = 0. \tag{10.30}$$

The H-covariant Lie derivative of the spin connection $\omega^{(L)}$ is given by

$$
\begin{aligned}
\mathbb{L}^{(L)}_{\pi_* A^\flat}\omega^{(L)} &= \Big[\Big(L(p), \left\{\pi_* A^\flat[\omega^{(L)}_A]\right\}\Big)\Big] \\
&= \Big[\Big(L(p), \left\{E^{(L)}_A[W^{(L)}_L(A)] - [W^{(L)}_L(A), \omega^{(L)}_A] + W^{(L)I}_L(A) f_{IA}{}^B\omega^{(L)}_B\right\}\Big)\Big],
\end{aligned}
\tag{10.31}
$$

where we have used the infinitesimal composition rule for the left and right H-compensators, Eq. (8.33b).

10.4 Comparison to the literature

H-covariant Lie derivatives have been mentioned in various places in the literature, see for example [1, 9]. In this section we shall show how our definition relates to those given in the literature. In order to do this we shall first derive a relation between the ordinary Lie derivative and the H-covariant Lie derivative on a general section of an associated bundle.

Recall the expression for the Lie derivative given in Eq. (10.14),

$$\mathcal{L}_{\pi_* A^\flat} s(p) = \Big[\Big(L(p)\, ,\, \pi_* A^\flat[\xi(p)] + \rho\big(W_L^{(L)}(A)\big)\xi(p) \Big)\Big].$$

Using the definition of the action of the algebra of H on a local section, see Eq. (7.7), and the expression for the H-covariant Lie derivative given in Eq. (10.27) we find from this

$$\mathcal{L}_{\pi_* A^\flat} s(p) = \big(\mathbb{L}^{(L)}_{\pi_* A^\flat} + \tilde{R}^{(L)}_{W_L^{(L)}(A)}\big) s(p). \tag{10.32}$$

From this we see that the Lie derivative consists of two parts, namely the H-covariant Lie derivative and a gauge transformation. Let us understand this in more detail.

First consider the H-covariant Lie derivative on a local section $s(p) = [(L(p), \xi(p))]$. This measures the difference between $\xi(l_{e^{tA}}(p))$ at $L(l_{e^{tA}}(p))$ and $\xi(p)$ at $L(p)$ by dragging $\xi(l_{e^{tA}}(p))$ back constantly along the curve $\gamma_1(s_1) = L^{(L)}_{e^{s_1 A}}(L(p))$, see Figure 1.

In contrast, the ordinary Lie derivative measures the difference between $\xi(l_{e^{tA}}(p))$ at $L(l_{e^{tA}}(p))$ and $\xi(p)$ at $L(p)$ by dragging $\xi(l_{e^{tA}}(p))$ back constantly along the curve $\gamma(s) = L_{e^{sA}}(L(p)\tilde{h}_L^{(L)}(p, e^{tA})^{-1})$. Alternatively this can be understood as dragging $\xi(l_{e^{tA}}(p))$ back constantly first along the local section $L(p)$, namely along the curve $\gamma_1(s_1)$, and then along the curve $\gamma_2(s_2) = R_{\tilde{h}_L^{(L)}(p, e^{s_2 A})}(L(p)\tilde{h}_L^{(L)}(p, e^{tA})^{-1})$, see Figure 1.

As such the Lie derivative splits up into two parts: The first one, the H-covariant Lie derivative, measures the change along the local section $L(p)$, the second one measures the change in the direction 'perpendicular' to the local section $L(p)$.

Now consider the H-covariant Lie derivative. This is given by, see Eq. (10.27),

$$\mathbb{L}^{(L)}_{\pi_* A^\flat} s(p) = [(L(p), \pi_* A^\flat[\xi(p)])].$$

Recall that $\xi(p) \in F$ stands for the components of the supertensor $s(p)$ with respect to the basis of the $E_A^{(L)}$. E.g., for $s(p) = X$ a vector, $\xi(p)$ stands for the components X^A of this vector in the basis $E_A^{(L)}$, see Section 7.3. As such $\pi_* A^\flat[\xi(p)]$ is equal to the Lie derivative of the *components* of the supertensor $s(p)$. E.g., in the case of $s(p) = X$ a vector, we have $\pi_* A^\flat[X^A(p)] = \ell_{\pi_* A^\flat} X^A(p)$, where we wrote the Lie derivative of the *scalar* quantity X^A as $\ell_{\pi_* A^\flat} X^A(p)$ in order to distinguish this from $(\mathcal{L}_{\pi_* A^\flat} X)^A$, i.e. from the Ath component of the Lie derivative of X. Note that the difference we make between the Lie derivative $\ell_{\pi_* A^\flat}$ and the usual Lie derivative $\mathcal{L}_{\pi_* A^\flat}$ is a purely notational one.

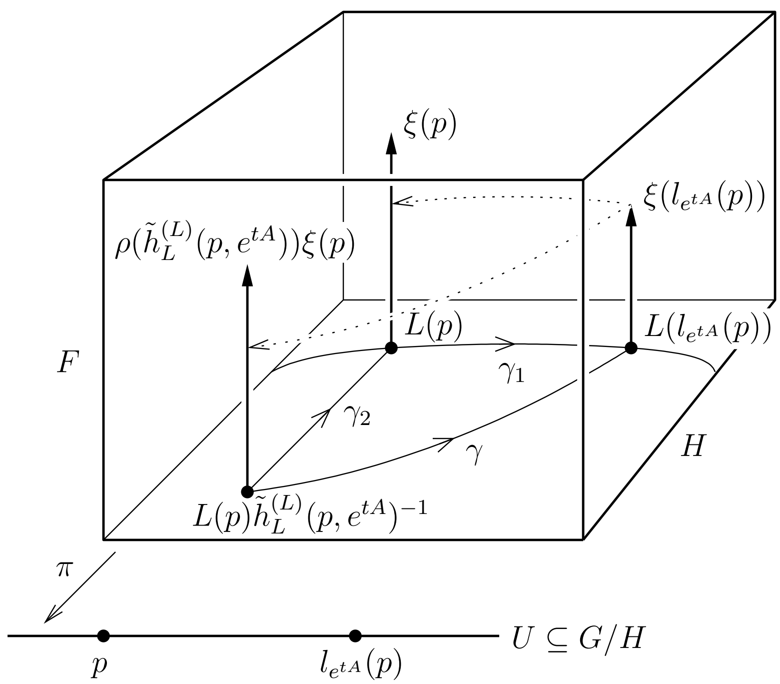

Figure 1: Illustration of the dragging operation associated with the Lie derivative and the H-covariant Lie derivative, respectively.

We can thus rewrite the H-covariant Lie derivative as

$$\mathbb{L}^{(L)}_{\pi_* A^\flat} s(p) = [(L(p), \ell_{\pi_* A^\flat}.\xi(p))] \tag{10.33}$$

Now let us consider the Lie derivative. Using the above notation we can define

$$\mathbb{l}^{(L)}_{\pi_* A^\flat} \stackrel{\text{def}}{=} \ell_{\pi_* A^\flat} + \rho\big(W_L^{(L)}(A)\big), \tag{10.34}$$

where $\rho\big(W_L^{(L)}(A)\big)$ transforms the tangent space indices in the way following from Eq. (7.18). This allows us to rewrite the Lie derivative of a local section $s(p)$ as

$$\mathcal{L}_{\pi_* A^\flat} s(p) = [(L(p), \mathbb{l}^{(L)}_{\pi_* A^\flat}\xi(p))]. \tag{10.35}$$

In the literature it is $\mathbb{l}^{(L)}_{\pi_* A^\flat}$, defined in Eq. (10.34), that is commonly referred to as the H-covariant Lie derivative. One should note the similarity of Eq. (10.34) to the expression for the Lie derivative in terms of the H-covariant Lie derivative, see Eq. (10.32).

As an example consider the Lie derivative and the H-covariant Lie derivative, respectively, of $s(p) = X$ a vector. We find

$$(\mathcal{L}_{\pi_* A^\flat} X)^A = \mathbb{l}^{(L)}_{\pi_* A^\flat}(X^A) \tag{10.36a}$$

$$(\mathbb{L}^{(L)}_{\pi_* A^\flat} X)^A = \ell_{\pi_* A^\flat}(X^A), \tag{10.36b}$$

where

$$\mathbb{I}^{(L)}_{\pi_* A^\flat}(X^A) = \pi_* A^\flat[X^A] + X^B W_L^{(L)I}(A) f_{IB}{}^A. \tag{10.36c}$$

Now consider the covariant derivative. We have, see Eq. (10.6),

$$\nabla^{(L)}_A s(p) = \Big[\Big(L(p)\,,\, E^{(L)}_A[\xi(p)] + \rho\big(\omega^{(L)}_A\big)\xi(p)\Big)\Big]$$

and setting

$$\mathcal{D}^{(L)}_A \stackrel{\text{def}}{=} E^{(L)}_A + \rho\big(\omega^{(L)}_A\big) \tag{10.37}$$

we can rewrite the covariant derivative as

$$\nabla^{(L)}_A s(p) = \Big[\Big(L(p)\,,\, \mathcal{D}^{(L)}_A \xi(p)\Big)\Big]. \tag{10.38}$$

Now, considering the commutator of the covariant derivative with the Lie derivative we have from Eq. (10.24)

$$[\mathcal{L}_{\pi_* A^\flat}, \nabla_{E_A}] s(p) = \Big[\Big(L(p), [\mathbb{I}^{(L)}_{\pi_* A^\flat}, \mathcal{D}^{(L)}_A]\xi(p)\Big)\Big] = 0$$

and hence

$$[\mathbb{I}^{(L)}_{\pi_* A^\flat}, \mathcal{D}^{(L)}_A] = 0. \tag{10.39}$$

11 Killing supervectors

We have, in Section 9, touched upon the notion of Killing supervectors, the supervectors associated to infinitesimal isometries. In this section we shall analyze in more detail how the Killing supervectors may be defined, particularly in terms of the derivations we discussed in Section 10.

11.1 The generalized Lie derivative

Let us first introduce what we shall call the *generalized Lie derivative*. Just as the Lie derivative and covariant derivative were constructed from the action of certain local bundle maps so will the generalized Lie derivative. The map we use will be a local bundle map of the form

$$\begin{aligned} F^{(L)}_t &: \pi^{-1}(U) \to \pi^{-1}(U) \\ &: L(p)h \mapsto L(f_t(p))\tilde{h}_t(p)^{-1}h. \end{aligned} \tag{11.1}$$

Here $f_t : U \to U$ is a one parameter family of maps on the base and the maps $\tilde{h}_t : U \to H$ are a one parameter family of H-valued functions. The map $F^{(L)}_t$ is required to satisfy $F^{(L)}_0 = \mathrm{id}_{\pi^{-1}(U)}$, and so, associated with the map is a supervector field on $\pi^{-1}(U)$ defined by

$$V|_g[f] = \frac{\mathrm{d}}{\mathrm{d}t} f(F^{(L)}_t(g))\Big|_{t=0}. \tag{11.2}$$

Following from the decomposition of the map $F^{(L)}_t$ as mentioned above we have a decomposition of the supervector field V as

$$V|_{L(p)} = L_*\big(X|_p\big) - H^\sharp|_{L(p)}. \tag{11.3}$$

Here $X|_p = \pi_*\big(V|_{L(p)}\big)$ is the supervector associated to the map f_t on the base and H is the algebra element associated to $\tilde{h}_t(p)$ given by $\tilde{h}_\epsilon(p) = 1 + \epsilon H(p) + \mathcal{O}(\epsilon^2)$. It can be shown that the supervector field V at points off the local section is given by

$$V|_{L(p)h} = R_{h*}\left(V|_{L(p)}\right). \tag{11.4}$$

As a consequence of this $\pi_* V$ is a well defined supervector field on the base manifold, without the need to specify from which point the supervector V was taken to originate, analogously to the situation for $\pi_* A^\flat$.

The map $F_t^{(L)}$ can then be extended to a map on an associated bundle analogously to Eqs. (8.9a–8.9c)

$$\tilde{F}_t^{(L)} : [(L(p), \xi(p))] \mapsto [(F_t^{(L)}(L(p)), \xi(p))]. \tag{11.5}$$

We then define the generalized Lie derivative of a (local) section $s(p)$ of the associated bundle by

$$\mathcal{K}_V s(p) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\left(\tilde{F}_\epsilon^{(L)}\right)^{-1} s(f_\epsilon(p)) - s(p)\right). \tag{11.6}$$

Note that here we have indexed the generalized Lie derivative with the supervector V. To see how $\mathcal{K}_V$ depends on the components X and H, first note that

$$\big(F_t^{(L)}\big)^{-1} L(f_t(p)) = L(p)\tilde{h}_t(p). \tag{11.7}$$

Using this we find from the definition of $\mathcal{K}_V$ that

$$\begin{aligned}
\mathcal{K}_V s(p) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\Big[\Big(L(p)\tilde{h}_\epsilon(p), \xi(f_\epsilon(p))\Big)\Big] - \Big[\Big(L(p), \xi(p)\Big)\Big]\right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(\Big[\Big(L(p), \xi(f_\epsilon(p)) - \xi(p)\Big)\Big] \\
&\qquad\qquad + \Big[\Big(L(p), (\rho(\tilde{h}_\epsilon(p)) - 1)\xi(f_\epsilon(p))\Big)\Big]\Big) \\
&= \big(\mathbb{L}_X^{(L)} + \tilde{R}_H^{(L)}\big)s(p).
\end{aligned} \tag{11.8}$$

In the last line we have introduced a generalization of the H-covariant Lie derivative, see Eq. (10.25), to act in the direction of an arbitrary supervector field X rather than just in the direction of a Killing supervector $\pi_* A^\flat$

$$\mathbb{L}_X^{(L)} s(p) \stackrel{\text{def}}{=} \big[\big(L(p), X[\xi(p)]\big)\big]. \tag{11.9}$$

Also we have used the map $\tilde{R}_H^{(L)}$ defined in Eq. (7.7) for the algebra element H defined by $\tilde{h}_\epsilon(p) = 1 + \epsilon H(p) + \mathcal{O}(\epsilon^2)$. Eq. (11.8) should be compared to the similar result for the Lie derivative, Eq. (10.32). Note that $\mathcal{K}_V$ may or may not depend on the local section, we will not explicitly indicate this dependence.

The generalized Lie derivatives can be shown to form an algebra

$$[\mathcal{K}_U, \mathcal{K}_V] = \mathcal{K}_{[U,V]}. \tag{11.10}$$

For the supervector U we will write the decomposition, Eq. (11.3), as $U = L_* X_U - H_U^\sharp$, and similarly for the supervectors V and $[U, V]$. Then, the algebra of the generalized Lie derivatives decomposes as

$$X_{[U,V]} = [X_U, X_V] \tag{11.11a}$$
$$H_{[U,V]} = X_U[H_V] - X_V[H_U] + [H_U, H_V]. \tag{11.11b}$$

These equations can be seen to encompass the various properties of the left and right H-compensators discussed in Section 8.1. For instance, if we take the supervectors $U = A^\flat$ and $V = B^\flat$ then $[U, V] = [B, A]^\flat$. The decomposition gives $X_U = \pi_* A^\flat$ and $H_U = W_L^{(L)}(A)$, and similarly for V and $[U, V]$. Using these values we see that Eqs. (11.11a, 11.11b) give precisely Eqs. (8.20, 8.21). Following a similar approach with more elaborate choices of the supervectors U and V we may likewise obtain Eqs. (8.30a, 8.30b) and Eqs. (8.33a, 8.33b).

We see from Eq. (11.8) that the action of $\mathcal{K}_V$ can be thought of as comprised of two parts. The first part, $\mathbb{L}_X^{(L)}$, can be thought of as arising from an infinitesimal general coordinate transformation in the direction of the supervector X. The second part, $\tilde{R}_H^{(L)}$, can be thought of as arising from an infinitesimal local gauge transformation. Thus $\mathcal{K}_V$ can be thought of as giving the general infinitesimal transformation of a section. This is also apparent from the form of the map $F_t^{(L)}$ that was used to define $\mathcal{K}_V$.

11.2 Definition of Killing supervectors

We seek a definition of Killing supervectors which is consistent with our notion of infinitesimal isometries as discussed in Section 9. Since $\mathcal{K}_V$ represents a general infinitesimal transformation we would like to find a condition on $\mathcal{K}_V$ such that it represents an infinitesimal isometry. Once we have this, then the supervector field X in the decomposition Eq. (11.3) will be the desired Killing supervector.

Recall from Section 9 that we defined isometries to be the group of transformations which leave the supergeometry, i.e. the frame and connection, invariant up to a single gauge transformation. Now the covariant derivative is a quantity which contains both the frame and the connection, thus we would suspect that there is some way of utilizing the covariant derivative to give a condition for $\mathcal{K}_V$ to be an infinitesimal isometry.

Now, recall that the Lie derivative in the direction $\pi_* A^\flat$ commutes with the covariant derivative, Eq. (10.24). With this in mind let us consider the commutator of the covariant derivative with the generalized Lie derivative. We have

$$\begin{aligned}
[\mathcal{K}_V, \nabla_A^{(L)}]s(p) &= \mathcal{K}_V\big[\big(L(p), (E_A^{(L)} + \omega_A^{(L)})\xi(p)\big)\big] \\
&\qquad - \nabla_A^{(L)}\big[\big(L(p), (X + H)\xi(p)\big)\big] \\
&= \Big[\Big(L(p), \Big([X, E_A^{(L)}] - V^I f_{IA}{}^B E_B^{(L)} + X[\omega_A^{(L)}] - E_A^{(L)}[H] \\
&\qquad + [H, \omega_A^{(L)}] - V^I f_{IA}{}^B \omega_B^{(L)}\Big)\xi(p)\Big)\Big],
\end{aligned} \tag{11.12}$$

where we have taken the algebra valued quantities $\omega_A^{(L)}$ and $H = V^I H_I$ to be acting in the appropriate representation. Thus supposing we impose the condition

$$[\mathcal{K}_V, \nabla_A^{(L)}] = 0 \tag{11.13}$$

then we see from Eq. (11.12) that this implies

$$[X, E_A^{(L)}] = V^I f_{IA}{}^B E_B^{(L)}, \tag{11.14a}$$

$$X[\omega_A^{(L)}] - V^I f_{IA}{}^B \omega_B^{(L)} = E_A^{(L)}[H] - [H, \omega_A^{(L)}]. \tag{11.14b}$$

Now, we note that[6] $\mathcal{L}_X E_A^{(L)} = [X, E_A^{(L)}]$, thus the first equation gives us an expression for the Lie derivative of the frame. Also we have

$$\begin{aligned} X[\omega_A^{(L)}] &= \ell_X\big(\omega^{(L)}(E_A^{(L)})\big) \\ &= \big(\mathcal{L}_X\omega^{(L)}\big)(E_A^{(L)}) + \omega^{(L)}\big(\mathcal{L}_X E_A^{(L)}\big) \\ &= \big(\mathcal{L}_X\omega^{(L)}\big)(E_A^{(L)}) + V^I f_{IA}{}^B \omega_B^{(L)}, \end{aligned} \tag{11.15}$$

where in the last line we have used the expression for $\mathcal{L}_X E_A^{(L)}$ given by Eq. (11.14a). Using Eq. (11.15) we see that Eq. (11.14b) gives us an expression for the Ath component of the Lie derivative of $\omega^{(L)}$. Thus in total we may rewrite Eqs. (11.14a, 11.14b) as

$$\mathcal{L}_X E_A^{(L)} = V^I f_{IA}{}^B E_B^{(L)} \tag{11.16a}$$

$$\mathcal{L}_X \omega^{(L)} = \mathrm{d}H - [H, \omega^{(L)}]. \tag{11.16b}$$

These equations are simply the first order contributions to

$$f_{\epsilon *}\left(E_A^{(L)}\big|_p\right) = \Lambda_A{}^B(\tilde{h}_\epsilon) E_B^{(L)}\big|_{f_\epsilon(p)} \tag{11.17a}$$

$$f_\epsilon{}^*\left(\omega^{(L)}\big|_{f_\epsilon(p)}\right) = \tilde{h}_\epsilon^{-1}\omega^{(L)}\big|_p \tilde{h}_\epsilon + \tilde{h}_\epsilon^{-1}\mathrm{d}\tilde{h}_\epsilon. \tag{11.17b}$$

If we compare this result to Eqs. (9.1a, 9.1b) we see that infinitesimally the transformation f_ϵ represents an isometry, as under its action the frame and connection both transform by the gauge transformation given by $\tilde{h}_\epsilon$.

Based on these considerations we take Eq. (11.13) as defining the Killing supervectors: The supervector $X = \pi_* V$ associated with a transformation $\mathcal{K}_V$ is a Killing supervector if the transformation $\mathcal{K}_V$ commutes with the covariant derivative $\nabla_A^{(L)}$. The advantage of this definition of Killing supervectors is that it is easily generalized to superspaces which are not written as a coset space. So long as we have a covariant derivative we just seek the infinitesimal combined coordinate and gauge transformation which commutes with it. This is the approach taken in [3], however there the derivations used are viewed as acting on the components of a section rather than on the section itself.

[6] Here we shall use the definition of the Lie derivative $\mathcal{L}_X$ (and ℓ_X) in an arbitrary direction X as opposed to $\pi_* A^\flat$, this is defined in the usual way when acting on vectors, scalars and forms.

11.3 Killing supervectors for the super coset space

We know from Section 9 that the supervector fields $\pi_* A^\flat$ are Killing supervectors of the coset space G/H. We would like to see how we may obtain these Killing supervectors from Eq. (11.13). Let us first start by rewriting Eq. (11.13) as, cf. Eq. (10.23),

$$[\mathcal{K}_V, \nabla_{\pi_* B^\sharp}] = \nabla_{(\mathcal{K}_V \pi_* B^\sharp)}, \qquad \forall B \in \mathfrak{k} \tag{11.18}$$

which is easily checked noting that $B = Y^A K_A$ for some constants Y^A. Note we again here use the shorthand of dropping the dependence on $L(p)$ in the vector $\pi_* B^\sharp$. The supervector V will be decomposed into components X and H as in Eq. (11.3). Then as the Y^A are constants we have that $\mathbb{L}_X^{(L)} \pi_* B^\sharp = 0$, hence

$$\mathcal{K}_V \pi_* B^\sharp = \tilde{R}_H^{(L)} \pi_* B^\sharp = \pi_*[H, B]^\sharp. \tag{11.19}$$

Then, either by direct calculation or by using Eqs. (11.11a, 11.11b), we see that Eq. (11.18) decomposes into the following two conditions

$$\pi_*[H, B]^\sharp = [X, \pi_* B^\sharp] \tag{11.20a}$$

$$\omega^{(L)}(\pi_*[H, B]^\sharp) = X[\omega^{(L)}(\pi_* B^\sharp)] - \pi_* B^\sharp[H] + [H, \omega^{(L)}(\pi_* B^\sharp)], \tag{11.20b}$$

which must be satisfied for all possible choices of the algebra element B. We know from Eqs. (8.33a, 8.33b) that a solution to these conditions is $X = \pi_* A^\flat$ and $H = W_L^{(L)}(A)$ for some algebra element A. We shall now show that all solutions to these equations can be expressed in this form. Stated differently, we will show that the unique solution to Eq. (11.13) is given by

$$\mathcal{K}_V = \mathcal{L}_{\pi_* A^\flat}, \tag{11.21}$$

and the Killing supervectors are thus of the form $\pi_* A^\flat$.

Let us expand $X = X^A E_A$ and $H = V^I H_I$, then Eqs. (11.20a, 11.20b) may be rewritten as

$$\begin{aligned}
\pi_* B^\sharp[X^A] &= X^B [E_B, \pi_* B^\sharp]^A - V^I (\pi_*[H_I, B]^\sharp)^A \\
\pi_* B^\sharp[V^I] &= X^A E_A[\omega^I(\pi_* B^\sharp)] + V^J [H_J, \omega(\pi_* B^\sharp)]^I - V^J \omega(\pi_*[H_J, B]^\sharp)^I,
\end{aligned}$$

which should be satisfied for all choices of $B \in \mathfrak{k}$. As usual we drop the dependence on $L(p)$ for the vectors $\pi_* B^\sharp$. Thus, if we group the components into a single object $Z^p = (X^A, V^I)$ these equations take the simple form

$$\pi_* B^\sharp[Z^p] + Z^q M_q{}^p(B) = 0, \qquad \forall B \in \mathfrak{k} \tag{11.22}$$

where the objects $M_q{}^p(B)$ depend linearly on the choice of B, they are also functions of the frame and connection. This equation is linear and homogeneous in Z, thus solutions may be combined linearly to give other solutions. Clearly also $Z^p = 0$ is a solution for all choices of B. In fact we already have a whole family of solutions parameterized by the algebra elements $A \in \mathfrak{g}$, we will denote these solutions by $Z^p(A)$ whose components are $X(A) = \pi_* A^\flat$ and $H(A) = W_L^{(L)}(A)$. We will now show that all solutions are of this type.

Suppose we have some solution Z^p to Eq. (11.22). Let us focus on a particular point, p, in the base manifold and consider a curve $\gamma(t)$ joining the point $p = \gamma(0)$ to some nearby point $q = \gamma(1)$. At each point $\gamma(t)$ along the curve the tangent to the curve can be expressed as $\pi_* B(t)^\sharp$ for some choice of algebra element $B(t)$, this can be viewed as simply the expansion of the tangent vector in the basis E_A. Now, our solution Z^p to Eq. (11.22) is a solution for all choices of B at all points, and thus in particular the choice $B(t)$ at the point $\gamma(t)$. Abbreviating $Z^p|_{\gamma(t)} = Z^p(t)$ and $M_q{}^p(B(t)) = M_q{}^p(t)$ Eq. (11.22) becomes, for this specific choice,

$$\frac{\mathrm{d}}{\mathrm{d}t} Z^p(t) + Z^q(t) M_q{}^p(t) = 0 \tag{11.23}$$

which should then be satisfied for all t along the curve.

Now, Eq. (11.23) is a linear first order differential equation. Thus, given the initial conditions, $Z^p(0)$, the solution $Z^p(t)$ is uniquely determined for $t > 0$. In particular $Z^p(1)$ is uniquely determined. In fact, since the curve γ and the point q are arbitrary we only need to specify Z^p at one single point in the base manifold from which Z^p will be uniquely determined for all other points. Thus, to show that all solutions to Eq. (11.22) can be expressed in the form $Z^p(A)$ for some algebra element A it is sufficient to show that at a single point p an arbitrary Z^p can be expressed as $Z^p(A)$ for some A.

So let us consider a solution Z^p at the point p composed of some supervector $X|_p$ and algebra element $H|_p \in \mathfrak{h}$. First note that it is always possible to choose an algebra element $A \in \mathfrak{g}$ such that

$$\pi_* \left(A^\flat|_{L(p)} \right) = X|_p. \tag{11.24}$$

For instance it is straightforward to show that the choice $A = X|_p[L]L(p)^{-1}$ satisfies this equation. However, this choice of A is not unique, there is a freedom in the choice of A under which $\pi_* A^\flat$ at the point p remains unchanged. This can be seen in the following way. Let $\tilde{H}$ be an arbitrary element of $\mathfrak{h}$, then

$$\begin{aligned}
L(l_{e^{tA}}(p))\tilde{h}_L^{(L)}(p, e^{tA}) &= e^{tA} L(p) \\
&= e^{tA} e^{tL\tilde{H}L^{-1}} L(p) e^{-t\tilde{H}} \\
&= e^{t(A+L\tilde{H}L^{-1})} L(p) e^{-t\tilde{H}} + \mathcal{O}(t^2) \\
&= L(l_{e^{t(A+L\tilde{H}L^{-1})}}(p))\tilde{h}_L^{(L)}\big(p, e^{t(A+L\tilde{H}L^{-1})}\big) e^{-t\tilde{H}} + \mathcal{O}(t^2).
\end{aligned}$$

From which we immediately deduce

$$\pi_*\big(A^\flat\big|_{L(p)}\big) = \pi_*\big((A + L(p)\tilde{H}L(p)^{-1})^\flat\big|_{L(p)}\big) \tag{11.25a}$$

$$W_L^{(L)}\big(p, A + L(p)\tilde{H}L(p)^{-1}\big) = W_L^{(L)}(p, A) + \tilde{H}. \tag{11.25b}$$

Therefore, after choosing the algebra element A so that at p we have $\pi_* A^\flat = X$ we still have the freedom to change $A \to A + L(p)\tilde{H}L(p)^{-1}$, and under such a transformation the H-compensator $W_L^{(L)}(A)$ at p can be changed arbitrarily. Thus as well as choosing A so that Eq. (11.24) is satisfied we may simultaneously choose A such that

$$W_L^{(L)}(p, A) = H|_p. \tag{11.26}$$

This concludes the proof.

12 Flat superspace

In this section we will apply the concepts discussed in the previous sections to the well known case of flat superspace. We start with the super Poincaré group $G = S\Pi$ and its subgroup of Lorentz transformations H. Flat superspace is defined to be the coset space $S\Pi/H$. The non-zero commutators of the super Poincaré algebra, $\mathfrak{g} = \mathfrak{s}\pi$, are

$$[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} \tag{12.1a}$$
$$[M_{ab}, P_c] = \eta_{cb}P_a - \eta_{ca}P_b \tag{12.1b}$$
$$[M_{ab}, Q_\alpha] = -(\sigma_{ab})_\alpha{}^\beta Q_\beta \tag{12.1c}$$
$$[Q_\alpha, Q_\beta] = 2k(\gamma^a C^{-1})_{\alpha\beta}P_a. \tag{12.1d}$$

Here the γ^a are the Dirac gamma matrices for the flat metric η_{ab}, $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, C is the charge conjugation matrix, and k is a phase factor. The indices may be grouped as $A = (a, \alpha)$ and the antisymmetric pair $I = [ab]$. We will distinguish between tangent space indices $A = (a, \alpha)$ and coordinate indices $M = (m, \mu)$, however both sets range over the same values. Comparing Eqs. (12.1a–12.1d) to Eqs. (4.3a–4.3c) we have that the even generators M_{ab} generate the Lorentz subgroup and play the role of the H_I whereas the even generators P_a generating translations and the odd generators Q_α generating supersymmetry transformations play the role of the K_A. We see that, further to the reductive property, $\mathfrak{k}$ forms a nilpotent subalgebra of $\mathfrak{g}$. We have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \tag{12.2a}$$
$$[\mathfrak{k}, [\mathfrak{k}, \mathfrak{k}]] = 0. \tag{12.2b}$$

In the expansion of a general element of the super Poincaré algebra, $X = X^A K_A + X^I H_I$, it is possible to restrict the X^I to be *ordinary* real numbers. This is a result of the reductive property of the super Poincaré algebra and the following: The H_I are all even, the structure constants $f_{IJ}{}^K$ are ordinary real numbers (i.e. H is conventional [3, 5]) and the coset space is flat, i.e. the structure constants $f_{AB}{}^I$ vanish, cf. Eq. (6.11). While such a choice for the super Poincaré algebra is not necessary it does allow one to deal simply with the ordinary Lorentz group.

As a consequence of the reductive property any element $g \in S\Pi$ can be written as $g = e^{x^M K_M} e^{y^I H_I}$ for some x^M and y^I [3]. Further, using Eq. (12.2a), one can show that x^M and y^I are uniquely determined by g. It follows that the principal bundle is trivial, admitting a global section

$$\begin{aligned} L &: S\Pi/H \to S\Pi \\ &: p \mapsto e^{x^M(p)K_M} \end{aligned} \tag{12.3}$$

where $x^M(p)$ are uniquely determined in terms of p and thus can be used as the coordinates of the point p. This section is horizontal and can be thought of as a special choice of gauge.

Let us calculate the action of an isometry on these coordinates. Consider left multiplication by $g = e^A$. For $A = Y^A K_A \in \mathfrak{k}$ we can use the Baker-Campbell-Hausdorff formula, which combined with Eq. (12.2b) gives

$$e^{Y^A K_A} e^{x^M K_M} = e^{Y^A K_A + x^M K_M + \frac{1}{2}[Y^A K_A, x^M K_M]}.$$

From Eq. (12.2a) we see that under this transformation we do not leave the section L. We thus find

$$x^M(l_{e^{Y^A K_A}}(p)) = x^M(p) + Y^A\Big(\delta_A{}^M - \frac{1}{2}x^N(p) f_{NA}{}^M\Big) \tag{12.4a}$$

$$\tilde{h}_L^{(L)}\big(p, e^{Y^A K_A}\big) = 0. \tag{12.4b}$$

For the algebra element $A = Y^I H_I \in \mathfrak{h}$ we have

$$e^{Y^I H_I} e^{x^M K_M} = e^{x^M \Lambda_M{}^N(e^{-Y^I H_I}) K_N} e^{Y^I H_I},$$

where $\Lambda_M{}^N(e^{-Y^I H_I})$ is the coadjoint representation of $e^{-Y^I H_I} \in H$, see Eq. (5.5). This gives

$$x^M(l_{e^{Y^I H_I}}(p)) = x^M(p) \Lambda_M{}^N(e^{-Y^I H_I}) \tag{12.5a}$$

$$\tilde{h}_L^{(L)}\big(p, e^{Y^I H_I}\big) = e^{Y^I H_I}. \tag{12.5b}$$

As in this case the normalizer of the Lorentz subgroup $N(H)$, see Section 9, is simply H we may determine all Killing supervectors by considering the infinitesimal versions of these transformations. We find

$$\pi_* P_a^\flat = \partial_a \tag{12.6a}$$

$$\pi_* Q_\alpha^\flat = \partial_\alpha - k x^\beta (\gamma^a C^{-1})_{\beta\alpha} \partial_a \tag{12.6b}$$

$$\pi_* M_{ab}^\flat = x_a \partial_b - x_b \partial_a - x^\alpha (\sigma_{ab})_\alpha{}^\beta \partial_\beta. \tag{12.6c}$$

It is standard to refer to these vectors as the differential operator representation of the super Poincaré algebra. Note, however, that due to Eq. (8.20) the algebra will have an extra minus sign compared to $\mathfrak{s}\pi$.

A similar analysis for right multiplication leads to an expression for the vectors $\pi_*\big(K_A^\sharp|_{L(p)}\big)$, i.e. the frame. We find

$$E_a^{(L)} = \partial_a \tag{12.7a}$$

$$E_\alpha^{(L)} = \partial_\alpha + k x^\beta (\gamma^a C^{-1})_{\beta\alpha} \partial_a. \tag{12.7b}$$

The coframe and the connection may be calculated from the pullback of the Maurer-Cartan form as in Eq. (6.1). We have

$$\begin{aligned} L^*\big(\zeta|_{L(p)}\big) &= L(p)^{-1}\mathrm{d}L(p) \\ &= e^{-x^M K_M}\mathrm{d}e^{x^M K_M} \\ &= \mathrm{d}x^M K_M + \frac{1}{2}[\mathrm{d}x^M K_M, x^N K_N] \\ &= \mathrm{d}x^M(\delta_M{}^A - \frac{1}{2}x^N f_{NM}{}^A)K_A. \end{aligned}$$

We may then read off the coframe and connection

$$E^a_{(L)} = \mathrm{d}x^a - k\,\mathrm{d}x^m x^n(\gamma^a C^{-1})_{nm} \tag{12.8a}$$

$$E^\alpha_{(L)} = \mathrm{d}x^\alpha \tag{12.8b}$$

$$\omega^{(L)} = 0. \tag{12.8c}$$

As $\omega^{(L)} = 0$ we see from Eq. (10.37) that $\mathcal{D}^{(L)}_A = E^{(L)}_A$. This explains how – as is usually stated in the literature – in flat superspace the covariant derivative is obtained directly in terms of right multiplication on the coset space. In general we must obviously consider the map $R^{(L)}_g$ of Eq. (8.1c).

The covariant derivative $\mathcal{D}^{(L)}_A$ is usually considered as the operator which (anti)commutes with the differential operator representation of the K_A, i.e. the $\pi_* K^\flat_A$, particularly for $A = \alpha$. This can be seen as a special case of Eq. (10.39) since the H-compensator $W^{(L)}_L(K_A)$ vanishes. Note that the covariant derivative will not commute with the $\pi_* M^\flat_{ab}$ as $W^{(L)}_L(M_{ab}) \neq 0$.

Starting with the frame and connection one can construct the Killing supervectors of flat superspace using the method discussed in Section 11, i.e. by imposing that the commutator of the generalized Lie derivative with the covariant derivative be zero. This procedure, in the four-dimensional case, is treated in [3].

The curvature, Eq. (6.2), of flat superspace is clearly zero. The only non-zero components of the torsion are given in terms of the algebra structure constants, see Eq. (6.11). They are

$$T_{\alpha\beta}{}^a = 2k(\gamma^a C^{-1})_{\alpha\beta}. \tag{12.9}$$

While flat superspace is a very useful example of a super coset space its geometry is relatively simple. The same techniques we have discussed here may, however, be applied directly to more complex geometries. A particularly well studied super coset space is the $AdS_5 \times S^5$ superspace which arises as a coset space of the super Lie group $SU(2,2|4)$, see for example [10, 12]. A lower dimensional example of a super coset space is provided by the supersphere [14, 15] which is a coset space of the super Lie group $UOSp(1|2)$.

13 Conclusions

We have discussed in detail the geometry of super coset spaces with the focus on how the geometric structures of the coset space G/H are inherited from G.

While the concepts and methods presented in this paper apply to coset spaces in general, our main aim has been to analyze the geometry of *super* coset spaces and their isometries. As such one important aspect of our work was to review and clarify the notion of Killing supervectors in the context of super coset spaces. Due to the fact that the notion of supermetric is not physically relevant for the construction of superspace the standard definition of isometries in terms of the metric cannot be applied and must be given in terms of the supergeometry – the frame and connection. Although the definition of Killing supervectors we give is derived from the understanding of the geometry of coset spaces it clearly extends to more general situations.

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A Appendix

A.1 Conventions for super differential forms

We define a super n-form ϕ on a supermanifold M at the point p to be a mapping from n copies of the tangent supervector space to $\mathbb{R}_\infty$, the real supernumbers, i.e.

$$\phi : T_pM \times \ldots \times T_pM \to \mathbb{R}_\infty. \tag{A.1}$$

The target space can be generalized to any vector space with a "multiplication", e.g. a super Lie algebra. While in this paper we write the supervectors on which a form acts on the right, its properties are more conveniently represented with the vectors on the left, we define

$$\phi(X_1, \ldots, X_n) \stackrel{\text{def}}{=} (X_n, \ldots, X_1) \cdot \phi. \tag{A.2}$$

We then require the following properties to be satisfied

$$(X + Y, Z, \ldots) \cdot \phi = (X, Z, \ldots) \cdot \phi + (Y, Z, \ldots) \cdot \phi \tag{A.3a}$$
$$(\lambda X, Y, \ldots) \cdot \phi = \lambda(X, Y, \ldots) \cdot \phi \tag{A.3b}$$
$$(\ldots, X, Y, \ldots) \cdot \phi = -(-1)^{XY} (\ldots, Y, X, \ldots) \cdot \phi, \tag{A.3c}$$

where in the last equation the supervectors X and Y must be pure (i.e. even or odd). From these relations we may further deduce

$$(\ldots, X + Y, Z, \ldots) \cdot \phi = (\ldots, X, Z, \ldots) \cdot \phi + (\ldots, Y, Z, \ldots) \cdot \phi \tag{A.4a}$$
$$(\ldots, X\lambda, Y, \ldots) \cdot \phi = (\ldots, X, \lambda Y, \ldots) \cdot \phi. \tag{A.4b}$$

Given a p-form ϕ and a q-form ψ we define the exterior (wedge) product

$\phi \wedge \psi$ by its action on a set of $p+q$ pure supervectors as

$$\begin{aligned}(X_{p+q},\ldots,X_1)\cdot\phi\wedge\psi \stackrel{\text{def}}{=} \frac{1}{p!q!}\sum_{\sigma}&(-1)^{\operatorname{sgn}\sigma}(-1)^{\nu_\sigma(X_{p+q},\ldots,X_1)}\\ &(-1)^{(X_{\sigma(p+q)}+\ldots+X_{\sigma(p+1)})(X_{\sigma(p)}+\ldots+X_{\sigma(1)}+\phi)}\\ &(X_{\sigma(p)},\ldots,X_{\sigma(1)})\cdot\phi\,(X_{\sigma(p+q)},\ldots,X_{\sigma(p+1)})\cdot\psi. \end{aligned}\tag{A.5}$$

Here σ is a permutation on $p+q$ elements. The quantity ν_σ is defined as

$$a_{\sigma(1)}\ldots a_{\sigma(n)} = (-1)^{\nu_\sigma(a_1,\ldots,a_n)}a_1\ldots a_n, \tag{A.6}$$

where the a_i, $i=1,\ldots,n$, are pure supernumbers. It is possible to show that the definition Eq. (A.5) does indeed define a $(p+q)$-form. Further, if ϕ and ψ are pure forms we have

$$\phi\wedge\psi = (-1)^{\phi\psi+pq}\psi\wedge\phi, \tag{A.7}$$

where the ϕ and ψ occurring in the exponent denote the parities of ϕ and ψ, respectively.

In the case when the target space in Eq. (A.1) is generalized to a super Lie algebra we have an *algebra valued form*. The product used in the definition Eq. (A.5) must be replaced by the Lie algebra bracket, to indicate this we denote the wedge product instead by $[\phi,\psi]$. The symmetry of this wedge product has an additional minus sign

$$[\phi,\psi] = -(-1)^{\phi\psi+pq}[\psi,\phi]. \tag{A.8}$$

If we work in a matrix representation of the algebra we could instead use matrix multiplication as the product and define $\phi\wedge\psi$. While this latter wedge product does not in general result in an algebra valued form we do however have the relation

$$[\phi,\psi] = \phi\wedge\psi - (-1)^{\phi\psi+pq}\psi\wedge\phi. \tag{A.9}$$

In particular $\phi\wedge\phi$ is algebra valued.

The exterior derivative is defined initially on 0-forms (functions) by

$$\mathrm{d}f(X) = X[f], \tag{A.10a}$$

for an arbitrary supervector X. This definition is then extended to n-forms by requiring the following properties to hold

$$\mathrm{d}(\phi+\chi) = \mathrm{d}\phi + \mathrm{d}\chi \tag{A.10b}$$
$$\mathrm{d}(\lambda\phi) = \lambda\mathrm{d}\phi \tag{A.10c}$$
$$\mathrm{d}(\phi\wedge\psi) = \mathrm{d}\phi\wedge\psi + (-1)^p\phi\wedge\mathrm{d}\psi \tag{A.10d}$$
$$\mathrm{d}^2 = 0. \tag{A.10e}$$

Here ϕ and χ are p-forms, ψ is a q-form, and λ is an arbitrary supernumber.

Let us suppose we have a basis of one-forms $E^A_{(L)}$, $A = 1, \ldots, \dim M$. Our conventions for expanding the form ϕ in this basis are, cf. Eq. (7.25)

$$\phi \stackrel{\text{def}}{=} \frac{1}{p!}(-1)^{\Delta_p(A,A)} E^{A_1}_{(L)} \wedge \ldots \wedge E^{A_p}_{(L)} \phi_{A_1 \cdots A_p}. \tag{A.11}$$

Here we have used the parity function, defined as

$$\Delta_p(A,B) = \sum_{\substack{t,u \\ t<u}}^{p} A_t B_u, \tag{A.12}$$

where A_t and B_u represent the parities of the indices. This function could in fact take as arguments any set of objects with parity, for example see Eq. (A.14) below. Note that this function is clearly linear in both arguments. Further, $\Delta_p(A,A)$ is just the sum over all non-equal pairs of indices and is invariant under any index permutation. Using this it is possible to show that the components of ϕ are given by

$$\phi_{A_1 \cdots A_p} = (-1)^{\Delta_p(A,A)} (E^{(L)}_{A_p}, \ldots, E^{(L)}_{A_1}) \cdot \phi. \tag{A.13}$$

We can then also show that

$$(X_p, \ldots, X_1) \cdot \phi = (-1)^{\Delta_p(X,A)} X_p^{A_p} \cdots X_1^{A_1} \phi_{A_1 \cdots A_p}. \tag{A.14}$$

A.2 Proof of Eq. (4.11) and Eq. (4.15)

Consider two curves $\gamma_1, \gamma_2 : [0,1] \to G$ in the group which both project down to the same curve $\gamma : [0,1] \to G/H$ in the base, i.e. $\pi \circ \gamma_i = \gamma$, for $i = 1, 2$. Clearly then we have

$$\gamma_2(t) = \gamma_1(t) h(t) \tag{A.15}$$

for some function $h(t) \in H$. Let X_1, X_2 denote the tangent vectors to the curves in the group, and X the tangent vector to the curve in the base. Working in a matrix representation we thus have

$$\begin{aligned}
X_2\big|_{\gamma_2(0)} &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_2(t)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)h(t))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_1(t)\Big|_{t=0} h(0) + \gamma_1(0)\frac{\mathrm{d}}{\mathrm{d}t}h(t)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)h(0))\Big|_{t=0} + \gamma_2(0)h(0)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}h(t)\Big|_{t=0}.
\end{aligned} \tag{A.16}$$

Now consider

$$\begin{aligned}
R_{h(0)*}(X_1\big|_{\gamma_1(0)})[f] &= X_1\big|_{\gamma_1(0)}[f \circ R_{h(0)}] \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f \circ R_{h(0)} \circ \gamma_1(t)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_1(t)h(0))\Big|_{t=0},
\end{aligned}$$

from which we deduce

$$R_{h(0)_*}(X_1\big|_{\gamma_1(0)}) = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)h(0))\Big|_{t=0}. \tag{A.17}$$

Also we have

$$\begin{aligned}(h(0)^{-1}\mathrm{d}h(X\big|_{\gamma(0)}))^\sharp\big|_{\gamma_2(0)} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma_2(0)e^{th(0)^{-1}\mathrm{d}h(X|_{\gamma(0)})}\big)\Big|_{t=0} \\ &= \gamma_2(0)h(0)^{-1}\mathrm{d}h(X\big|_{\gamma(0)}) \\ &= \gamma_2(0)h(0)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}h(t)\Big|_{t=0}. \end{aligned} \tag{A.18}$$

Thus if we use both Eqs. (A.17, A.18) in Eq. (A.16) we find

$$X_2\big|_{\gamma_2(0)} = R_{h(0)_*}(X_1\big|_{\gamma_1(0)}) + (h(0)^{-1}\mathrm{d}h(X\big|_{\gamma(0)}))^\sharp\big|_{\gamma_2(0)}.$$

Note that the point $t = 0$ is not special, and in general we have

$$X_2\big|_{\gamma_2(t)} = R_{h(t)_*}(X_1\big|_{\gamma_1(t)}) + (h(t)^{-1}\mathrm{d}h(X\big|_{\gamma(t)}))^\sharp\big|_{\gamma_2(t)}. \tag{A.19}$$

Eq. (4.11) follows immediately from this result by choosing $\gamma_1(t) = L(\gamma(t))$ and $\gamma_2(t) = \tilde{\gamma}(t)$, whereas Eq. (4.15) follows by choosing $\gamma_1(t) = L(\gamma(t))$ and $\gamma_2(t) = L'(\gamma(t))$.

References

[1] N. Alonso-Alberca, E. Lozano-Tellechea, and T. Ortín, "Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes," *Class. Quant. Grav.* **19** (2002) 6009–6024, `hep-th/0208158`.

[2] L. Andrianopoli, S. Ferrara, M. A. Lleó, and O. Maciá, "Integration of massive states as contractions of non-linear σ-models," *J. Math. Phys.* **46** (2005) 072307, `hep-th/0503196`.

[3] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity.* Institute of Physics Publishing, 1995.

[4] P. West, *Introduction to supersymmetry and supergravity.* World Scientific, 1990.

[5] B. DeWitt, *Supermanifolds.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1992.

[6] M. Nakahara, *Geometry, Topology and Physics.* Institute of Physics Publishing Bristol and Philadelphia, 2002.

[7] N. Steenrod, *The Topology of Fibre Bundles.* Princeton University Press, 1951.

[8] M. Göckeler and T. Schücker, *Differential Geometry, Gauge Theories, and Gravity.* Cambridge University Press, 1987.

[9] L. Castellani, R. D'Auria, and P. Fré, *Supergravity and Superstrings, A Geometric Perspective*, vol. 1: Mathematical Foundations. World Scientific, 1991.

[10] P. Claus, J. Rahmfeld, H. Robins, J. Tannenhauser, and Y. Zunger, "Isometries in anti-de Sitter and conformal superspaces," *JHEP* **0007** (2000) 047, `hep-th/0007099`.

[11] L. Castellani, "On G/H geometry and its use in M-theory compactifications," *Annals Phys.* **287** (2001) 1–13, `hep-th/9912277`.

[12] P. Claus and R. Kallosh, "Superisometries of the $adS \times S$ superspace," *JHEP* **9903** (1999) 014, `hep-th/9812087`.

[13] L. Castellani, L. J. Romans, and N. P. Warner, "Symmetries of coset spaces and Kaluza-Klein supergravity," *Annals Phys.* **157** (1984) 394–407.

[14] A. F. Schunck and C. Wainwright, "A geometric approach to scalar field theories on the supersphere," *J. Math. Phys.* **46** (2005) 033511, `hep-th/0409257`.

[15] G. Landi, "Projective modules of finite type over the supersphere $S^{2,2}$," *Diff. Geom. Appl.* **14** (2001) 95–111, `math-ph/9907020`.