Special metrics and Triality

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ABSTRACT: We investigate a new 8–dimensional Riemannian geometry which arises as a critical point of Hitchin's variational principle and is defined by a generic closed and coclosed 3–form with stabiliser $PSU(3)$, along with well–known almost quaternionic structures with $Sp(1) \cdot Sp(2)$–invariant closed 4–form. We give a Riemannian characterisation of these groups in terms of invariant isometries in $\Delta_{\pm} \otimes \Lambda^1$. We prove that the integrability condition on the forms is equivalent to the harmonicity of the corresponding isometry with respect to the twisted Dirac operator and thereby derive integrability conditions on the Ricci tensor. We also show how $Spin(7)$–structures fit into this picture and provide thus a unified treatment to $Spin(7)$–, $PSU(3)$– and almost–quaternionic geometry. We establish various obstructions to the existence of topological reductions to $PSU(3)$ and for compact manifolds we also give sufficient conditions for $PSU(3)$–structures that can be lifted to $SU(3)$–structures. Finally, we construct the first known compact integrable non–symmetric $PSU(3)$–structures.

KEYWORDS: spin geometry, G–structures, special Riemannian metrics, triality, Dirac operator.

Contents

1. Introduction

The motivation for the present work was to investigate a new 8–dimensional Riemannian geometry which arose as a critical point of Hitchin's variational principle [7]. In terms of G–structures, it is associated with $PSU(3)$ acting on the tangent space via its adjoint representation $Ad : PSU(3) \hookrightarrow SO(8)$. Equivalently, and in close analogy to the better known G_2–geometry, this structure can be (topologically) characterised by a special 3–form coming from the $PSU(3)$–invariant 3–form $B([X,Y],Z)$, where B denotes the Killing form and $[\cdot,\cdot]$ the Lie bracket of $\mathfrak{su}(3)$. As in the G_2–case, the $PSU(3)$–forms are *stable* [7] and to make the analogy complete, a $PSU(3)$–form defines a critical point if and only if it is closed and coclosed with respect to the metric it induces. The key difference between these two geometries is that close– and cocloseness imply the G_2–invariant 3–form to be parallel with respect to the Levi–Civita connection, in other words, the holonomy reduces to G_2, while for $PSU(3)$ this can only happen for (locally) flat or symmetric metrics in virtue of Berger's theorem.

The inclusion of $PSU(3)$ into $SO(8)$ can be lifted to $Spin(8)$ and as was already shown in [7], the vector representation Λ^1 and the two irreducible spin representations $\Delta_\pm$ of $Spin(8)$ restricted to $PSU(3)$ all coincide. By the triality principle, Λ^1, Δ_+ and Δ_- carry the same Euclidean structure, so there exist two $PSU(3)$–equivariant isometries $\Lambda^1 \to \Delta_\pm$. Moreover, these are elements of the kernel of Clifford multiplication, since as a $Spin(8)$–module, $\Delta_\pm \otimes \Lambda^1 = \Delta_\mp \oplus \Lambda^3\Delta_\mp$ and $\Lambda^3\Delta_\mp = \ker\mu$. In view of a Riemannian characterisation of $PSU(3)$–structures, that is, a description of the coset space $Spin(8)/PSU(3)$, we are naturally led to ask: What is the stabiliser of a *supersymmetric map*, that is, an isometry $\Gamma_+ : \Lambda^1 \to \Delta_-$ which lies in an irreducible $Spin(8)$–component of $\Delta_- \otimes \Lambda^1$? If $\Gamma_+ \in \Delta_+ \subset \Delta_- \otimes \Lambda^1$, then Γ_+ is induced by a spinor of unit norm and the stabiliser inside $Spin(8)$ is $Spin(7)$. For $\Gamma_+ \in \Lambda^3\Delta_+$ to define an isometry $\Lambda^1 \to \Delta_-$ our first result asserts this to be equivalent to the existence of a Lie bracket $[\cdot,\cdot]$ on Δ_+ such that the adjoint group preserves the spin invariant metric q (Thm. 3.1). Then as a 3–form over Δ_+, $\Gamma_+(X,Y,Z) = q([X,Y],Z)$. Since this involves only the metric structure, analogous statements hold for isometries $\Lambda^1 \to \Delta_-$ and $\Delta_+ \to \Delta_-$ by the triality principle. We can classify the resulting Lie algebra structures by observing that they are necessarily reductive (Prop. 3.2) and consequently determine the orbit structure of $Spin(8)$ on supersymmetric maps in $\Lambda^3\Delta_+$ (Thm. 3.4). The occurring stabiliser groups are given as follows:

- $SU(2) \cdot SU(2) \times U(1)$ and $Sp(1) \cdot Sp(2)$ as stabiliser of one orientation–preserving isometry $\Gamma_+ : \Lambda^1 \to \Delta_-$ or $\Gamma_- : \Lambda^1 \to \Delta_+$, where the inclusion covers the canonical embedding of $SO(3) \times SO(3) \times SO(2)$ and $SO(3) \times SO(5)$ into $SO(8)$.

- $PSU(3)$ as stabiliser of a pair of orientation–reversing isometries $\Gamma_\pm : \Lambda^1 \to \Delta_\mp$.

Remarkably, $Sp(1) \cdot Sp(2)$ shows up in this context. It is one of the possible Riemannian holonomy groups on Berger's list which so far has been characterised by the existence of a special self–dual 4–form [12], thereby similar to $Spin(7)$. In our approach, this group appears as the stabiliser of a supersymmetric map, rendering it akin to $PSU(3)$.

The hybrid nature of almost–quaternionic structures defined by $Sp(1) \cdot Sp(2)$ also persists when it comes to integrability. Although the orbits associated with $Spin(7)$ and $Sp(1) \cdot Sp(2)$ are not open, the provenance of $PSU(3)$ from Hitchin's variational principle suggests close– and cocloseness of the corresponding invariant differential form as a natural integrability condition. However, this has quite distinct implications for the resulting geometric properties. As in the G_2–case, closeness of the $Spin(7)$–invariant self–dual 4–form is equivalent for the holonomy to be contained in $Spin(7)$. This is also true for the $Sp(1) \cdot Sp(k)$–invariant self–dual 4–form provided $k \geq 3$ [16], but Salamon's counterexample [14] shows this to fail for $k = 2$, begging thus for a suitable geometric interpretation of this integrability condition. Put differently, integrable $Spin(7)$–structures are *torsion–free*, while $Sp(1) \cdot Sp(2)$– and $PSU(3)$–structures allow for non–trivial components of the intrinsic torsion in some irreducible $Sp(1) \cdot Sp(2)$– and $PSU(3)$–modules. The intrinsic torsion can also be captured by looking at the corresponding invariant supersymmetric map. It turns out that the invariant $Sp(1) \cdot Sp(2)$– and $PSU(3)$–forms are closed and coclosed if and only if the corresponding supersymmetric map is harmonic for the twisted Dirac operator, that is, it lies in the kernel of $\mathbf{Đ} : \Gamma(\Delta_\pm \otimes \Lambda) \to \Gamma(\Delta_\mp \otimes \Lambda)$ (Thm. 6.2 and Thm. 6.4). In the $PSU(3)$–case, the implication was already asserted in [7]. However, as we will explain, there are problems with the proof, so we have to establish both the implication and the converse. As a matter of fact, for $Spin(7)$ this harmonicity condition is actually a reformulation of the usual holonomy condition (Proposition 6.1). As a result, we obtain a unified treatment of integrable $Spin(7)$–, $Sp(1) \cdot Sp(2)$– and $PSU(3)$–structures in terms of supersymmetric maps. Moreover, this spinorial approach has practical consequences, too. Using a formula of Weitzenböck kind for the twisted Dirac operator [20] readily yields geometrical obstructions on the Ricci tensor. Indeed, five respectively eight components of the Ricci tensor of an integrable $Sp(1) \cdot Sp(2)$– or $PSU(3)$–metric have to vanish (Proposition 6.5), contrasting sharply with integrable $Spin(7)$–metrics which are Ricci flat.

This leaves us with finding integrable examples of $PSU(3)$–structures with non–trivial torsion, in particular compact ones, a problem risen by Hitchin [7], [8]. We first examine the obstructions to the existence of a topological reduction to $PSU(3)$ acting in its adjoint representation. These turn out to be quite severe. Indeed, among other obstructions, it implies the existence of four linearly independent vector fields (Prop. 5.8). As a consequence, the only homogeneous compact example of the form G/H where G is simple is $SU(3)$ (Prop. 5.3). Due to the non–trivial $Spin(8)$–orbit structure on supersymmetric maps in $\Lambda^3\Delta_\pm$, sufficient conditions for existence are hard to find and we will content ourselves with a special case, namely $PSU(3)$–structures with vanishing *triality class*. This class is the cohomological obstruction for lifting the structure group from $PSU(3)$ to $SU(3)$ which can be thought of as the analogue of lifting an orthonormal frame bundle to a spin structure. With the complex structure at hand we can derive a complete set of necessary and sufficient conditions for such a $PSU(3)$–structure to exist by invoking a K–theoretic argument. As a result, it is not surprising that the examples we found are topologically trivial. A family of local integrable examples with non–trivial torsion is built out of a 4–dimensional hyperkähler manifold times flat Euclidean 4–space and a compact example is obtained out of a 6–dimensional nilmanifold times a 2–torus. By computing the diagonal of the Ricci tensor, these examples also show that we cannot improve the integrability condition on the Ricci tensor, nor that Ricci flatness implies the vanishing of the torsion. We complement our discussion of $PSU(3)$ with the corresponding results for $Sp(1) \cdot Sp(2)$ where we mainly draw on the existing literature [19] and [14].

The paper is organised as follows. We first discuss the triality principle in Section 2 before we classify the $Spin(8)$–orbit structure of supersymmetric maps in Section 3. Since much of the techniques are representation theoretic, Section 4 is dedicated to a thorough discussion of the linear algebra of $Sp(1) \cdot Sp(2)$– and $PSU(3)$–structures. We then move on to global issues and investigate necessary and sufficient conditions for the existence of topological reductions to $Sp(1) \cdot Sp(2)$ and $PSU(3)$ in Section 5. Integrability issues are discussed in Section 6, while in Section 7 we construct local and compact examples.

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2. Triality

The interplay between vectors and spinors is at the heart of the special geometric features of various low–dimensional geometries like the exceptional geometries associated with G_2 or $Spin(7)$. In this respect, dimension 8 plays a special rôle as the space of vectors and spinors carry the same internal structure by the *triality principle.*

To see what this means recall that the vector representation Λ^1 and the two irreducible spin representations $\Delta_\pm$ of $Spin(8)$ are all 8–dimensional and real. A convenient way to think of these spaces is to adopt the octonions $\mathbb{O}$ as the underlying Euclidean vector space. More concretely, let us fix an orthonormal basis $e_1, \ldots, e_8$ in Λ^1 and identify these vectors with the standard basis $1, i, \ldots, e \cdot k$ of $(\mathbb{O}, \| \cdot \|)$. If R_u denotes right multiplication by $u \in \mathbb{O}$, the map

$$u \in \mathbb{O} \mapsto \begin{pmatrix} 0 & R_u \\ -R_{\bar{u}} & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{O} \oplus \mathbb{O}) \tag{2.1}$$

extends to an isomorphism *Cliff*$(\mathbb{O}) \cong \mathrm{End}(\mathbb{O} \oplus \mathbb{O})$ where $\Delta = \mathbb{O} \oplus \mathbb{O}$ is the (reducible) space of spinors for $Spin(8)$. These two summands can be distinguished after fixing an orientation, since a volume form acts on these by $\pm id$. We thus obtain the irreducible spin representations $\Delta_\pm$. The explicit matrix representation (2.1) we will use throughout this paper is given in Appendix A. Moreover, the inner product on $\mathbb{O}$ can be adopted as the $Spin(8)$–invariant inner product q on Δ_+ and Δ_-. Consequently the three irreducible representations $\pi_0 : Spin(8) \to SO(\Lambda^1)$, $\pi_+ : Spin(8) \to SO(\Delta_+)$ and $\pi_- : Spin(8) \to SO(\Delta_-)$, albeit non-equivalent as representation spaces of $Spin(8)$, carry the same Euclidean structure. In fact, they are related by two outer $Spin(8)$–automorphisms κ and λ of order two and three respectively, namely

$$\pi_0 = \pi_+ \circ \kappa \circ \lambda \text{ and } \pi_- = \pi_+ \circ \lambda^2.$$

Morally this means that we can exchange any two of the representations Λ^1, Δ_+ and Δ_- by an outer automorphism, while the remaining third one is fixed.

Example: Consider Clifford multiplication $\mu_- : \Delta_+ \otimes \Lambda^1 \to \Delta_-$ which as a $Spin(8)$–equivariant map gives rise to a decomposition into irreducibles $\Delta_- \oplus \ker \mu_-$. The kernel of Clifford multiplication is isomorphic to $\Lambda^3\Delta_-$ and we will use both representations interchangeably. Moreover, the unit sphere in Δ_- is isomorphic to $Spin(8)/Spin(7)_-$, where the subscript "$-$" stresses the fact that $Spin(7)$ stabilises a spinor of negative chirality and not a vector in the vector representation of $Spin(8)$. By exchanging via triality Δ_+ with Δ_- while keeping fixed Λ^1, we see that $\Delta_- \otimes \Lambda^1 = \Delta_+ \oplus \Lambda^3\Delta_+$ and the unit sphere in Δ_+ is isomorphic to $Spin(8)/Spin(7)_+$. Similarly, $\Delta_- \otimes \Delta_+ = \Lambda^1 \oplus \Lambda^3$ and the unit sphere in Λ^1 is isomorphic to $Spin(8)/Spin(7)_0$. Summarising, the stabiliser of a vector or a spinor of positive/negative chirality is isomorphic to $Spin(7)$, but lives in distinct conjugacy classes inside $Spin(8)$ permuted by the outer triality automorphisms.

3. Supersymmetric maps

Let us take up the previous example and consider a unit spinor $\Psi_+ \in \Delta_+$. It can be seen as an element in $\Delta_- \otimes \Lambda^1$ giving rise to the isometry $X \in \Lambda^1 \mapsto X \cdot \Psi_+ \in \Delta_-$.

Definition 3.1. *A* supersymmetric map *is an isometry between two of the three spaces Λ^1, Δ_+ or Δ_+, which lies in an irreducible $Spin(8)$–submodule.*

The jargon has its origin in particle physics where a supersymmetry is supposed to transform bosons (particles which are elements in a vector representation of the spin group) into fermions (particles which are elements in a spin representation of the spin group).

One easily verifies that a supersymmetric map in $\Delta_\pm$ boils down to a unit vector or spinor. Hence, there is only one $Spin(8)$–orbit which is isomorphic to the 7–sphere. More interesting is the case of supersymmetric maps which are induced by a 3–form over Λ^1, Δ_+ or Δ_-. As we are only concerned with the metric structure of the spaces Λ^1, Δ_+, and Δ_-, triality implies that we are free to consider the module $\Delta_+ \otimes \Delta_-$ rather than $\Delta_\pm \otimes \Lambda^1$ and we subsequently do so for various reasons. We first try to exhibit the orbit structure of $Spin(8)$ on the set of supersymmetric maps in Λ^3 which we denote by $\mathfrak{I}_g$. A first step is the following characterisation.

Theorem 3.1. *If a $\rho \in \Lambda^3$ lies in $\mathfrak{I}_g$, then ρ is of unit length and there exists a Lie bracket $[\cdot\,,\cdot]$ on Λ^1 such that*

$$\rho(x, y, z) = g([x, y], z). \tag{3.1}$$

Consequently, the adjoint group of this Lie algebra acts as a group of isometries on Λ^1.

Conversely, if there exists a Lie algebra structure on Λ^1 whose adjoint group leaves g invariant, then the 3–form defined by (3.1) and divided by its norm belongs to $\mathfrak{I}_g$.

Proof: Because of the skew–symmetry of ρ, the metric g is necessarily invariant under the adjoint action of the induced Lie algebra, for

$$g([x, y], z) = \rho(x, y, z) = -g([x, z], y).$$

We are left to show with that an isometry induces a Lie bracket and vice versa. In fact, inducing an isometry and defining a Lie bracket through (3.1) are both quadratic conditions on the coefficients of ρ which we show to coincide. To begin with we define the linear map

$$Jac : \Lambda^3 \otimes \Lambda^3 \to \Lambda^4$$

by skew–symmetrising the contraction to $\Lambda^2 \otimes \Lambda^2$. This is most suitably expressed in index notation with respect to some orthonormal basis $\{e_i\}$, namely

$$\begin{aligned} Jac(\rho_{ijk}\tau_{lmn}) &= \rho_{[ij}{}^{k}\tau_{lm]k} \\ &= \frac{1}{6}(\rho_{ij}{}^{k}\tau_{lmk} + \rho_{il}{}^{k}\tau_{mjk} + \rho_{im}{}^{k}\tau_{jlk} + \rho_{jl}{}^{k}\tau_{imk} + \rho_{jm}{}^{k}\tau_{lik} + \rho_{lm}{}^{k}\tau_{ijk}) \end{aligned}$$

and in particular

$$Jac(\rho_{ijk}\rho_{lmn}) = \frac{1}{3}(\rho_{ij}{}^{k}\rho_{klm} + \rho_{li}{}^{k}\rho_{kjm} + \rho_{jl}{}^{k}\rho_{kim}). \tag{3.2}$$

If we are given a 3–form ρ and define a skew–symmetric map $[\cdot,\cdot]:\Lambda^2\to\Lambda^1$ by (3.1), then the Jacobi identity holds, i.e. we have defined a Lie bracket, if and only if $Jac(\rho\otimes\rho)=0$.

Next we analyse the conditions for ρ to induce an isometry. For a p–form ρ we have $q(\rho^p\cdot\Psi_1,\Psi_2)=(-1)^{p(p+1)/2}q(\Psi_1,\rho^p\cdot\Psi_2)$, so ρ defines an isometry $\Delta_\pm\to\Delta_\mp$ if and only if for any pair of spinors of equal chirality $q(\rho\cdot\rho\cdot\Psi_1,\Psi_2)=q(\Psi_1,\Psi_2)$ holds. Considering thus the $Spin(8)$–equivariant maps

$$\Gamma_\pm:\rho\otimes\tau\in\Lambda^3\otimes\Lambda^3\mapsto\rho\cdot\tau\in \mathit{Cliff}(\Lambda^1)\cong\mathrm{End}(\Delta)\overset{pr_\pm}{\mapsto}\rho\cdot\tau_{|\Delta_\pm}\in\Delta_\pm\otimes\Delta_\pm,$$

this condition reads $\rho\in\mathfrak{I}_g$ if and only if $\Gamma_\pm(\rho\otimes\rho)=\mathrm{Id}_{\Delta_\pm}$. Using the algorithm in [13] and labeling irreducible representations by their highest weight (expressed in the basis of fundamental roots), we decompose both the domain and the target space into irreducible components to find

$$\begin{aligned}\Lambda^3\otimes\Lambda^3 &\cong \mathbf{1}\oplus 2\Lambda^2\oplus\Lambda^4_+\oplus\Lambda^4_-\oplus[1,4,3,3]\oplus[2,4,2,3]\oplus[2,4,3,2]\oplus\\ &\qquad [2,4,2,2]\oplus 2[2,3,2,2]\oplus[2,2,1,1]\\ \Delta_\pm\otimes\Delta_\pm &= \Lambda^2\Delta_\pm\oplus\odot^2\Delta_\pm\cong\Lambda^2\oplus\mathbf{1}\oplus\Lambda^4_\pm.\end{aligned}$$

The modules $\Lambda^4_+=[1,2,2,1]$ and $\Lambda^4_-=[1,2,1,2]$ are the spaces of self–dual and anti–self–dual 4–forms respectively. Note that $\Gamma_+(\rho\otimes\tau)^{tr}=\Gamma_-(\tau\otimes\rho)$ and so it suffices to consider the map Γ_+ only. Since the map induced by ρ is symmetric, it follows that $\Gamma_\pm(\rho\otimes\rho)\in\odot^2\Delta_\pm=\mathbf{1}\oplus\Lambda^4_\pm$. Moreover the image clearly contains $\Lambda^4_\pm$. As a result, $\Gamma_\pm(\rho\otimes\rho)_{\odot^2_0\Delta_\pm}=0$ is a necessary condition for ρ to lie in $\mathfrak{I}_g$.

Next we identify this obstruction in Λ^4_+ with $Jac(\rho\otimes\rho)$ by showing

$$\Gamma_+(\rho\otimes\rho)\oplus\Gamma_-(\rho\otimes\rho)=-3Jac(\rho\otimes\rho)_{\Lambda^4_+}+\parallel\rho\parallel^2\,\mathrm{Id}\,. \tag{3.3}$$

We first remark that Clifford multiplication induces a map

$$\rho\otimes\tau\in\Lambda^3\otimes\Lambda^3\mapsto\rho\cdot\tau\in\Lambda^0\oplus\Lambda^2\oplus\Lambda^4\oplus\Lambda^6$$

if we regard the product $\rho\cdot\tau$ as an element of $\mathit{Cliff}(\Lambda^1,g)\cong\Lambda^*$ under the natural isomorphism. The various components of $\rho\cdot\tau$ under this identification are accounted for by the "coinciding pairs" (c.p.) in the expression $\rho_{ijk}\tau_{lmn}e_{ijklmn}$, $i<j<l$, $l<m<n$. For instance, having three coinciding pairs implies $i=l$, $j=m$ and $k=n$, hence $e_{ijklmn}=1$. Then $\rho=\sum_{i<j<k}c_{ijk}e_{ijk}$ gets mapped to

$$\rho\cdot\rho=\sum_{\substack{i<j<k\\ l<m<n}}\rho_{ijk}\rho_{lmn}e_{lmnijk}=\sum_{\substack{i<j<k\\ l<m<n}\text{, 3 c.p.}}\rho_{ijk}\rho_{lmn}e_{lmnijk}+\sum_{\substack{i<j<k\\ l<m<n}\text{, 1 c.p.}}\rho_{ijk}\rho_{lmn}e_{lmnijk}.$$

There is no contribution by the sum of two c.p. as $\rho\cdot\rho$ is symmetric. Now the first sum is just

$$\parallel\rho\parallel^2\cdot\mathbf{1}=\sum\nolimits_{i<j<k}\rho^2_{ijk}\mathbf{1}$$

which leaves us with the contribution of the sum with one pair of equal indices. No matter which indices of the two triples $(i < j < k)$ and $(l < m < n)$ coincide, the skew–symmetry of the c_{ijk} and e_{ijk} allows us to rearrange and rename the indices in such a way that the second sum equals

$$\sum_a \sum_{\substack{j<k,\, m<n \\ j,k,m,n \text{ dist.}}} \rho_{ajk}\rho_{amn} e_{amnajk} = -\sum_a \sum_{\substack{j<k,\, m<n \\ j,k,m,n \text{ dist.}}} \rho_{ajk}\rho_{amn} e_{mnjk}$$
$$= -3Jac(\rho \otimes \rho),$$

whence (3.3) and consequently the assertion of the theorem. ■

The 3–forms in $\mathfrak{I}_g$ thus encapsulate the data of a Lie algebra structure whose adjoint action preserves the metric on Λ^1. We also say that the Lie structure is *adapted* to the metric g and write $\mathfrak{l}$ if we think of Λ^1 as a Lie algebra. In order to exhibit the $Spin(8)$–orbit structure on $\mathfrak{I}_g$, our next task is to classify the resulting Lie algebras.

Let us recall some basic notions (see for instance [11]). A Lie algebra $\mathfrak{g}$ is said to be *simple* if it contains no non–trivial ideals. A *semi–simple* Lie algebra is a direct sum of simple ones which is to say that it does not possess any non–trivial abelian ideal. Equivalently, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. On the other hand, if the derived series defined inductively by $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$ becomes actually trivial from some integer k on, then $\mathfrak{g}$ is *solvable*. Any abelian Lie algebra is solvable and so is any sub–algebra of a solvable one. Moreover, every Lie algebra contains a maximal solvable ideal, the so–called *radical* $\mathfrak{r}(\mathfrak{g})$ of $\mathfrak{g}$. In particular, the centre $\mathfrak{z}(\mathfrak{g})$ is contained in $\mathfrak{r}(\mathfrak{g})$. If there is equality, then $\mathfrak{g}$ is said to be *reductive*. Reductive Lie algebras are a direct Lie algebra sum of their centre and a semi–simple Lie algebra.

Proposition 3.2. *An adapted Lie algebra $\mathfrak{l}$ is reductive.*

Proof: By the lemma below, $\mathfrak{r}(\mathfrak{g})$ is abelian, therefore $g([R_1, X], R_2) = -g(X, [R_1, R_2]) = 0$ for any $X \in \mathfrak{l}$ and R_1, $R_2 \in \mathfrak{r}(\mathfrak{g})$. It follows that $[X, R_2] \in \mathfrak{r}(\mathfrak{g}) \cap \mathfrak{r}(\mathfrak{g})^{\perp} = \{0\}$ and consequently, $\mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$. ■

Lemma 3.3. *Let $\mathfrak{s}$ be a solvable Lie algebra which is adapted to some metric g. Then $\mathfrak{s}$ is abelian.*

Proof: We proceed by induction over n, the dimension of $\mathfrak{s}$. If $n = 1$, then $\mathfrak{s}$ is abelian and the assertion is trivial. Now assume that the assumption holds for all $1 \leq m < n$. Let $\mathfrak{a}$ be a non–trivial abelian ideal of $\mathfrak{s}$. This, of course, does exist, for otherwise $\mathfrak{s}$ would be semi–simple. The ad–invariance of g implies

$$g([A, X], Y) = 0 \quad \text{for all } X,\, Y \in \mathfrak{s},\, A \in \mathfrak{a},$$

for if $X \in \mathfrak{a}$, then $[A, X] = 0$ and if $X \in \mathfrak{a}^{\perp}$, then $g([A, X], Y) = -g(X, [A, Y]) = 0$ since $[A, Y] \in \mathfrak{a}$. Hence $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{s})$. We can therefore split $\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{h}$ into a direct sum of vector spaces with $\mathfrak{h}$ an orthogonal complement to $\mathfrak{z}$ of dimension strictly less than n. Now for all $X \in \mathfrak{s}$, $Z \in \mathfrak{z}$ and $H \in \mathfrak{h}$ we have $g([X,H],Z) = -g(H,[X,Z]) = 0$, so $[X,H] \in \mathfrak{z}^\perp = \mathfrak{h}$, or equivalently, $\mathfrak{h}$ is an ideal of $\mathfrak{s}$. As such, it is adapted and solvable since $\mathfrak{s}$ is adapted and solvable. Our induction hypothesis applies and we deduce that $\mathfrak{s}$ is abelian. ∎

As a result, we are left to determine the semi–simple part of an adapted Lie algebra $\mathfrak{l}$ of dimension 8. Appealing to Cartan's classification of simple Lie algebras, we obtain the following possibilities where $\mathfrak{z}^p$ denotes the centre of dimension p:

1. $\mathfrak{l}_1 = \mathfrak{su}(3)$
2. $\mathfrak{l}_2 = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{z}^2$
3. $\mathfrak{l}_3 = \mathfrak{su}(2) \oplus \mathfrak{z}^5$.

Hence there is a disjoint decomposition of $\mathfrak{I}_g$ into the sets $\mathfrak{I}_{g1}$, $\mathfrak{I}_{g2}$ and $\mathfrak{I}_{g3}$ acted on by $Spin(8)$ and pooling together the forms which induce the Lie algebra structure $\mathfrak{l}_1$, $\mathfrak{l}_2$ or $\mathfrak{l}_3$.

Theorem 3.4. *The sets $\mathfrak{I}_{g1}$, $\mathfrak{I}_{g2}$ and $\mathfrak{I}_{g3}$ can be described as follows:*

1. $\mathfrak{I}_{g1} = Spin(8)/\big(PSU(3) \times \mathbb{Z}_2\big)$
2. $\mathfrak{I}_{g2} = (0,1) \times Spin(8)/\big(SU(2) \cdot SU(2) \times U(1)\big)$
3. $\mathfrak{I}_{g3} = Spin(8)/Sp(1) \cdot Sp(2)$,

where $SU(2) \cdot SU(2) = SU(2) \times SU(2)/\mathbb{Z}_2$ and $Sp(1) \cdot Sp(2) = Sp(1) \times Sp(2)/\mathbb{Z}_2$ cover the inclusions $SO(3) \times SO(3) \hookrightarrow SO(8)$ and $SO(3) \times SO(5) \hookrightarrow SO(8)$. Furthermore, consider the $Spin(8)$–invariant decomposition $\mathfrak{I}_g = \mathfrak{I}_{g+} \cup \mathfrak{I}_{g-}$ into 3–forms whose induced isometry is orientation–preserving or reversing respectively. Then $\mathfrak{I}_{g-} = Spin(8)/\big(PSU(3) \times \mathbb{Z}_2\big)$ and $\mathfrak{I}_{g+}$ foliates over the circle S^1 with principal orbits $Spin(8)/\big(SU(2) \cdot SU(2) \times U(1)\big)$ over $S^1 - \{pt\}$ and a degenerate orbit $Spin(8)/Sp(1) \cdot Sp(2)$ at $\{pt\}$.

Proof: We remark that the stabiliser of $\rho_i \in \mathfrak{I}_{gi}$ in $SO(8)$ is $SO(8) \cap Aut(\mathfrak{l}_i)$. Consider then the case of a 3–form $\rho_1 \in \mathfrak{I}_{g1}$, that is, ρ_1 induces an $\mathfrak{su}(3)$–structure on Λ. Since the fixed Riemannian metric g is ad–invariant it must up to a negative constant c coincide with the (negative definite) Killing form $B(X,Y) = \mathrm{Tr}(ad_X \circ ad_Y)$. It is well known (cf. for instance [5]) that there exists an orthogonal basis $e_1, \ldots, e_8$ such that the totally anti–symmetric structure constants c_{ijk} are given by

$$c_{123} = 1, \quad c_{147} = -c_{156} = c_{246} = c_{257} = c_{345} = -c_{367} = \frac{1}{2}, \quad c_{458} = c_{678} = \frac{\sqrt{3}}{2}$$

and $B(e_i, e_i) = -3$. Hence $f_i = e_i/\sqrt{-3c}$ is g–orthonormal. The relation (3.1) and the requirement to be of unit norm then implies that

$$\begin{aligned}\rho_1 = {} & \frac{1}{\sqrt{-3c}} f_{123} + \frac{1}{2\sqrt{-3c}} f_1(f_{47} - f_{56}) + \frac{1}{2\sqrt{-3c}} f_2(f_{46} + f_{57}) + \frac{1}{2\sqrt{-3c}} f_3(f_{45} - f_{67}) + \\ & \frac{1}{2\sqrt{-c}} f_8(f_{45} + f_{67}) \\ = {} & \frac{1}{2} f_{123} + \frac{1}{4} f_1(f_{47} - f_{56}) + \frac{1}{4} f_2(f_{46} + f_{57}) + \frac{1}{4} f_3(f_{45} - f_{67}) + \frac{\sqrt{3}}{4} f_8(f_{45} + f_{67}),\end{aligned}$$

where as usual, the notation f_{ijk} will be shorthand for $f_i \wedge f_j \wedge f_k$ and vectors are identified with their dual in presence of a metric. Any 3–form of $\mathfrak{I}_{g1}$ being representable in this way, it follows that $SO(8)$ acts transitively on $\mathfrak{I}_{g1}$. The stabiliser of ρ_1 in $SO(8)$ is the adjoint group $SU(3)/\mathbb{Z}_3 = PSU(3)$ which is covered by $PSU(3) \times \mathbb{Z}_2$ in $Spin(8)$ (note that $\pi_1\big(PSU(3)\big) = \mathbb{Z}_3$), hence $\mathfrak{I}_{g1} = Spin(8)/\big(PSU(3) \times \mathbb{Z}_2\big)$. Using the matrix representation of *Cliff*(Λ^1) given in Appendix A with respect to some ordered basis $\Psi_{i\pm}$ of $\Delta_\pm$ the isometry $A_{\rho_1} : \Delta_- \to \Delta_+$ induced by ρ_1 is

$$A_{\rho_1} = \frac{1}{4} \begin{pmatrix} \sqrt{3} & 0 & 0 & 3 & -\sqrt{3} & 0 & 0 & 1 \\ 2 & -\sqrt{3} & -1 & 0 & 2 & -\sqrt{3} & -1 & 0 \\ 0 & 3 & -\sqrt{3} & 0 & 0 & -1 & -\sqrt{3} & 0 \\ -1 & 0 & 2 & \sqrt{3} & 1 & 0 & -2 & -\sqrt{3} \\ -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} & 0 & 0 & 3 \\ -2 & -\sqrt{3} & -1 & 0 & -2 & -\sqrt{3} & -1 & 0 \\ 0 & -1 & -\sqrt{3} & 0 & 0 & 3 & -\sqrt{3} & 0 \\ 1 & 0 & 2 & -\sqrt{3} & -1 & 0 & -2 & \sqrt{3} \end{pmatrix}, \tag{3.4}$$

hence $\det(A_{\rho_1}) = -1$. Moreover, we have $\det \pi_\pm(a) = 1$ for any $a \in Spin(8)$ as the generators $e_i \cdot e_j$ square to $-Id$ and are therefore of determinant 1. The $Spin(8)$–equivariance of the embedding $\Lambda^3 \to \Delta \otimes \Delta$ entails $A_{\pi_0(a)^*\rho_1} = \pi_+(a) \circ A_{\rho_1} \circ \pi_-(a)^{-1}$, whence $\mathfrak{I}_{g1} \subset \mathfrak{I}_{g-}$.

Next we turn to the Lie algebras $\mathfrak{l}_2$ and $\mathfrak{l}_3$ where the latter can be seen as a degeneration of the former. So assume ρ_2 to be an element of $\mathfrak{I}_{g2}$ inducing an $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{z}^2$–structure. The restriction to g to any copy of $\mathfrak{su}(2)$ must be as above a negative multiple of the Killing form of $\mathfrak{su}(2)$, so $g = c_1 B_1 \oplus c_2 B_2 \oplus g_{|\mathfrak{z}^p}$. There exists a basis e_i of $\mathfrak{su}(2)$ such that $[e_i, e_j] = \epsilon_{ijk} e_k$ (where ϵ_{ijk} is totally anti–symmetric) and $B(e_i, e_i) = -2$. Choosing such a basis for each copy of $\mathfrak{su}(2)$ and extending this to an orthonormal basis f_i of Λ^1 by normalising, the requirement on ρ_2 to be of unit norm implies

$$\begin{aligned} \rho_2 &= \frac{1}{\sqrt{-2c_1}} f_{123} + \frac{1}{\sqrt{-2c_1}} f_{456} \\ &= \frac{1}{\sqrt{-2c_1}} f_{123} + \sqrt{\frac{2c_1 + 1}{2c_1}} f_{456}, \end{aligned}$$

where $c_1 = -\sin(\pi\alpha)/2$, $\alpha \in (0, 1)$ is the only $SO(8)$–invariant of ρ_2. It follows that $\mathfrak{I}_{g2}$ foliates in $SO(8)$–orbits over $(0, 1)$. The automorphism group is $SU(2)/\mathbb{Z}_2 \times SU(2)/\mathbb{Z}_2 \times GL(2) = SO(3) \times SO(3) \times GL(2)$ and since the Lie algebra structure is adapted to g, the stabiliser of ρ_2 in $SO(8)$ is given by $SO(3) \times SO(3) \times SO(2)$. This is covered twice by $SU(2){\cdot}SU(2) \times U(1) \subset Spin(8)$ and we obtain $\mathfrak{I}_{g2} = (0, 1) \times Spin(8)/\big(SU(2){\cdot}SU(2) \times U(1)\big)$.

The induced isometry $\Delta_- \to \Delta_+$ is

$$A_{\rho_2} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{2c_1+1}{2c_1}} & \frac{1}{\sqrt{-2c_1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{-2c_1}} & \sqrt{\frac{2c_1+1}{2c_1}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{2c_1+1}{2c_1}} & \frac{1}{\sqrt{-2c_1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{-2c_1}} & \sqrt{\frac{2c_1+1}{2c_1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2c_1+1}{2c_1}} & \frac{1}{\sqrt{-2c_1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{-2c_1}} & \sqrt{\frac{2c_1+1}{2c_1}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2c_1+1}{2c_1}} & \frac{1}{\sqrt{-2c_1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{-2c_1}} & \sqrt{\frac{2c_1+1}{2c_1}} & 0 & 0 \end{pmatrix}$$

and thus of positive determinant. We conclude as above that $\mathfrak{I}_{g2} \subset \mathfrak{I}_{g+}$.

We obtain the last case for $c_2 = 0$, i.e. $c_1 = -1/2$. Here the stabiliser in $SO(8)$ is isomorphic to $SO(3) \times SO(5)$ whose double covering to $Spin(8)$ is $Sp(1) \cdot Sp(2)$ (using the isomorphism between $SU(2) = Sp(1)$ and $Spin(5) = Sp(2)$). Moreover, $\mathfrak{I}_{g3} \subset \mathfrak{I}_{g+}$, whence the theorem. ■

By the triality principle, we can exchange Δ_+ or Δ_- with Λ^1 while leaving Δ_- or Δ_+ fixed. Hence we get an analogous orbit decomposition for $\Delta_\pm \otimes \Lambda^1$ where the stabiliser subgroups sit now in $SO(\Delta_\mp)$ and lift via $\pi_\mp$ to $Spin(8)$. Note however that the characterisation of $\mathfrak{I}_{g\pm}$ does depend on the module under consideration as the outer triality morphisms reverse the orientation. In any case, the covering group in $Spin(8)$ acts on all three representations and we analyse now this action in detail, where again it suffices to discuss the case where the stabiliser of the isometry lifts through π_0.

We start with the group $PSU(3) \times \mathbb{Z}_2$ which projects to $PSU(3)$ in $SO(\Lambda^1)$, $SO(\Delta_+)$ and $SO(\Delta_-)$. Hence $PSU(3) \subset SO(\Lambda^1)$ gives also rise to $PSU(3)$–invariant isometries in $\Delta_\pm \otimes \Lambda^1$. We immediately deduce that restricted to $PSU(3)$ in $Spin(8)$, the representation spaces Λ^1, Δ_+ and Δ_- are equivalent. In particular, Clifford multiplication $\mu : \Lambda^1 \otimes \Delta_\pm \to \Delta_\mp$ induces an orthogonal product

$$\times : \Lambda^1 \otimes \Delta_+ \cong \Lambda^1 \otimes \Lambda^1 \to \Delta_- \cong \Lambda^1, \tag{3.5}$$

a fact previously noticed in [7].

Next we analyse the case of $SU(2) \cdot SU(2) \times U(1)$. As before, we label irreducible representations by their highest weight expressed in the basis of fundamental roots. Recall that the irreducible representations of $SU(2)$ are given by the symmetric power $\sigma^n = \odot^n \mathbb{C}^2$ of the complex vector representation $\mathbb{C}^2$ and labeled by the half–integer $l = n/2$. They are real for n even and quaternionic for n odd. Consequently, the irreducible representations of $SU(2) \cdot SU(2) \times U(1)$ can be labeled by $(l_1, l_2, m) = (l_1) \otimes (l_2) \otimes (m)$, where the third factor denotes the irreducible S^1–representation $S_m : \theta(z) \mapsto e^{im\theta} \cdot z$ which is one–dimensional and complex. Moreover we will use, as we already did in Theorem 3.1, the notation from [13] and denote a real module V by $[n_1, \ldots, n_l]$ if its complexification $V \otimes \mathbb{C} = (n_1, \ldots, n_l)$ is self–dual (that is, $V \otimes \mathbb{C}$ is complex irreducible). Otherwise, we write $[\![n_1, \ldots, n_l]\!]$, which means that $V \otimes \mathbb{C} = W \oplus \overline{W}$ where W is an irreducible complex module non–equivalent to

$\overline{W}$. By assumption, we then have

$$\Lambda^1 = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2 = [1,0,0] \oplus [0,1,0] \oplus [\![0,0,2]\!].$$

Hence, $SU(2) \cdot SU(2) \times U(1)$ acts with weights 0, α_1, α_2, $2m$ with α_1 and α_2 being the fundamental roots of $SU(2) \times SU(2)$. Substituting

$$x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = 2m,\ x_4 = 0$$

into the $Spin(8)$–weights $\pm x_1, \ldots, \pm x_4$ given by the parameters of the standard Cartan sub–algebra of $Spin(8)$ shows that as a $SU(2) \cdot SU(2) \times U(1)$–space

$$\Delta_+ = \Delta_- = [\![\frac{1}{2}, \frac{1}{2}, 1]\!] = [\![\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S_1]\!]$$

(the $Spin(8)$–weights on $\Delta_\pm$ are $(x_1 \pm \ldots \pm x_4)/2$ with an even (respectively odd) number of minus signs). In particular, the action of $SU(2) \cdot SU(2) \times U(1)$ on $\Delta_\pm$ preserves a complex structure. Note however that this structure does not reduce to $SU(4)$ as the torus component acts non–trivially on $\lambda^{4,0}\Delta_\pm$. Permuting with the triality automorphisms yields a complex structure on Λ^1 and $\Delta_\pm$ if the isometry is an element of $\Lambda^3\Delta_\mp$.

Finally we consider the group $Sp(1) \cdot Sp(2)$, that is

$$\Lambda^1 = \mathfrak{su}(2) \oplus \mathfrak{z}^5 = [1,0,0] \oplus [0,2,-1].$$

Here the first component refers to the representation labeled by α, the fundamental root of $\mathfrak{sp}(1) \otimes \mathbb{C} = \mathfrak{su}(2) \otimes \mathbb{C}$, while the last two indices (m_1, m_2) designate the irreducible $Sp(2)$–representation with respect to the basis of fundamental roots β_1 and β_2. The weights of the action on Λ^1 are 0, α, $\beta_1 + \frac{1}{2}\beta_2$, $\beta_1 + \frac{3}{2}\beta_2$ and substituting as above, we obtain

$$\Delta_+ = \Delta_- = [1/2, 1, 1] = [\mathbb{C}^2 \otimes \mathbb{H}^2]$$

where the quaternionic space $\mathbb{H}^2$ serves as a model for the irreducible spin representation of $Sp(2) = Spin(5)$.

In this paper we focus on the groups $PSU(3)$ and $Sp(1) \cdot Sp(2)$ stabilising a supersymmetric map $\Gamma_\pm \in \Lambda^3\Delta_\pm \subset \Delta_- \otimes \Lambda^1$, thus acting irreducibly on Λ^1. Before we can continue, a thorough discussion of these is in order.

4. Algebraic reductions to $Sp(1) \cdot Sp(2)$ and $PSU(3)$

Definition 4.1. *Let G be a Lie group. The choice of a subgroup $H \subset G$ is called a(n)* (algebraic) reduction *of G to H. One also says that G* (algebraically) reduces *to H.*

Reductions are equivalent to the choice of an H–invariant in the coset space G/H, that is, H arises as the stabiliser of an object acted on by G. For instance, choosing a supersymmetric map reduces $G = Spin(8)$ to $Spin(7)$, $PSU(3)$, $SU(2) \cdot SU(2) \times U(1)$ or $Sp(1) \cdot Sp(2)$. This example stresses the importance of the embedding of H into G as the induced representations depend crucially on how H sits inside G.

First, we briefly review some facts about $Spin(7)$ (cf. for instance [4]). Here, the embedding is induced by a chiral unit spinor which we can always assume, possibly after changing the orientation accordingly, to be of positive chirality. The induced representation on Δ_+ can be decomposed into $\Delta_+ = \mathbf{1} \oplus \mathbb{R}^7$, where $\mathbb{R}^7$ is the vector representation on $\mathbb{R}^7$. It follows that regarded as a $Spin(7)$–module $\Lambda^1 = [1/2, 1, 3/2]$, that is, Λ^1 becomes the irreducible $Spin(7)$–spin representation Δ. Decomposing Λ^4 we also see that $Spin(7)$ stabilises a 4–form ρ which induces a reduction from $GL(8)$ to $Spin(7)$. To understand where it comes from, equip Λ^1 with a quaternionic structure, i.e. $\Lambda^1 = \mathbb{H}^2$. This induces, in particular, the three Kähler 2–forms ω_i, ω_j and ω_k. The $Spin(7)$–invariant 4–form is then

$$\rho = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j - \omega_k \wedge \omega_k.$$

Expressed in a suitable orthonormal frame we have $\omega_i = e_{12} - e_{34} + e_{56} - e_{78}$, $\omega_j = e_{13} + e_{24} + e_{57} + e_{68}$ and $\omega_k = e_{14} - e_{23} + e_{58} - e_{67}$, hence ρ is given by

$$\begin{aligned}\rho = & -e_{1234} + e_{1256} - e_{1278} + e_{1357} + e_{1368} - e_{1458} + e_{1467} \\ & +e_{2358} - e_{2367} + e_{2457} + e_{2468} - e_{3456} + e_{3478} - e_{5678}.\end{aligned}$$

Its stabiliser gives rise to the decomposition $\mathfrak{so}(7) \oplus \mathfrak{so}(7)^\perp$ of irreducible $Spin(7)$–modules inside $\mathfrak{so}(8) = \Lambda^2$. This determines the spinor Ψ_+ as a solution of the equation $a \cdot \Psi_+ = 0$ for all $a \in \mathfrak{so}(7) \subset \Lambda^2$.

This form point of view closely relates to $Sp(1) \cdot Sp(2)$–structures to which we turn next. Here the vector representation of $GL(8)$ restricted to this group gives $\Lambda^1 = [\mathbb{C}^2 \otimes \mathbb{H}^2]$. Elevating this to the fourth exterior power yields an invariant 4–form which is obtained in a similar way as in the $Spin(7)$–case, namely by

$$\rho = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k$$

so that in the orthonormal basis given above

$$\begin{aligned}\rho = & -3e_{1234} + e_{1256} - e_{1278} + e_{1357} + e_{1368} + e_{1458} - e_{1467} \\ & -e_{2358} + e_{2367} + e_{2457} + e_{2468} - e_{3456} + e_{3478} - 3e_{5678}.\end{aligned} \tag{4.1}$$

We refer to any orthonormal basis such that ρ takes the form (4.1) as an $Sp(1)\cdot Sp(2)$*–frame* and whenever we are dealing with this group, we assume such a frame to be chosen unless otherwise stated. The invariant 4–form induces a splitting of $\mathfrak{so}(8)$ into the Lie algebra of the stabiliser and its orthogonal complement. For later applications we need to investigate this decomposition further. If $a^*\Omega = 0$ for $\sum_{i<j} a_{ij} e_i \wedge e_j$ where a^* denotes the usual action of $\mathfrak{gl}(8)$ on exterior forms, then

$$\begin{aligned}
&a_{68} - a_{13} - a_{24} + a_{57} = 0, \quad a_{46} - a_{17} = 0, \quad a_{47} - a_{25} = 0, \quad a_{38} + a_{25} = 0, \quad a_{16} + a_{25} = 0 \\
&a_{23} - a_{14} - a_{67} + a_{58} = 0, \quad a_{35} + a_{17} = 0, \quad a_{28} + a_{17} = 0, \quad a_{27} - a_{18} = 0, \quad a_{36} + a_{18} = 0, \\
&a_{34} - a_{78} + a_{56} - a_{12} = 0, \quad a_{45} + a_{18} = 0, \quad a_{26} - a_{48} = 0, \quad a_{15} - a_{48} = 0, \quad a_{37} - a_{48} = 0.
\end{aligned}$$

A maximal torus is spanned for instance by $a_1 = (e_{12} - e_{34} + e_{56} - e_{78})/2$, $a_2 = e_{12} + e_{34} + e_{56} + e_{78}$ and $a_3 = e_{12} + e_{34} - e_{56} - e_{78}$ with corresponding fundamental roots $\alpha = ia^1$, $\beta_1 = 2i(a^2 - a^3)$ and $\beta_2 = 2ia^5$. The weights of $\Lambda^1 = [\mathbb{C}^2 \otimes \mathbb{H}^2]$ are

$$\pm\tfrac{1}{2}(\alpha+\beta_1),\quad \pm\tfrac{1}{2}(\alpha-\beta_1),\quad \pm\tfrac{1}{2}(\alpha+\beta_1)+\beta_2,\quad \pm\tfrac{1}{2}(\alpha-\beta_1)-\beta_2 \tag{4.2}$$

and the weight vectors are given by $x_{(\alpha+\beta_1)/2} = e_5 - ie_6$, $x_{(\alpha-\beta_1)/2} = e_7 + ie_8$, $x_{(\alpha+\beta_1)/2+\beta_2} = e_1 - ie_2$ and $x_{(\alpha-\beta_1)/2-\beta_2} = e_3 + ie_4$. A different characterisation of the decomposition $\Lambda^2 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp$ is given by the equivariant map $\alpha \mapsto \alpha \llcorner \rho$. A straightforward application of Schur's Lemma yields

Proposition 4.1. *We have* $\mathfrak{sp}(1) := \{\alpha \in \Lambda^2 \,|\, (\alpha\llcorner\Omega) = 5\alpha\}$, $\mathfrak{sp}(2) := \{\alpha \in \Lambda^2 \,|\, (\alpha\llcorner\Omega) = -3\alpha\}$ *and* $\big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp := \{\alpha \in \Lambda^2 \,|\, (\alpha\llcorner\Omega) = \alpha\}$. *Moreover, the projection operators onto these modules are* $\pi^2_3(\alpha) = \big(-3\alpha + 2\alpha\llcorner\Omega + (\alpha\llcorner\Omega)\llcorner\Omega\big)/32$, $\pi^2_{10}(\alpha) = \big(5\alpha - 6\alpha\llcorner\Omega + (\alpha\llcorner\Omega)\llcorner\Omega\big)/32$ *and* $\pi^2_{15}(\alpha) = \big(15\alpha + 2\alpha\llcorner\Omega - (\alpha\llcorner\Omega)\llcorner\Omega\big)/16$ *respectively.*

We also need to investigate the action of $Sp(1)\cdot Sp(2)$ on the remaining exterior powers. As already introduced in the previous section, $\sigma = (1/2) = \mathbb{C}^2$ denotes the vector representation of $Sp(1) = SU(2)$ and $\mathbb{H}^2 = [1/2, 1]$ the vector representation of $Sp(2)$. More generally, there is a kind of Clebsch–Gordan decomposition

$$\Lambda^r \cong \Lambda^r[\sigma \otimes \mathbb{H}^2] \cong \bigoplus_{s=0}^{[r/2]} [\sigma^{r-2s} \otimes V^r_s], \quad 0 \le r \le 8,$$

where the irreducible $GL(2,\mathbb{H})$–module V^r_s is the direct sum of irreducible $Sp(2)$–modules

$$\lambda^r_s = (\frac{r}{2}, [\frac{3+s}{2}]), \quad 0 \le r - 2s \le k$$

([16] – note that our choice of a basis differs from [13]). In particular, $\Lambda^1_0 = (\frac{1}{2}, 1) = \mathbb{H}^2$. We then obtain

Proposition 4.2.

1. $\Lambda^1 = [S \otimes \Lambda^1_0] = [\frac{1}{2}, \frac{1}{2}, 1]$ *is irreducible.*
2. $\Lambda^2 = [S^2] \oplus [\lambda^2_1] \oplus [S^2 \otimes \lambda^2_1] = [1,0,0] \oplus [0,1,2] \oplus [1,1,1]$.
3. $\Lambda^3 = [S \otimes \Lambda^1_0] \oplus [S \otimes \lambda^3_1] \oplus [S^3 \otimes \Lambda^1_0] = [\frac{1}{2}, \frac{1}{2}, 1] \oplus [\frac{1}{2}, \frac{3}{2}, 2] \oplus [\frac{3}{2}, \frac{1}{2}, 1]$
4. $\Lambda^4 = \mathbb{R} \oplus [\lambda^2_0] \oplus [\lambda^4_2] \oplus [S^2 \otimes \lambda^2_0] \oplus [S^2 \otimes \lambda^2_1] \oplus [S^4] = [0,0,0] \oplus [0,1,1] \oplus [0,2,2] \oplus [1,1,1] \oplus [1,1,2] \oplus [2,0,0]$

The decomposition of the remaining modules follow from the Hodge–duality $\Lambda^p_q \cong \Lambda^{n-p}_q$.

As proven in the previous section, $Sp(1)\cdot Sp(2)$ also stabilises a supersymmetric map Γ_+ in $\Lambda^3\Delta_+$ which describes the reduction from $Spin(8)$ to $Sp(1)\cdot Sp(2)$. It is determined (up to a scalar) by the equation

$$a(\Gamma_+) \;=\; \frac{1}{2}\kappa(a)\cdot\Gamma_+ - \Gamma_+ \circ a \;=\; 0.$$

supposed to hold for all $a \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ and can be computed using the matrix representation κ in Appendix A. Given as a matrix with respect to some $Sp(1) \cdot Sp(2)$–frame and fixed orthonormal basis of $\Delta_\pm$ we find

$$\Gamma_+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.3}$$

Its determinant is -1 in accordance with Theorem 3.4 since the outer triality morphisms reverse the orientation.

Finally, we investigate the group $PSU(3) = SU(3)/Z\big(SU(3)\big)$ [21]. It is the compact, 8–dimensional identity component of the automorphism group of $\mathfrak{su}(3)$. In particular, the adjoint representation $Ad : SU(3) \to SO(8)$ descends to an embedding $PSU(3) \hookrightarrow SO(8) \subset GL(8)$, so that $\Lambda^1 = \mathfrak{su}(3)$. The group $PSU(3)$ arises as the stabiliser of the 3–form

$$\rho(X, Y, Z) = -\frac{1}{6}B([X, Y], Z)$$

inside $GL_+(8)$. Since $\dim GL_+(8) - \dim PSU(3) = \dim \Lambda^3$, the $GL_+(8)$–orbit of $PSU(3)$–invariant forms is *open* which means that these forms are *stable* following the language of [7]. As we have already used above, a $PSU(3)$–invariant form ρ can be expressed in a suitable frame as

$$\rho = \frac{1}{2}e_{123} + \frac{1}{4}e_1(e_{47} - e_{56}) + \frac{1}{4}e_2(e_{46} + e_{57}) + \frac{1}{4}e_3(e_{45} - e_{67}) + \frac{\sqrt{3}}{4}e_8(e_{45} + e_{67}). \tag{4.4}$$

When dealing with $PSU(3)$, we will always assume to work with such a frame (to which we also refer as a $PSU(3)$*–frame*) unless otherwise stated.

Next we will discuss some elements of the representation theory for $PSU(3)$. The Lie algebra of its stabiliser is given by the vectors $x \llcorner \rho \in \Lambda^2$ and a maximal torus is spanned by $x_3 = e_3 \llcorner \rho$ and $x_8 = e_8 \llcorner \rho$ with roots are given by $\pm\alpha_1 = \pm i(x^3 + \sqrt{3}x^8)/2$, $\pm\alpha_2 = \pm i(x^3 - \sqrt{3}x^8)/2$ and $\pm(\alpha_1 + \alpha_2) = \pm ix^3$ and weight vectors (in $\Lambda^1 \cong \mathfrak{su}(3)$) $x_{\alpha_1} = e_4 - ie_5$, $x_{\alpha_1} = e_5 + ie_6$ and $x_{\alpha_1+\alpha_2} = e_1 - ie_2$. For the exterior algebra we find the following decomposition.

Proposition 4.3.

1. $\Lambda^1 = \mathfrak{su}(3) = [1, 1]$ *is irreducible.*
2. $\Lambda^2 = \Lambda^2_8 \oplus \Lambda^2_{20} = [1, 1] \oplus [\![1, 2]\!]$.

3. $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_8 \oplus \Lambda^3_{20} \oplus \Lambda^3_{27} = \mathbf{1} \oplus [1,1] \oplus [\![1,2]\!] \oplus [2,2]$

4. $\Lambda^4 = 2\Lambda^4_8 \oplus 2\Lambda^4_{27} = 2[1,1] \oplus 2[2,2]$

We used the standard notation whereby Λ^p_q represents a q–dimensional irreducible subspace of Λ^p.

In the situation of $PSU(3)$, we can approach this decomposition also from a cohomological point of view well–suited for our later purposes. The Lie algebra structure on $\Lambda^1 = \mathfrak{su}(3)$ induces a $PSU(3)$–invariant operator $b_k : \Lambda^k \to \Lambda^{k+1}$ by extension of

$$be_i = \sum_{j<k} c_{ijk} e_j \wedge e_k.$$

Since b is built out of the structure constants, it is just the exterior differential operator restricted to the left–invariant differential forms of $SU(3)$ with adjoint $b^* = d^* = -\star d\star$. The resulting elliptic complex is isomorphic to the de Rham cohomology $H^*(SU(3),\mathbb{R})$ which is trivial except for the Betti numbers $b_0 = b_3 = 1 = b_5 = b_8$. Hence, $\operatorname{im} b_k = \ker b_{k+1}$ for $k = 0,\ 1,\ 3,\ 5,\ 6$ and $\operatorname{im} b_k = \ker b_{k+1} \oplus \mathbb{R}$ for $k = -1,\ 2,\ 4,\ 7$. Schematically, we have

$$\begin{array}{ccccccccccccccccc}
\Lambda^0_1 & & & & & & \Lambda^3_1 & & & & \Lambda^5_1 & & & & & & \Lambda^8_1 \\
& & \Lambda^1_8 & \xrightarrow{b} & \Lambda^2_8 & & \Lambda^3_8 & \xrightarrow{b} & \Lambda^4_8 & & & & \Lambda^6_8 & \xrightarrow{b} & \Lambda^7_8 & & \\
& & & & & & & & \Lambda^4_8 & \xrightarrow{b} & \Lambda^5_8 & & & & & & \\
& & & & \Lambda^2_{20} & \xrightarrow{b} & \Lambda^3_{20} & & & & \Lambda^5_{20} & \xrightarrow{b} & \Lambda^6_{20} & & & & \\
& & & & & & \Lambda^3_{27} & \xrightarrow{b} & \Lambda^4_{27} & & & & & & & & \\
& & & & & & & & \Lambda^4_{27} & \xrightarrow{b} & \Lambda^5_{27} & & & & & &
\end{array} \tag{4.5}$$

with an arrow indicating the non–trivial maps. In particular, we will use the more natural splitting of Λ^4 into $\Lambda^4_o = \ker b_3$ and $\Lambda^4_i = \operatorname{im} b^*$ instead of the $SO(8)$–equivariant splitting into self– and anti–self–dual forms.

Using the b–operator and its co–differential, we can easily construct the projection operators for Λ^2.

Proposition 4.4. *For any $\alpha \in \Lambda^2$ we have $b(\alpha) = -\alpha^*\rho$. Moreover, $\Lambda^2_8 = \ker b_2$ and the projection operator on the complement is $\pi^2_{20}(\alpha) = \frac{4}{3} b^*_3 b_2(\alpha)$. For the complexification, we find $\Lambda^2_{20} \otimes \mathbb{C} = \Lambda^2_{10+} \oplus \Lambda^2_{10-} = (1,2) \oplus (2,1)$ where*

$$\Lambda^2_{10\pm} = \{\alpha \in \Lambda^2 \otimes \mathbb{C} \mid \star(\rho \wedge \alpha) = \pm i\sqrt{3}\alpha^*\rho\}.$$

The projection operators are $\pi^2_{10\pm}(\alpha) = \frac{2}{3} b^*_3 b_2(\alpha) \mp \frac{8\sqrt{3}}{9} i \star (b_2(\alpha) \wedge \rho)$.

The proof can be readily verified by applying Schur's Lemma with the sample vectors $x_{\alpha_2} \wedge x_{\alpha_1+\alpha_2} \in (1,2)$ and $x_{\alpha_1} \wedge x_{\alpha_1+\alpha_2} \in (2,1)$.

Finally, we want to characterise $PSU(3)$ in Riemannian terms. It stabilises two supersymmetric maps $\Gamma_\pm \in \Lambda^3\Delta_\pm$. These are characterised (up to a scalar) by the equations

$$x \llcorner \rho(\Gamma_\pm) \;=\; \frac{1}{2}\kappa(x \llcorner \rho) \cdot \Gamma_\pm - \Gamma_\pm \circ x \llcorner \rho \;=\; 0.$$

For their matrices with respect to a $PSU(3)$–frame and a fixed orthonormal basis of $\Delta_{\pm}$, we find

$$\Gamma_+ = \begin{pmatrix} -\tfrac{1}{2} & 0 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & 0 \\ 0 & -\tfrac{1}{2} & 0 & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & 0 \\ 0 & \tfrac{1}{2} & 0 & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & 0 \\ 0 & \tfrac{1}{2} & 0 & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & -\tfrac{1}{2} \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & 0 \\ 0 & \tfrac{1}{2} & 0 & -\tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \end{pmatrix}, \quad \Gamma_- = \begin{pmatrix} 0 & -\tfrac{1}{2} & 0 & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & 0 \\ 0 & \tfrac{1}{2} & 0 & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & \tfrac{1}{2} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & 0 \\ 0 & -\tfrac{1}{2} & 0 & -\tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{2} \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & 0 \\ 0 & -\tfrac{1}{2} & 0 & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & \tfrac{1}{2} & -\tfrac{\sqrt{3}}{4} & -\tfrac{1}{4} & \tfrac{\sqrt{3}}{4} & \tfrac{1}{4} & 0 \end{pmatrix}.$$

They are of determinant 1 in accordance with what we found for the $Sp(1) \cdot Sp(2)$–case.

5. Topological reductions to $Sp(1) \cdot Sp(2)$ and $PSU(3)$

Definition 5.1. *Let G be a Lie group and M^n be a manifold endowed with a principal G–fibre bundle P_G. The choice of a subgroup $H \subset G$ together with an principal H–fibre subbundle $P_H \subset P_G$ such that the action of H on P_H coincides with the restriction of the action of G on P_G to H and P_H is called a* (topological) reduction *of P_G to P_H. One also says that P_G* (topologically) reduces *to P_H.*

We usually drop the adjective "topological" if there is no risk of confusion with algebraic or geometric reductions (where a fixed G–connection restricts on P_H to an H–connection).

Definition 5.2. *Let M^8 be an 8–fold. A topological reduction from the ($GL(8)$–) frame bundle to an $Sp(1) \cdot Sp(2)$– or $PSU(3)$–principal fibre bundle, where $\iota : Sp(1) \cdot Sp(2) \hookrightarrow GL(8)$ is the natural inclusion and $Ad : PSU(3) \hookrightarrow GL(8)$, is called an $Sp(1) \cdot Sp(2)$– or* $PSU(3)$–structure.

Akin to its algebraic analogue, topological reductions from G to H are in 1-1–correspondence with sections of the associated bundle with fibre G/H. Therefore, an $Sp(1) \cdot Sp(2)$– or $PSU(3)$–structure exists if and only if there is a global 4– or 3–form which locally can always be written as in (4.1) or (4.4) for some basis of T. The existence of such a section is usually topologically obstructed. Since both groups live in a group conjugated to $SO(8)$, an $Sp(1) \cdot Sp(2)$– or $PSU(3)$–structure induces a principal $SO(8)$–fibre bundle by extension and thus a globally defined metric as well as an orientation. Moreover this inclusion can be lifted to $Spin(8)$, so we also get a canonic spin structure. In particular, the manifold is spinnable, that is, the first two Stiefel–Whitney classes vanish, $w_1 = w_2 = 0$. The problem of reducing $GL(8)$ to one of these groups then boils down to the problem of reducing $Spin(8)$ to $PSU(3)$ or $Sp(1) \cdot Sp(2)$, or equivalently, to the existence of a supersymmetric map. For $Spin(7)_{\pm}$–structures, this obstruction can be identified with the Euler class of $\chi(\Delta_{\pm})$ as we are asking for a section in the bundle with associated fiber $Spin(8)/Spin(7)_{\pm} \cong S^7$. The following result is classical [9] and is an easy consequence of the Borel–Hirzebruch formalism [3] illustrated below.

Proposition 5.1. *The $Spin(8)$–structure of a spinnable 8–fold reduces to $Spin(7)_\pm$ if and only if* $16e(\Delta_\pm) = \pm 8e + p_1^2 - 4p_2 = 0$.

This proposition stresses once more the importance of the embedding $H \hookrightarrow G$ as the topological obstructions are not invariant under the outer triality morphism.

To tackle this question in the case of $Sp(1) \cdot Sp(2)$– and a $PSU(3)$–structures, let $\pm x_1, \ldots, \pm x_4$ denote the weights of the vector representation of $Spin(8)$. Formally, the total Pontrjagin class p and the Euler class e of M are expressed as the product

$$p = \prod (1 + x_i^2), \quad e = \prod x_i.$$

Now consider $Sp(1) \cdot Sp(2)$. Here, the weights of $\mathbb{R}^8 = [\sigma \otimes \Lambda_0^1]$ are given by (4.2). Substituting this into the $Spin(8)$–weights

$$x_1 = \frac{1}{2}(\alpha - \beta_1), \quad x_2 = \frac{1}{2}(\alpha + \beta_1), \quad x_3 = \frac{1}{2}(\alpha - \beta_1) - \beta_2, \quad x_4 = \frac{1}{2}(\alpha + \beta_1) + \beta_2$$

yields $p_1 = \alpha^2 + \beta_1^2 + 2\beta_1\beta_2 + 2\beta_2^2$, $16p_2 = 6\alpha^4 + 4\alpha^2\beta_1^2 + 8\alpha^2\beta_1\beta_2 + 8\alpha^2\beta_2^2 + 6\beta_1^4 + 32\beta_1\beta_2^3 + 40\beta_1^2\beta_2^2 + 24\beta_1^3\beta_2 + 16\beta_2^4$ and $16e = \alpha^4 - 2\alpha^2\beta_1^2 - 4\alpha^2\beta_1\beta_2 - 4\alpha^2\beta_2^2 + \beta_1^4 + 4\beta_1^3\beta_2 + 4\beta_1^2\beta_2^2$. Consequently, a necessary condition for the existence of an $Sp(1) \cdot Sp(2)$–structure is $8e + p_1^2 - 4p_2 = 0$. An exhaustive treatment of all topological obstructions was given in [18]. By using Moore–Postnikov factorisations, the authors also found sufficient conditions for existence.

Theorem 5.2. [18] *Let M be an oriented closed connected spinnable manifold of dimension 8. If M carries an $Sp(1) \cdot Sp(2)$–structure, then $8e + p_1^2 - 4p_2 = 0$. Moreover, provided that $H^2(M, \mathbb{Z}_2) = 0$, we have $w_6 = 0$ and there exists an $R \in H^4(M, \mathbb{Z})$ such that $Sq^2\rho_2 R = 0$, $(Rp_1 - 2R^2)[M] \equiv 0 \bmod 16$ and $(R^2 + Rp_1 - e)[M] \equiv 0 \bmod 4$, where $[M]$ denotes the fundamental class of M. Conversely, these conditions are sufficient (regardless of $H^2(M, \mathbb{Z}_2) = 0$) to ensure the existence of an $Sp(1) \cdot Sp(2)$–structure.*

In the $PSU(3)$–case, the tangent space is associated with the adjoint representation, so the $PSU(3)$–weights are just the roots. Substituting

$$x_1 = \alpha, \quad x_2 = \beta, \quad x_3 = \alpha + \beta, \quad x_4 = 0,$$

a reduction to $PSU(3)$ implies $p_1 = 2(\alpha^2 + \alpha\beta + \beta^2)$ and $p_2 = \alpha^4 + 2\alpha^3\beta + 3\alpha^2\beta^2 + 2\alpha^3\alpha + \beta^4$, hence $4p_2 = p_1$. Moreover, we obviously have $e = 0$. A first consequence is the following classification result.

Proposition 5.3. *Let $(G/H, g)$ be a compact Riemannian homogeneous space with G simple. If $M = G/H$ admits a topological $PSU(3)$–structure, then G/H is diffeomorphic to $SU(3)$.*

Proof: Since G sits inside the isometry group of (M, g), its dimension is less than or equal to $9 \cdot 8/2 = 36$. If we had equality, then M would be diffeomorphic to a torus or, up to a finite covering, to an 8–sphere. While the first case is ruled out for G has to be simple,

G	H up to a covering	$dim(H)$	$rk(H)$
A_2	$\{1\}$	0	0
A_3	$A_1 \times A_1 \times S^1$	7	3
A_4	$A_3 \times S^1,\ G_2 \times S^1 \times S^1,\ A_2 \times A_2$	16	4
A_5	$A_1 \times A_4,\ A_1 \times A_1 \times B_3,\ A_1 \times A_1 \times C_3$	27	5
B_2	$S^1 \times S^1$	2	2
B_3	$A_1 \times A_2$	13	3
C_3	$A_1 \times A_2$	13	3
D_4	$A_1 \times A_1 \times G_2$	20	4
G_2	$A_1 \times A_1$	6	2

Table 1:

the second case is excluded since $e(S^8) \neq 0$. Hence G must be, up to a covering, a group of type $A_1, \dots, A_5$, B_2, B_3, C_3, D_4 or G_2. As a closed subgroup of G, H is compact and hence reductive. Therefore H is covered by a direct product of simple Lie groups and a torus, that is the Lie algebra of H is isomorphic to $\mathfrak{h} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \oplus \mathfrak{t}^l$. If we denote by $rk(G)$ the rank of the Lie group G, we get the following necessary conditions:

$$
\begin{aligned}
& k \leq rk(G) \\
& l + \sum rk(\mathfrak{g}_i) \leq rk(G) \\
& l + \sum dim(\mathfrak{g}_i) = dim(G) - 8,
\end{aligned}
$$

which yields the possibilities displayed in Table 1. It follows that H is of maximal rank, that is $rk(H) = rk(G)$, unless $G = SU(3)$ and $H = \{1\}$. By [15] the first case however implies that $e(G/H) \neq 0$. ■

By a straightforward computation using the definition of the $\hat{A}$–genus and the signature we derive the following.

Lemma 5.4. *If M is a compact spin manifold such that $p_1^2 = 4p_2$, then $sgn(M) = 16\hat{A}[M]$. In particular, $sgn(M) \equiv 0 \mod 16$.*

Let (b_4^+, b_4^-) be the signature of the Poincaré pairing on $H^4(M, \mathbb{Z})$ i.e. $sgn(M) = b_4^+ - b_4^-$.

Corollary 5.5. *Let M be a compact simply–connected manifold with a $PSU(3)$–structure. If $\hat{A}[M] = 0$ (e.g. if there exists a metric with strictly positive scalar curvature), then $1 + b_2 + b_4^+ = b_3$.*

Example: As we have already used in Section 4, $H^*(SU(3), \mathbb{R})$ is isomorphic to the space of ad–invariant forms in $\Lambda^*\mathfrak{su}(3)$ which is spanned by $\{1, \rho, \star\rho, vol\}$. Therefore, $b_2 = 0$, $b_3 = 1$ and $b_4^+ = 0$ in accordance with the corollary.

As $e = 0$ and $sgn(M) \equiv 0 \bmod 4$, we can assert the existence of two linearly independent vector fields [17]. The orthogonal product $\times$ in (3.5) produces a third one. In particular, this causes the sixth Stiefel-Whitney class of M to vanish. Taking $k = 0$ in the following proposition establishes the existence of four linearly independent vector fields.

Proposition 5.6. [19] *Let M be a closed connected smooth spin manifold of dimension 8. If $w_6(M) = 0$, $e(M) = 0$ and $\{4p_2(M) - p_1^2(M)\}[M] \equiv 0 \bmod 128$, and if there is a $k \in \mathbb{Z}$ such that $4p_2(M) = (2k-1)^2 p_1^2(M)$ and $k(k+2)p_2(M)[M] \equiv 0 \bmod 3$, then M has four linearly independent vector fields.*

As a consequence, the fifth Stiefel-Whitney class has to vanish.

Proposition 5.7. *We have $w_4^2 = 0$. In particular, all Stiefel-Whitney numbers vanish.*

Proof: By Wu's formula,

$$\mathrm{Sq}^k(w_m) = w_k w_m + \binom{k-m}{1} w_{k-1} w_{m+1} + \ldots + \binom{k-m}{k} w_0 w_{m+k}.$$

A further theorem of Wu asserts that

$$w_k = \sum_{i+j=k} \mathrm{Sq}^i(v_j),$$

where the elements $v_k \in H^k(M, \mathbb{Z}_2)$ are defined through the identity $v_k \cup x[M] = \mathrm{Sq}^k(x)[M]$ which holds for any $x \in H^{n-k}(M, \mathbb{Z}_2)$. In particular, we have $v_i = 0$ for $i > 4$. It follows that $v_1 = v_2 = v_3 = 0$, $w_4 = v_4$ and $w_8 = \mathrm{Sq}^4 w_4 = w_4^2 = 0$. ■

We summarise our results in the following proposition.

Proposition 5.8. *If a closed and oriented 8–manifold M carries a topological $PSU(3)$–structure, then $w_i = 0$ for all Stiefel–Whitney classes except for $i = 4$ where $w_4^2 = 0$. Moreover, we have $e = 0$ and $p_1^2 = 4p_2$. There exist four linearly independent vector fields on M and all Stiefel-Whitney numbers vanish.*

The question of sufficient conditions occupies us next. Again, let M^8 be again a connected, closed and spinnable 8-manifold. The idea will be to derive conditions for an $SU(3)$–structure and to ask when this arises as the 3–fold covering of an $PSU(3)$–structure. More concretely, consider the complex rank 3 vector bundle E associated with the vector representation of $SU(3)$. If the adjoint bundle $\mathfrak{su}(E) = P_{SU(3)} \times \mathfrak{su}(3)$ is isomorphic to the tangent bundle, then T is associated with a $PSU(3)$–structure coming from the projection $SU(3) \to PSU(3) = SU(3)/\ker Ad$. However, not every $PSU(3)$-structure arises in this way. A basic Čech cohomology argument implies that principal G-fibre bundles over M are classified by $H^1(M, G)$ (see, for example, [9] Appendix A). The exact sequence (where $\mathbb{Z}_3$ is central)

$$1 \to \mathbb{Z}_3 \to SU(3) \xrightarrow{p} PSU(3) \to 1$$

gives rise to an exact sequence

$$\ldots \to H^1(M, \mathbb{Z}_3) \to \mathrm{Prin}_{SU(3)}(M) \xrightarrow{p_*} \mathrm{Prin}_{PSU(3)}(M) \xrightarrow{t} H^2(M, \mathbb{Z}_3).$$

(where $\mathrm{Prin}_G(M)$ denotes the set of G-principal bundles over M). Hence, a principal $PSU(3)$-bundle P is induced by an $SU(3)$–bundle if and only if the obstruction class $t(P) \in$

$H^2(M, \mathbb{Z}_3)$ vanishes. Following [2] where the authors consider $PSU(3)$–structures over 4–manifolds, we call this class the *triality class*. By the universal coefficients theorem this obstruction vanishes trivially if $H^2(M, \mathbb{Z}) = 0$ and $H^3(M, \mathbb{Z})$ has no torsion elements of order divisible by three.

If $f : M \to BPSU(3)$ is a classifying map for P, then $t(P) = f^*t$ for the *universal* triality class $t \in H^2(BPSU(3), \mathbb{Z}_3)$. It is induced by $c_1(E_{U(3)})$, the first Chern class of the universal $U(3)$-bundle $E_{U(3)}$ [23]. Concretely, let $\overline{p} : U(3) \to PU(3)$ denote the natural projection. The inclusion $SU(3) \subset U(3)$ induces an isomorphism between $PSU(3)$ and $PU(3)$ and therefore identifies $BPSU(3)$ with $BPU(3)$. Since $BPU(3)$ is simply connected and $\pi_2\big(BU(3)\big) = \mathbb{Z} \to \pi_2\big(BPU(3)\big) = \mathbb{Z}_3$ is the reduction mod 3 map $\rho_3 : \mathbb{Z} \to \mathbb{Z}_3$, the Hurewicz isomorphism theorem and the universal coefficients theorem imply $H^2(BPU(3), \mathbb{Z}_3) = \mathbb{Z}_3$ and that $B\overline{p}^* : H^2(BPU(3), \mathbb{Z}_3) \to H^2(BU(3), \mathbb{Z}_3)$ is an isomorphism. Then

$$t = (B\overline{p}^*)^{-1} \rho_{3*} c_1(E_{U(3)}).$$

If the triality class vanishes the problem of finding sufficient conditions for the existence of a $PSU(3)$–structure reduces to the problem of the existence of a complex rank 3 vector bundle E with $\mathfrak{su}(E) = T$. This question is settled in the next theorem.

Theorem 5.9. *Suppose that M is a connected, closed and spinnable 8–manifold. Then the frame bundle reduces to a principal $PSU(3)$–fibre bundle P with $t(P) = 0$ if and only if $e = 0$, $4p_2 = p_1^2$, $w_6 = 0$, p_1 is divisible by 6 and $p_1^2[M] \in 216\mathbb{Z}$.*

Proof: Let us first assume that there exists a $PSU(3)$–fibre bundle P coming from a reduction $Ad : SU(3) \to SO(8)$ with principal bundle $\widetilde{P}$. In virtue of Proposition 5.8 we only need to establish the last two conditions. We define the complex vector bundle $E = \widetilde{P} \times_{SU(3)} \mathbb{C}^3$ so that $\mathfrak{su}(E) = T$, and compute the Pontrjagin classes of M. We have $\mathfrak{su}(3) \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$, hence $T \otimes \mathbb{C}$ equals $\mathrm{End}_0(E)$, the bundle of trace–free complex endomorphisms. The Chern character of $T \otimes \mathbb{C}$ equals (see for instance [12])

$$ch(T \otimes \mathbb{C}) = 8 + p_1 + \frac{1}{12}(p_1^2 - 2p_2).$$

On the other hand,

$$ch\big(End(E)\big) = ch(E \otimes \overline{E}) = 1 + ch\big(End_0(E)\big).$$

Now for a complex vector bundle with $c_1(E) = 0$,

$$ch(E) = 3 - c_2(E) + \frac{1}{2}c_3(E) + \frac{1}{12}c_2(E)^2$$

and $c_i(E) = (-1)^i c_i(\overline{E})$ which implies

$$ch(E \otimes \overline{E}) = ch(E) \cup ch(\overline{E}) = 9 - 6c_2(E) + \frac{3}{2}c_2(E)^2.$$

As a consequence

$$p_1 = p_1\big(\mathfrak{su}(E)\big) = -6c_2(E), \quad p_2 = p_2\big(\mathfrak{su}(E)\big) = 9c_2(E)^2. \tag{5.1}$$

In particular, p_1 is divisible by 6 (we also rederive the relation $4p_2 = p_1^2$). Moreover M is spinnable, hence the spin index $\hat{A} \cup \mathrm{ch}(E)[M]$ is an integer. Since

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) = 1 - \frac{1}{24}p_1 + \frac{1}{960}p_1^2$$

it follows

$$\hat{A} \cup ch(E)[M] = 3\hat{A}[M] + \frac{1}{24}p_1 c_2(E) + \frac{1}{12}c_2(E)^2 = 3\hat{A}[M] - \frac{1}{216}p_1^2[M] \in \mathbb{Z},$$

which means $p_1^2[M] \in 216\mathbb{Z}$, proving the necessity of the conditions.

For the proof of the converse I am indebted to ideas of M. Crabb. Let $B \subset M$ be an embedded open disc in M and consider the exact sequence of K–groups

$$K(M, M - B) \to K(M) \to K(M - B).$$

We have $K(M, M - B) = \widetilde{K}(S^8) \cong \mathbb{Z}$ and the sequence is split by the spin index

$$x \in K(M) \mapsto \hat{A} \cup \mathrm{ch}(x)[M] \in \mathbb{Z}$$

which therefore classifies the stable extensions over $M - B$ to M. The first step consists in finding a stable complex vector bundle ξ over $M - B$ such that $c_1(\xi) = 0$ and $\mathfrak{su}(3)$ is stably equivalent to $T_{|M-B}$. To that end, let $[(M - B)_+, BSU(\infty)] \subset K(M - B)$ denote the set of pointed homotopy classes, the subscript $+$ indicating a disjoint basepoint. Let (c_2, c_3) be the map which takes an equivalence class of $[(M - B)_+, BSU(\infty)]$ to the second and third Chern class of the associated bundle.

Lemma 5.10. *The image of the map*

$$(c_2, c_3) : [(M - B)_+, BSU(\infty)] \to H^4(M, \mathbb{Z}) \oplus H^6(M, \mathbb{Z})$$

is the set $\{(u, v) \mid Sq^2\rho_2 u = \rho_2 v\}$, *where* $\rho_2 : \mathbb{Z} \to \mathbb{Z}_2$ *is reduction* mod 2.

Proof: We first prove that for a complex vector bundle ξ with $c_1(\xi) = 0$, we have

$$Sq^2\rho_2 c_2(\xi) = \rho_2 c_3(\xi). \tag{5.2}$$

Now if W_i denote the Stiefel–Whitney classes of the *real* vector bundle underlying ξ this is equivalent to $Sq^2 W_4 = W_6$. On the other hand, Wu's formula implies

$$Sq^2 W_4 = W_2 W_4 + W_6$$

and thus (5.2) since $W_2 = \rho_2 c_1 = 0$. Next let $i : F \hookrightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ denote the homotopy fibre of the induced map

$$Sq^2 \circ \rho_2 + \rho_2 : K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6) \to K(\mathbb{Z}_2, 6).$$

The relation (5.2) implies that the map $(c_2, c_3) : BSU(\infty) \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ is null–homotopic. Consequently, (c_2, c_3) lifts to a map $k : BSU(\infty) \to F$, thereby inducing an isomorphism of homotopy groups $\pi_i\big(BSU(\infty)\big) \to \pi_i(F)$ for $i \le 7$ and a surjection for $i = 8$. By the exact homotopy sequence for fibrations we conclude on one hand that $\pi_4(F) = \mathbb{Z}$, $\pi_6(F) = 2\mathbb{Z}$ and $\pi_i(F) = 0$ for i otherwise. On the other hand, the Chern class $c_2 : \pi_4\big(BSU(\infty)\big) \cong \widetilde{K}(S^4) = \mathbb{Z} \to \pi_4\big(K(\mathbb{Z},4) \times K(\mathbb{Z},6)\big) \cong H^4(S^4,\mathbb{Z}) = \mathbb{Z}$ is an isomorphism and $c_3 : \pi_4\big(BSU(\infty)\big) \cong \widetilde{K}(S^6) = \mathbb{Z} \to \pi_6\big(K(\mathbb{Z},4) \times K(\mathbb{Z},6)\big) \cong H^6(S^6,\mathbb{Z}) = \mathbb{Z}$ is multiplication by 2. Since $M - B$ is 8–dimensional, it follows that the induced map $k_* : [(M-B)_+, BSU(\infty)] \to [(M-B)_+, F]$ is surjective. The horizontal row in the commutative diagram

$$\begin{array}{ccccc} [(M-B)_+, F] & \xrightarrow{\ i_*\ } & H^4(M,\mathbb{Z}) \oplus H^6(M,\mathbb{Z}) & \xrightarrow{Sq^2\rho_2 + \rho_2} & H^6(M,\mathbb{Z}_2) \\ & \nwarrow^{k_*} & \uparrow (c_2, c_3) & & \\ & & [(M-B)_+, BSU(\infty)] & & \end{array}$$

is exact, hence $\operatorname{im}(c_2, c_3) = \operatorname{im} i_* = \ker(Sq^2 \circ \rho_2 + \rho_2)$. □

By assumption $p_1 \in H^4(M,\mathbb{Z})$ is divisible by 6 and therefore we can write $p_1 = -6u$ for $u \in H^4(M,\mathbb{Z})$. On the other hand, $p_1 = 2q_1$, where q_1 is the first spin characteristic class that satisfies $\rho_2(q_1) = w_4$. Hence $Sq^2\rho_2(u) = Sq^2 w_4 = w_2 w_4 + w_6 = 0$, and the previous lemma implies the existence of a stable complex vector bundle ξ such that $c_1(\xi) = 0$, $c_2(\xi) = u$ and $c_3(\xi) = 0$. From (5.1) it follows that $p_1(\mathfrak{su}(\xi)) = p_1$ and since $w_2(\mathfrak{su}(\xi)) = 0$, $\mathfrak{su}(\xi)$ and T are stably equivalent over the 4–skeleton $M^{(4)}$ [22]. Then $\mathfrak{su}(\xi)$ and T are stably equivalent over $M - B$ as the restriction map $KO(M-B) \to KO(M^{(4)})$ is injective. This follows from the exact sequence

$$KO(M^{(i+1)}, M^{(i)}) \to KO(M^{(i+1)}) \to KO(M^{(i)}).$$

By definition $KO(M^{(i+1)}, M^{(i)}) = \widetilde{KO}(M^{(i+1)}/M^{(i)})$ and $M^{(i+1)}/M^{(i)}$ is a disjoint union of spheres S^{i+1}. But $\widetilde{KO}(S^{i+1}) = 0$ for $i = 4$, 5 and 6 and therefore the map $KO(M^{(i+1)}) \to KO(M^{(i)})$ is injective. Since $M = M^{(8)}$ is the disjoint union of $M^{(7)}$ and a finite number of open embedded discs, the assertion follows. Next we extend ξ over B to a stable bundle on M. The condition to be represented by a complex vector bundle E of rank 3 is $c_4(\xi) = 0$. As pointed out above, such a bundle exists if the spin index

$$\hat{A} \cup \mathrm{ch}(\xi)[M] = 3\hat{A}[M] + p_1 u/24 + u^2/12$$

is an integer, but this holds by assumption. Next $p_2(\mathfrak{su}(\xi)) = 9u^2 = p_2$ and as a consequence, $\mathfrak{su}(\xi)$ is stably isomorphic to T [22]. Finally, two stably isomorphic oriented real vector bundles of rank 8 are isomorphic as $SO(8)$–bundles if they have the same Euler class. Since $e(\mathfrak{su}(E)) = 0$, we conclude $T = \mathfrak{su}(E)$. ■

Corollary 5.11. *If M is closed and carries a $PSU(3)$–structure with vanishing triality class, then $\hat{A}[M] \in 40\mathbb{Z}$ and $sgn(M) \in 640\mathbb{Z}$.*

6. The Dirac equation

For $Spin(7)$–structures we have a natural integrability condition, namely for the invariant spinor to be parallel with respect to the Levi–Civita connection it induces. This, on the other hand, turns out to be equivalent for the invariant self–dual 4–form to be closed. While the latter point of view can also be adopted for $PSU(3)$– and $Sp(1)\cdot Sp(2)$–structures, parallelity of the invariant supersymmetric maps is to strong a requirement. Indeed, $PSU(3)$ does not appear on Berger's list; any irreducible $PSU(3)$–structure with parallel supersymmetric map is either locally symmetric or flat. Moreover, closeness of the $Sp(1)\cdot Sp(2)$–invariant 4–form does not imply for the holonomy of the induced metric to be contained in that group [14] and consequently, the corresponding supersymmetric will not be parallel either. In order to find a natural integrability condition on the supersymmetric map, we first reformulate the integrability condition in the $Spin(7)$–case by introducing the twisted Dirac operator Đ on $\Delta\otimes\Lambda^1$, locally given by

$$Đ(\Psi\otimes X)=\sum e_i\cdot\Psi\otimes\nabla_{e_i}X+e_i\cdot\nabla_{e_i}\Psi\otimes X,$$

where ∇ denotes the Levi–Civita connection as well as its lift to the spin bundle. With respect to the splitting $\Delta\otimes\Lambda^1=\Delta\oplus\ker\mu$ the Dirac operator Đ takes the shape [20]

$$Đ=\begin{pmatrix}-\frac{3}{4}\iota\circ D\circ\iota^{-1} & 2\iota\circ đ\\ \frac{1}{4}P\circ\iota^{-1} & Q\end{pmatrix}. \tag{6.1}$$

As in Section 2, the map $\iota:\Delta\to\Delta\otimes\Lambda^1$ is given by $\iota(\Psi)(X)=-X\cdot\psi/8$. Moreover, $D:\Delta\to\Delta$ and $P:\Delta\to\ker\mu$ denote the usual Dirac– and Twistor–operator. Locally,

$$D(\Psi)=\sum e_i\cdot\nabla_{e_i}\Psi,\quad P(X\otimes\Psi)=\sum e_i\otimes\nabla_{e_i}\Psi-\frac{1}{8}e_i\otimes e_i\cdot D(\Psi),$$

and $đ:\Lambda^1\otimes\Delta\to\Delta$ is the twisted co-differential,

$$đ(X\otimes\Psi)=-2\sum(e_i\llcorner\nabla_{e_i}X)\cdot\Psi+(e_i\llcorner X)\cdot\nabla_{e_i}\Psi.$$

The operator $Q:\ker\mu\to\ker\mu$ is the so–called *Rarita–Schwinger operator.*

Now assume to be given a $Spin(7)_+$–structure with associated spinor Ψ_+, so $\Gamma_+=\sum_i e_i\cdot\Psi_+\otimes e_i\in\Delta_+\leqslant\Delta_-\otimes\Lambda^1$. Then Γ_+ solves the Dirac equation $Đ(\Gamma_+)=0$, i.e. Γ_+ is *harmonic* if and only if $D(\Psi_+)=0$ and $P(\Psi_+)=0$. This is equivalent to the parallelity of Ψ_+ (and thus of Γ_+) as follows from substituting the first condition into the local definition of P and contracting with e_i, whence $\nabla_{e_i}\Psi=0$ for all i and thus $\nabla\Psi=0$. The well–known theory of manifolds with holonomy $Spin(7)$ (see, for instance, [4]) then implies the

Proposition 6.1. *For a $Spin(7)$–structure, the following statements are equivalent.*
(i) *The supersymmetric map Γ_+ is harmonic with respect to the twisted Dirac–operator, i.e. $Đ(\Gamma_+)=0$.*
(ii) *The $Spin(7)$–invariant 4–form ρ^4 is closed, $d\rho^4=0$.*
(iii) *The holonomy of the induced metric is contained in $Spin(7)$.*

Definition 6.1. *Let* Γ *be a supersymmetric map on* M^8*. The corresponding topological structure is said to be* harmonic *if and only if*

$$Ð(\Gamma) = 0. \tag{6.2}$$

Examples of integrable structures will be given in the next section. For the moment being, we want to establish an analogue of Proposition 6.1 for $Sp(1) \cdot Sp(2)$– and $PSU(3)$–structures. To that end we will apply the standard representation theoretic machinery.

For a general Riemannian G–structure, the Levi–Civita connection form acts through $\mathfrak{so}(n) \cong \Lambda^2$ on any G–invariant γ. Since it is acted on trivially by its stabiliser algebra $\mathfrak{g}$,

$$\nabla\gamma = T(\gamma), \tag{6.3}$$

where T is the so–called *intrinsic torsion* of the G–structure, a tensor contained in $\Lambda^1 \otimes \mathfrak{g}^\perp$, subsequently referred to as the *torsion module*. It vanishes if and only if the holonomy of the metric is contained in G, so Theorem 6.1 could be rephrased by saying that harmonicity of the $Spin(7)$–structure is equivalent to the vanishing of its torsion, i.e. $T = 0$.

In our case, the invariants of $G = Sp(1) \cdot Sp(2)$ and $PSU(3)$ live in $\ker \mu_\mp \subset \Delta_\pm \otimes \Lambda^1$. We define the maps

$$\mathbf{Đ}: \; X \otimes a \in \Lambda^1 \otimes \mathfrak{g}^\perp \mapsto \mu\big(X \otimes a(\Gamma_\pm)\big) \in \Delta_\pm \otimes \Lambda^1,$$

where a acts via the induced action of Λ^2 on $\Delta_\pm \otimes \Lambda^1$, i.e.

$$X \wedge Y(\Psi_\pm \otimes V) = \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \Psi_\pm \otimes V + \Psi_\pm \otimes (X(V)Y - Y(V)X)$$

and Clifford multiplication takes $X \otimes \Psi_\pm \otimes V \in \Lambda^1 \otimes \Delta_\pm \otimes \Lambda^1$ to $X \cdot \Psi_\pm \otimes V \in \Delta_\mp \otimes \Lambda^1$. From this definition and (6.3) we deduce for $T = \sum e_i \otimes a_i$ that $\mathbf{Đ}(T) = \sum e_i \cdot A_i(\gamma) = Ð(\gamma)$. Hence the G–structure is harmonic in the sense of Definition 6.1 if and only if $T \in \ker \mathbf{Đ}$. On the other hand, we can also consider the equivariant maps defined by the invariant forms ρ^p, namely

$$\mathbf{d} : X \otimes a \in \Lambda \otimes \mathfrak{g}^\perp \mapsto X \wedge a(\rho^p) \in \Lambda^{p+1}, \quad \mathbf{d}^* : X \otimes a \in \Lambda \otimes \mathfrak{g}^\perp \mapsto X \llcorner a(\rho^p) \in \Lambda^{p-1}.$$

Since the differential operators d and d^* are induced by the skew–symmetrisation and (minus the) contraction of the Levi–Civita connection, we see that the G–invariant p–form ρ^p is closed or coclosed if and only if $\rho^p \in \ker \mathbf{d}$ or $\rho^p \in \ker \mathbf{d}^*$. Consequently, our task consists in showing that $\ker \mathbf{Đ} = \ker \mathbf{d} \cap \ker \mathbf{d}^*$. The maps $\mathbf{Đ}$, $\mathbf{d}$ and $\mathbf{d}^*$ are all G–equivariant, so their kernel can be computed by using Schur's lemma and G–representation theory. From a technical point of view, the $Sp(1) \cdot Sp(2)$–case is a lot easier to deal with, so we start with this one. Here, the invariant 4–form ρ^4 is self–dual, so we only need to show $\ker \mathbf{Đ} = \ker \mathbf{d}$.

For an application of Schur's lemma, we first have to decompose the torsion module $\Lambda^1 \otimes \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp = [1,0,1] \otimes [1,1,1]$ into $Sp(1) \cdot Sp(2)$–irreducibles which yields

$$\Lambda^1 \otimes \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp = [\tfrac{1}{2}, \tfrac{1}{2}, 1] \oplus [\tfrac{1}{2}, \tfrac{3}{2}, 2] \oplus [\tfrac{3}{2}, \tfrac{1}{2}, 1] \oplus [\tfrac{3}{2}, \tfrac{3}{2}, 2].$$

On the other hand, we find for the target spaces of $\mathbf{Đ}$ and $\mathbf{d}$

$$\Delta_+ \otimes \Lambda^1 = 2[\tfrac{1}{2},\tfrac{1}{2},1] \oplus [\tfrac{1}{2},\tfrac{3}{2},2] \oplus [\tfrac{3}{2},\tfrac{1}{2},1], \quad \Lambda^5 = [\tfrac{1}{2},\tfrac{1}{2},1] \oplus [\tfrac{1}{2},\tfrac{3}{2},2] \oplus [\tfrac{3}{2},\tfrac{1}{2},1].$$

We are now in a position to prove the following theorem.

Theorem 6.2. *For an $Sp(1)\cdot Sp(2)$–structure over M^8 with invariants $\rho \in \Omega^4(M)$ and $\Gamma_+ \in \Gamma(\Delta_- \otimes \Lambda^1)$, the following statements are equivalent.*
(i) *The $Sp(1)\cdot Sp(2)$–structure is harmonic, i.e. $Đ(\Gamma_+) = 0$.*
(ii) *The $Sp(1)\cdot Sp(2)$–invariant 4–form ρ is closed, i.e. $d\rho = 0$.*
(iii) *The intrinsic torsion T of the $Sp(1)\cdot Sp(2)$–structure takes values in $[\tfrac{3}{2},\tfrac{3}{2},2]$.*

Proof: By Schur's lemma, it is enough to evaluate the maps $\mathbf{Đ}$ and $\mathbf{d}$ for a sample vector of a given module in order to check whether or not it maps non–trivially. The operator $\mathbf{d}$ is known to be surjective [13], that is, $\ker \mathbf{d} = [\tfrac{3}{2},\tfrac{3}{2},2]$, hence we are left to with $\mathbf{Đ}$. Obviously, $[\tfrac{3}{2},\tfrac{3}{2},2] \subset \ker \mathbf{Đ}$ and the assertion follows if we can show that the remaining irreducible modules in $\Lambda^1 \otimes \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp$ map non–trivially. To that end it will be convenient to consider the operator $L : \Lambda^3 \to \Lambda^3$ built out of $\mathbf{Đ}$, followed by composition with $id \otimes \Gamma_+$ and the projection $\Delta_+ \otimes \Delta_- \to \Lambda^3$ which expressed in an orthonormal basis is given by $\sum q(\Psi_+, e_I \cdot \Psi_-) e_I$ (q being the $Spin(8)$–invariant inner product on Δ). Since all the representations involved are of real type, L restricted to an irreducible module of Λ^3 is multiplication by a real scalar, possibly zero. To begin with, we map the element $4e_1 \llcorner \rho \in [\tfrac{1}{2},\tfrac{1}{2},1]$ into $\Lambda^1 \otimes \Lambda^2$ in the natural way, namely $x \wedge y \wedge z \mapsto (x \otimes y \wedge z + cyc.\ perm.)/3$. Projecting the second factor onto $[1,1,1] = \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)\big)^\perp$ by means of the projection operator given in Proposition 4.1, yields the element

$$\begin{aligned} t_1 = {} & e_2 \otimes (e_{12} - e_{34} - e_{56} + e_{78}) + e_3 \otimes (e_{13} + e_{24} - e_{57} - e_{68}) + \\ & e_4 \otimes (e_{14} - e_{23} - e_{58} + e_{67}) + e_5 \otimes (3e_{15} - e_{26} - e_{37} - e_{48}) + \\ & e_6 \otimes (3e_{16} + e_{25} - e_{38} + e_{47}) + e_7 \otimes (3e_{17} + e_{28} + e_{35} - e_{46}) + \\ & e_8 \otimes (3e_{18} - e_{27} + e_{36} + e_{45}) \in [\tfrac{1}{2},\tfrac{1}{2},1] \subset \Lambda^1 \otimes [1,1,1]. \end{aligned}$$

With these choices, the matrix of Γ_+ is given by (4.3) and evaluating L on t_1 yields

$$L(t_1) = -6e_{234} + 2e_{256} - 2e_{278} + 2e_{357} + 2e_{368} + 2e_{458} - 2e_{467} = 2t_1.$$

In particular, t_1 maps non–trivially under $\mathbf{Đ}$. Next we turn to the module $[\tfrac{3}{2},\tfrac{1}{2},1]$. Using the weight vectors provided in (4.2), we see that the vector

$$\begin{aligned} x_{(\alpha+\beta_1)/2+\beta} \wedge x_{(\alpha-\beta_1)/2} \wedge x_{(\alpha+\beta_1)/2} = {} & e_{157} + ie_{158} - ie_{167} + e_{168} - \\ & ie_{257} + e_{258} - e_{267} - ie_{268} \end{aligned}$$

is of the highest weight occurring in $\Lambda^3 \otimes \mathbb{C}$. Hence it is actually the weight vector of $(\tfrac{3}{2},\tfrac{1}{2},1) \subset \Lambda^3 \otimes \mathbb{C}$. Proceeding as before yields then the sample vector $t_2 \in (\tfrac{3}{2},\tfrac{1}{2},1) \subset \Lambda^1 \otimes [1,1,1] \otimes \mathbb{C}$ and this is mapped to $20t_2$ under L. For the remaining module $[\tfrac{1}{2},\tfrac{3}{2},2]$ we consider the vector $-e_{134}/3 + e_{178}$. Wedging with ρ yields zero, so it is necessarily contained in $[\frac{1}{2},\frac{3}{2},2]\oplus[\frac{3}{2},\frac{1}{2},1]$. Injecting it into $\Lambda^1\otimes[1,1,1]$ gives the vector t_3 whose projections on $[\frac{1}{2},\frac{3}{2},2]$ and $[\frac{3}{2},\frac{1}{2},1]$ we denote by t_{3a} and t_{3b}. The image under L is

$$\begin{aligned} L(t_{3a}+t_{3b}) &= c\cdot t_{3a}+20\cdot t_{3b} \\ &= (-12e_{134}-8e_{156}+44e_{178}+4e_{358}-4e_{367}-4e_{457}-4e_{468})/3 \end{aligned}$$

Since $L(t_3)-20t_3\neq 0$, we see that $[\frac{1}{2},\frac{3}{2},2]$ maps also non–trivially under q (applying L again to $L(t_3)-20t_3$ shows that restricted to this module, L is actually multiplication by 12). Hence $\ker \mathbf{Đ}=[\frac{3}{2},\frac{3}{2},2]$ which proves the theorem. ■

Corollary 6.3. *Let $L:\Lambda^3\to\Lambda^3$ denote the $Sp(1)\cdot Sp(2)$–invariant map defined in the proof of Theorem 6.2. The irreducible $Sp(1)\cdot Sp(2)$–modules can be characterised as follows:*

$$\begin{aligned} [\sigma\otimes\Lambda^1_0] &= [1/2,1/2,1] = \{\alpha\in\Lambda^3\,|\,L(\alpha)=2\alpha\} \\ [\sigma\otimes\lambda^3_1] &= [1/2,3/2,2] = \{\alpha\in\Lambda^3\,|\,L(\alpha)=12\alpha\} \\ [\sigma^3\otimes\Lambda^1_0] &= [3/2,1/2,1] = \{\alpha\in\Lambda^3\,|\,L(\alpha)=20\alpha\} \end{aligned}$$

Next we turn to $PSU(3)$. Here, the invariant form ρ is of degree 3 so we also need to take the operator $\mathbf{d}^*$ into account. This situation is also more involved due to the presence of modules with multiplicities greater than one.

Again we begin by decomposing the torsion module. Let $\wedge:\Lambda^1\otimes\Lambda^2_{20}\to\Lambda^3$ denote the natural skewing map. Then $\Lambda^1\otimes\Lambda^2_{20}\cong\ker\wedge\oplus\rho^\perp$, where $\rho^\perp=[1,1]\oplus[\![1,2]\!]\oplus[2,2]$ is the orthogonal complement of ρ in Λ^3. Moreover, the natural contraction map $\llcorner:\ker\wedge\subset\Lambda^1\otimes\Lambda^2_{20}\to\Lambda^1$ splits $\ker\wedge$ into a direct sum isomorphic to $\ker\llcorner\oplus\Lambda^1$ where $\ker\llcorner\cong[2,2]\oplus[\![2,3]\!]$. Consequently, the complexification of $\Lambda^1\otimes\Lambda^2_{20}$ is the direct sum of

$$\Lambda^1\otimes\Lambda^2_{20}\otimes\mathbb{C}=\begin{array}{l}\Lambda^1\otimes\Lambda^2_{10+}=(1,1)_+\oplus(1,2)\oplus(2,2)_+\oplus(2,3)\\ \qquad\oplus\\ \Lambda^1\otimes\Lambda^2_{10-}=(1,1)_-\oplus(2,1)\oplus(2,2)_-\oplus(3,2).\end{array}$$

The modules $(1,1)_\pm$ and $(2,2)_\pm$ have non–trivial projections to both $\ker\wedge$ and $\rho^\perp$. In particular, they map non–trivially under $\wedge$. With the decomposition of the target spaces of $\mathbf{Đ}$, $\mathbf{d}$ and $\mathbf{d}^*$, namely

$$\Delta_\pm\otimes\Lambda^1=\mathbf{1}\oplus2[1,1]\oplus[\![1,2]\!]\oplus[2,2],\quad \Lambda^4=2[1,1]\oplus2[2,2],\quad \Lambda^2=[1,1]\oplus[\![1,2]\!],$$

we can now prove the analogue of Proposition 6.1.

Theorem 6.4. *For a $PSU(3)$–structure over M^8 with invariants $\rho^3\in\Omega^3(M)$ and $\Gamma=\Gamma_+\oplus\Gamma_-\in\Gamma(\Delta\otimes\Lambda^1)$, the following statements are equivalent.*
(i) *The $PSU(3)$–structure is harmonic, i.e. $D(\Gamma)=0$.*
(ii) *The $PSU(3)$–invariant 3–form ρ is closed and coclosed, i.e. $d\rho=0$, $d^*\rho=0$.*
(iii) *The intrinsic torsion T of the $PSU(3)$–structure takes values in $[\![2,3]\!]$.*

Proof: We first establish the equivalence between (i) and (iii) and start by determining the kernel of $\mathbf{d}^*$. Recall that $a_\pm(\rho)=\pm\sqrt{3}i\star(a_\pm\wedge\rho)$ for any $a_\pm\in\Lambda^2_{10\pm}$ (Proposition 4.4). By complexifying, it follows that restricted to the $PSU(3)$–invariant modules $\Lambda^1\otimes\Lambda^2_{10\pm}$,

$$\mathbf{d}^*(\sum e_j\otimes a_j^\pm)=\mp\sqrt{3}i\sum e_j\llcorner\star(a_j^\pm\wedge\rho)=\pm\sqrt{3}i\sum\star(e_j\wedge a_j^\pm\wedge\rho).\qquad(6.4)$$

In virtue of the remarks preceding the theorem, the kernel of the skewing map $\Lambda^1 \otimes \Lambda^2_\pm$ is isomorphic to $(2,3)$ and $(3,2)$, so this vanishes if and only if $\sum e_i \wedge a_i^\pm$ lies in $\mathbf{1} \oplus [2,2]$, the kernel of the map which wedges 3–forms with ρ. Invoking Schur's Lemma, $\ker \mathbf{d}^* \cong [1,1] \oplus 2[2,2] \oplus [\![2,3]\!]$, where the precise embedding of $[1,1]$ will be of no importance to us.

Next we consider the operator $\mathbf{d}$. If we can show that it is surjective, then $\ker \mathbf{d} \cong [\![1,2]\!] \oplus [\![2,3]\!]$ and consequently, the kernels of $\mathbf{d}^*$ and $\mathbf{d}$ intersect in $[\![2,3]\!]$. Let $\iota_{\rho^\perp}$ denote the injection of $\rho^\perp$ into $\Lambda^1 \otimes \Lambda^2_{20}$ obtained by projecting the natural embedding of Λ^3 into $\Lambda^1 \otimes \Lambda^2$. We first prove the relation

$$b_3(\alpha) = \frac{1}{2}\mathbf{d}\big(\iota_{\rho^\perp}(\alpha)\big), \quad \alpha \in \rho^\perp \subset \Lambda^3 \tag{6.5}$$

which shows that $\ker b_4 \subset \operatorname{im} \mathbf{d}$. By (4.5), the kernel of b_3 is isomorphic to $\mathbf{1} \oplus [\![1,2]\!]$, so the claim needs only to be verified for the module $[1,1] \oplus [2,2]$ in Λ^3. A sample vector is obtained by

$$p_3(e_{128}) = \alpha_8 \oplus \alpha_{27} = \tfrac{1}{8}(5e_{128} + \sqrt{3}e_{345} + \sqrt{3}e_{367} - 2e_{458} + 2e_{678}), \tag{6.6}$$

where $p_3 = b_4^* b_3$. That both components α_8 and α_{27} are non–trivial can be seen as follows. Restricting p_3 to Λ^3_8 and Λ^3_{27} is multiplication by real scalars x_1 and x_2 since the modules are representations of real type. If one, say x_1, vanished, then $p_3^2(e_{128}) = p_3(\alpha_{27}) = x_2 \cdot \alpha_{27}$. However

$$p_3^2(e_{128}) = \tfrac{1}{64}(39e_{128} + 7\sqrt{3}e_{345} + 7\sqrt{3}e_{367} - 18e_{458} + 18e_{678})$$

which is not a multiple of (6.6). Moreover, we have indeed

$$\begin{aligned} b_3 p_3(e_{128}) &= \tfrac{1}{32}(7\sqrt{3}e_{1245} + 7\sqrt{3}e_{1267} - 9e_{1468} - 9e_{1578} + 9e_{2478} - 9e_{2568}) \\ &= \tfrac{1}{2}\mathbf{d}\big(\iota_{\rho^\perp} p_3(e_{128})\big) \end{aligned}$$

which proves (6.5). For the inclusion $\operatorname{im} b_5^* \subset \operatorname{im} \mathbf{d}$ we consider the vector $e_1 \otimes e_{18}$ in $\ker \wedge$. Then $\mathbf{d}(e_1 \otimes e_{18}) = -e_{1238}/2 - e_{1478}/4 + e_{1568}/4$ takes values in both components of $\operatorname{im} b_5^* \subset \Lambda^4$ since $b_4^* \mathbf{d}(e_1 \otimes e_{18}) = 0$ and otherwise

$$b_5^* b_4 \mathbf{d}(e_1 \otimes e_{18}) = \tfrac{1}{32}(-10e_{1238} - 5e_{1478} + 5e_{1568} + 3e_{2468} + 3e_{2578} + 3e_{3458} - 3e_{3678})$$

would be a multiple of $\mathbf{d}(e_1 \otimes e_{18})$. Hence $\mathbf{d}$ is surjective and the equivalence between (i) and (iii) is established.

Finally, we turn to the Dirac equation. Since the $PSU(3)$–invariant supersymmetric map $\Gamma = \Gamma_+ \oplus \Gamma_-$ has now components in both $\Delta_- \otimes \Lambda^1$ and $\Delta_+ \otimes \Lambda^1$, we will split the Dirac operator $\mathbf{Đ} = \mathbf{Đ}_+ \oplus \mathbf{Đ}_-$ accordingly, i.e. $\mathbf{Đ}_\pm\big(X \otimes a\big) = \mu_\pm(X \cdot a(\Gamma_\pm)) \in \Delta_\pm \otimes \Lambda^1$. We now have to show that

$$\ker \mathbf{Đ}_+ \cap \ker \mathbf{Đ}_- = [\![2,3]\!] = \ker \mathbf{d} \cap \ker \mathbf{d}^*.$$

The intersection $\ker \mathbf{Đ}_+ \cap \ker \mathbf{Đ}_-$ contains at least the module $[\![2,3]\!]$. First we show that $[\![1,2]\!]$ is not contained in this intersection by taking the vector

$$\begin{aligned}\tau_{[\![1,2]\!]} &= id \otimes \pi^2_{20}\big(\iota_{\rho^\perp} b_2(4e_{18})\big)\\ &= -\sqrt{3}e_1 \otimes e_{45} - \sqrt{3}e_1 \otimes e_{67} + 2e_2 \otimes e_{38} - 2e_3 \otimes e_{28} + \sqrt{3}e_4 \otimes e_{15} +\\ &\quad e_4 \otimes e_{78} - \sqrt{3}e_5 \otimes e_{14} - e_5 \otimes e_{68} + \sqrt{3}e_6 \otimes e_{17} + e_6 \otimes e_{58} - \sqrt{3}e_7 \otimes e_{16} -\\ &\quad e_7 \otimes e_{48} + 2e_8 \otimes e_{23} + e_8 \otimes e_{47} - e_8 \otimes e_{56}.\end{aligned}$$

A straightforward, if tedious, computation shows $\mathbf{Đ}(\tau_{[\![1,2]\!]}) \neq 0$. For the remainder of the proof, it will again be convenient to complexify the torsion module $\Lambda^1 \otimes \Lambda^2_{20}$ and to consider $(1,1)_\pm$ and $(2,2)_\pm$. The invariant 3–form ρ induces equivariant maps $\rho_\mp : \Delta_\pm \to \Delta_\mp$ whose matrices with respect to the choices made in (A.1) are given by (3.4) for ρ_+ and by its transpose for ρ_-. By Schur's Lemma, we have

$$\mathbf{Đ}_-\big((2,2)_+\big) = z \cdot \rho_- \otimes id \circ \mathbf{Đ}_+\big((2,2)_+\big) \tag{6.7}$$

for a complex scalar z. Since the operators $\mathbf{Đ}_\pm$ are real and $(2,2)_-$ is the complex conjugate of $(2,2)_+$, the same relation holds for $(2,2)_-$ with $\bar{z}$. The vector $\tau_0 = 6(e_1 \otimes e_{18} - e_2 \otimes e_{28})$ is clearly in $\ker \llcorner \subset \ker \wedge$ and projecting the second factor to Λ^2_{10+} yields

$$\begin{aligned}id \otimes \pi^2_{10+}(\tau_0) &= e_1 \otimes (3e_{18} + i\sqrt{3}e_{23} - i\sqrt{3}e_{47} + i\sqrt{3}e_{56}) +\\ &\quad e_2 \otimes (i\sqrt{3}e_{13} - 3e_{28} + i\sqrt{3}e_{46} + i\sqrt{3}e_{57}).\end{aligned}$$

Since any possible component in $(2,3)$ gets killed under $\mathbf{Đ}$, we can plug this into (6.7) to find $z = (1 + i\sqrt{3})/8$ which shows that $(2,2)_\pm$ map non–trivially under $\mathbf{Đ}$. On dimensional grounds, $\ker \mathbf{Đ}_\pm$ therefore contains the module $(2,2)$ with multiplicity one. Their intersection, however, is trivial, for suppose otherwise. Let $(2,2)_0$ denote the corresponding copy in $\ker \mathbf{Đ}_+$. It is the graph of an isomorphism $P : (2,2)_+ \to (2,2)_-$ since it intersects $(2,2)_\pm$ trivially. Now if $\tau = \tau_+ \oplus P\tau_+ \in (2,2)_0$ were in $\ker \mathbf{Đ}_-$, then

$$\begin{aligned}\mathbf{Đ}_-(\tau_+ \oplus P\tau_+) &= z \cdot \rho \otimes id \circ \mathbf{Đ}_+(\tau_+) \oplus \bar{z} \cdot \rho \otimes id \circ \mathbf{Đ}_+(P\tau_+)\\ &= \rho \otimes id \circ \mathbf{Đ}_+(z \cdot \tau_+ \oplus \bar{z} \cdot P\tau_+)\\ &= 0.\end{aligned}$$

Consequently, $z \cdot \tau_+ \oplus \bar{z} \cdot P\tau_+ \in \ker \mathbf{Đ}_+$, that is, $\bar{z} \cdot P\tau_+ = Pz \cdot \tau_+$ or $\bar{z} = z$ which is a contradiction. This shows that the kernels of $\mathbf{Đ}_\pm$ intersect at most in $2(1,1) \oplus [\![2,3]\!]$ and furthermore, that the condition $Đ(\Gamma_+) = 0$ or $Đ(\Gamma_-) = 0$ on its own is not sufficient to guarantee the close– and cocloseness of ρ. This argument also applies to $(1,1)_\pm$. However, since $(1,1)$ appears twice in $\Delta_\pm \otimes \Lambda$, we first need to project onto $\Delta_\mp \cong (1,1)$ via Clifford multiplication before asserting the existence of a complex scalar z such that

$$\mu_+ \circ \mathbf{Đ}_-\big((1,1)_+\big) = z \cdot \rho_+\big(\mu_- \circ \mathbf{Đ}_+((1,1)_+)\big).$$

For the computation of z, we can use the vector

$$2\sqrt{3}ie_1 \otimes \pi^2_{10+}(e_{18}) = e_1 \otimes (\sqrt{3}ie_{18} - e_{23} + e_{47} - e_{56}) \in (1,1)_+ \oplus (2,2)_+ \oplus (2,3),$$

as possible non–trivial components in $(2,2)_+ \oplus (2,3)$ get killed under $\mu_\mp$. We find $z = 2(1-\sqrt{3}i)$ which as above shows that $(1,1)$ occurs with multiplicity at most one in $\ker \mathbf{Đ}_\pm$ and that it is not contained in their intersection. Consequently, $\ker \mathbf{Đ}_+ \cap \ker \mathbf{Đ}_- = [\![2,3]\!]$, which proves the equivalence between (ii) and (iii). ■

Remark: The implication $(i) \Rightarrow (ii)$ was already asserted in [7]. However, the proof is inconclusive. Firstly, some of the sample vectors provided in the proof are not contained in the right module. For instance, $x_{\alpha_1} \otimes x_{\alpha_1} \wedge x_{\alpha_2}$ is not contained in $(2,1) \subset \Lambda^1 \otimes \Lambda^2_{10-}$ as claimed for from the general properties of root vectors it follows that application of $[x_{\alpha_1}, \cdot]$ yields $x_{\alpha_1} \otimes x_{\alpha_1} \wedge x_{\alpha_1+\alpha_2}$. Moreover, due to the presence of modules with multiplicity two, the modules $(1,1)_\pm$ and $(2,2)_\pm$ can map non–trivially under say $\mathbf{đ}$ while $\ker \mathbf{đ}$ still contains a component isomorphic to $(1,1)$ or $(2,2)$.

As in the case of harmonic $Spin(7)$–structures, there are geometrical obstructions to harmonic $Sp(1)\cdot Sp(2)$– and $PSU(3)$–structures imposed upon the Ricci tensor. It is well–known that metrics whose holonomy is contained in $Spin(7)$ are necessarily Ricci flat; a weaker statement is still true in the case of $Sp(1)\cdot Sp(2)$– or $PSU(3)$–structures. According to Proposition 2.8 in [20], we have

$$(D \circ đ - đ \circ Đ)(\Gamma) = \frac{1}{2} p(\Gamma \circ Ric)$$

for any $\Gamma \in \Gamma(T^*M \otimes \Delta)$. Now write $\Gamma = \sum_i e_i \otimes \Gamma_i$ and regard Ric as an endomorphism of T so that

$$\Gamma \circ Ric = \sum_{i,j} Ric_{ij} e_i \otimes \Gamma_j.$$

Integrability implies

$$p(\Gamma \circ Ric) = \sum_{i,j} Ric_{ij} e_i \cdot \Gamma_j = 0.$$

This means that Ric is in the kernel of the map

$$A \in \odot^2 \mapsto p(A \circ \Gamma) = \sum_{i,j} A_{ij} e_i \Gamma_j \in \Delta,$$

which is invariant under the stabiliser of Γ. In the case of an $Sp(1)\cdot Sp(2)$–structure, $\Gamma = \Gamma_+ \in \Delta_- \otimes \Lambda$, hence $\odot^2 = \mathbf{1} \oplus [0,1,1] \oplus [1,1,2]$ and $\Delta_+ \cong [2,0,0] \oplus [0,1,1]$. Since this map is non-trivial, Ric vanishes on the module $[0,1,1]$. For a $PSU(3)$-structure, we have $\odot^2 = \mathbf{1} \oplus [1,1] \oplus [2,2]$ and $\Delta_\pm \cong [1,1]$, so Ric vanishes on the module $[1,1]$, a fact previously established in [7].

Proposition 6.5. *If g is a metric induced by a harmonic $Sp1(1)\cdot Sp(2)$– or $PSU(3)$–structure, then Ric vanishes on the 5–dimensional component $[0,1,1]$ or the 8–dimensional component $[1,1]$.*

Other than for special cases we cannot hope to strengthen this statement as the examples in the next section will show.

7. Examples

If B denotes the Killing form of the Lie algebra $\mathfrak{su}(3)$, trivial examples of $PSU(3)$–structures are provided by the 3–form $\rho(X,Y,Z) = -B(X,[Y,Z])/6$ on $\mathfrak{su}(3)$, the symmetric space $SU(3) = SU(3)\times SU(3)/SU(3)$ and its non-compact dual $SL(3,\mathbb{C})/SU(3)$. Since ρ and $\star\rho$ are *ad*–invariant forms, they are closed and hence they define integrable $PSU(3)$–structures. The metric is Einstein and of zero, positive and negative scalar curvature respectively. Its holonomy is contained in $\mathfrak{su}(3)$ since $\nabla\rho = 0$. As a consequence, the torsion vanishes and the example in some sense "trivial". The aim of this section is to construct non–trivial examples.

Local examples

The first example is built out of a hyperkähler 4-manifold M^4 with a triholomorphic vector field. Let $U \equiv U(x,y,z)$ be a strictly positive harmonic function defined on some domain $D \subset \mathbb{R}^3$ and θ a 1-form on $\mathbb{R}^3$ with $dU = \star d\theta$. By the Gibbons-Hawking ansatz [6], [1], the metric on $D \times \mathbb{R}$

$$g = U(dx^2 + dy^2 + dz^2) + \frac{1}{U}(dt+\theta)^2 \tag{7.1}$$

is hyperkähler with associated Kähler forms given by

$$\begin{aligned}
\omega_{-1} &= U dy \wedge dz + dx \wedge (dt+\theta)\\
\omega_{-2} &= U dx \wedge dy + dz \wedge (dt+\theta)\\
\omega_{-3} &= U dx \wedge dz - dy \wedge (dt+\theta).
\end{aligned}$$

The vector field $X = \frac{\partial}{\partial t}$ is triholomorphic, that is it defines an infinitesimal transformation which preserves any of the three complex structures induced by ω_{-1}, ω_{-2} or ω_{-3}. Conversely, a hyperkähler metric on a 4-dimensional manifold which admits a triholomorphic vector field is locally of the form (7.1).

Next we define the 2–form ω_{+3} by changing the sign in ω_{-3}, that is

$$\omega_{+3} = U dx \wedge dz + dy \wedge (dt+\theta).$$

This 2-form is closed if and only if $U \equiv U(x,z)$ for $d\omega_{+3} = 0$ implies

$$d(U dx \wedge dz) = d\big(dy \wedge (dt+\theta)\big), \tag{7.2}$$

so that

$$d\omega_{+3} = 2d(U dx \wedge dz) = 2\frac{\partial U}{\partial y} dy \wedge dx \wedge dz.$$

Pick such a U and take the standard coordinates $x_1,\ldots,x_4$ of the Euclidean space $(\mathbb{R}^4, g_0)$. Put

$$\begin{array}{llll}
e^1 = dx_1, & e^2 = dx_2, & e^3 = dx_3 & e^8 = dx_4\\
e^4 = \sqrt{U}dy, & e^5 = -\frac{1}{\sqrt{U}}(dt+\theta), & e^6 = -\sqrt{U}dx, & e^7 = \sqrt{U}dz
\end{array}$$

which we take as an orthonormal basis on $M^4 \times \mathbb{R}^4$. Endowed with the orientation defined by $(e_4,\ldots,e_7)$, the forms ω_{-i} are anti-self-dual on M^4, while the forms $\omega_{+1} = U dy \wedge dz - dx \wedge (dt+\theta)$, $\omega_{+2} = U dx \wedge dy - dz \wedge (dt+\theta)$ and ω_{+3} are self-dual, that is

$$\omega_{i\pm} \wedge \omega_{j\mp} = 0, \quad \omega_{i\pm} \wedge \omega_{j\pm} = \pm 2\delta_{ij} e_{4567}. \tag{7.3}$$

The 3–form

$$\rho = \frac{1}{2}e_{123} + \frac{1}{4}e_1\wedge\omega_{-1} + \frac{1}{4}e_2\wedge\omega_{-2} + \frac{1}{4}e_3\wedge\omega_{-3} + \frac{\sqrt{3}}{4}e_8\wedge\omega_{+3} \tag{7.4}$$

defines a $PSU(3)$–structure which is closed by design. Moreover, the same holds for

$$\begin{aligned}\star\rho = & \frac{1}{2}e_{45678} - \frac{1}{4}\omega_{-1}\wedge e_{238} + \frac{1}{4}\omega_{-2}\wedge e_{138} - \frac{1}{4}\omega_{-3}\wedge e_{128} + \frac{\sqrt{3}}{4}\omega_{+3}\wedge e_{123} \\ = & \frac{1}{2}U dx\wedge dz\wedge dy\wedge(dt+\theta)\wedge dx_4 - \frac{1}{4}\omega_{-1}\wedge dx_2\wedge dx_3\wedge dx_4 \\ & +\frac{1}{4}\omega_{-2}\wedge dx_1\wedge dx_3\wedge dx_4 - \frac{1}{4}\omega_{-3}\wedge dx_1\wedge dx_2\wedge dx_4 + \frac{\sqrt{3}}{4}\omega_{+3}\wedge dx_1\wedge dx_2\wedge dx_3.\end{aligned} \tag{7.5}$$

To check that this family of $PSU(3)$–structures is rich enough to provide also non–trivial examples, we consider the specific ansatz defined by $U(x,y,z) = x$ on $\{x>0\}$ and $\theta = ydz$ and show that $\nabla\rho_i \neq 0$. The metric g on $M^4\times\mathbb{R}^4$ is given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + xdx^2 + xdy^2 + (x+\frac{y^2}{x})dz^2 + \frac{1}{x}dt^2 + 2\frac{y}{x}dzdt$$

with orthonormal basis

$$\begin{aligned} &e_1 = \partial_{x_1},\quad e_2 = \partial_{x_2},\quad e_3 = \partial_{x_3},\quad e_8 = \partial_{x_4},\\ &e_4 = \tfrac{1}{\sqrt{x}}\partial_y,\ e_5 = -\sqrt{x}\partial_t,\ e_6 = -\tfrac{1}{\sqrt{x}}\partial_x,\ e_7 = \tfrac{1}{\sqrt{x}}(\partial_z - y\partial_t).\end{aligned}$$

The only non–trivial brackets are

$$\begin{aligned} &[e_4,e_6] = -\tfrac{1}{2\sqrt{x}^3}e_4 \quad [e_5,e_6] = \tfrac{1}{2\sqrt{x}^3}e_5\\ &[e_4,e_7] = \tfrac{1}{\sqrt{x}^3}e_5 \qquad [e_6,e_7] = \tfrac{1}{2\sqrt{x}^3}e_7.\end{aligned}$$

Since the anti–self–dual 2–forms ω_1, ω_2 and ω_3 are the associated Kähler forms of the hyperkähler structure on M, we have $\nabla\omega_i = 0$. From this and Koszul's formula

$$2g(\nabla_{e_i}e_j, e_k) = g([e_i,e_j],e_k) + g([e_k,e_i],e_j) + g([e_k,e_j],e_i) = c_{ijk} + c_{kij} + c_{kji}$$

we deduce

$$\nabla(e^6\wedge e^7) = \nabla(e^4\wedge e^5) = -\frac{1}{12}\cdot\frac{1}{\sqrt{x^3}}(e_4\otimes\omega_{+1} + e_5\otimes\omega_{+2})$$

and thus

$$\nabla\rho = -\frac{1}{8\sqrt{3}}\cdot\frac{1}{\sqrt{x^3}}(e_4\otimes\omega_{+1}\wedge e_8 + e_5\otimes\omega_{+2}\wedge e_8).$$

Note that this metric is Ricci–flat despite non–vanishing torsion.

Compact examples

Consider the nilpotent Lie algebra $\mathfrak{g} = \langle e_2,\ldots,e_8\rangle$ whose structure constants are determined by

$$de_i = \begin{cases} 0, & i = 2,\ldots,7 \\ e_{47} + e_{56} = \omega_{1+}, & i = 8\end{cases}, \tag{7.6}$$

that is the only non-trivial structure constants are $c_{478} = -c_{748} = c_{568} = -c_{658} = 1$. Let G be the associated simply-connected Lie group. The rationality of the structure constants guarantees the existence of a lattice Γ for which $N = \Gamma\backslash G$ is compact [10]. We let $M = T^2 \times N$ with $e_i = dt_i$, $i = 1, 2$ on the torus, hence $de_i = 0$. We take the basis $e_1, \ldots, e_8$ to be orthonormal on M and denote by g the corresponding metric.

As in (7.4), the 3–form $\rho = e_{123}/2 + \sum e_i \wedge \omega_{-i}/4 + \sqrt{3} e_8 \wedge \omega_{+3}/4$ defines a $PSU(3)$–structure whose 4–form invariant is given by (7.5). Then (7.3) and (7.6) imply

$$d\rho = \frac{\sqrt{3}}{2} de_8 \wedge \omega_{3+} = \frac{\sqrt{3}}{2} \omega_{1+} \wedge \omega_{3+} = 0$$

and

$$\begin{aligned} d \star \rho &= e_{4567} \wedge de_8 - \frac{1}{2}\omega_{1-} \wedge e_{23} \wedge de_8 + \frac{1}{2}\omega_{2-} \wedge e_{13} \wedge de_8 - \frac{1}{2}\omega_{3-} \wedge e_{12} \wedge de_8 \\ &= e_{4567} \wedge \omega_{1+} - \frac{1}{2}\omega_{1-} \wedge e_{23} \wedge \omega_{1+} + \frac{1}{2}\omega_{2-} \wedge e_{13} \wedge \omega_{1+} - \frac{1}{2}\omega_{3-} \wedge e_{12} \wedge \omega_{1+} \\ &= 0, \end{aligned}$$

so the $PSU(3)$–structure is harmonic. To show its non–triviality, we compute the covariant derivatives $\nabla_{e_i} e_j$ for which we obtain

$$\nabla e_i = \begin{cases} 0, & i = 1,\ 2,\ 3 \\ -\frac{1}{2}(e_7 \otimes e_8 + e_8 \otimes e_7), & i = 4 \\ -\frac{1}{2}(e_6 \otimes e_8 + e_8 \otimes e_6), & i = 5 \\ \frac{1}{2}(e_5 \otimes e_8 + e_8 \otimes e_5), & i = 6 \\ \frac{1}{2}(e_4 \otimes e_8 + e_8 \otimes e_4), & i = 7 \\ \frac{1}{2}(-e_4 \otimes e_7 + e_7 \otimes e_4 - e_5 \otimes e_6 + e_6 \otimes e_5), & i = 8 \\ . \end{cases}$$

Now $\nabla_{e_4}(e_8 \wedge \omega_{3+}) = e_{457}$ and since the coefficient of $e_8 \wedge \omega_{3+}$ is irrational while all the remaining ones are rational, we deduce $\nabla_{e_4}\rho \neq 0$. Moreover, a straightforward computation shows the diagonal of the Ricci–tensor $Ric_{ii} = \sum_j g(\nabla_{[e_i,e_j]} e_i - [\nabla_{e_i}, \nabla_{e_j}] e_i, e_j)$ to be given by

$$Ric_{ii} = \begin{cases} 0, & i = 1,\ 2,\ 3,\ 8 \\ -\frac{1}{2}, & i = 4,\ 5,\ 6,\ 7 \end{cases}$$

In particular, it follows that (M, g) is of negative scalar curvature, but not Einstein, that is, Ric has a non–trivial **1**– and $[2,2]$–component.

A non-trivial compact example of a harmonic $Sp(1) \cdot Sp(2)$–structure was given in [14], where Salamon constructed a compact almost quaternionic 8–manifold M whose structure form Ω is closed, but not parallel. The example is of the form $M = N^6 \times T^2$, where N^6 is a compact nilmanifold associated with the Lie algebra given by

$$de_i = \begin{cases} 0, & i = 1,\ 2,\ 3,\ 5 \\ e_{15}, & i = 4 \\ e_{13}, & i = 6 \end{cases}$$

Therefore, the structure constants are trivial except for $c_{154} = -c_{514} = c_{136} = -c_{316} = 1$ which implies

$$\nabla e_i = \begin{cases} 0, & i = 2,\ 4,\ 6,\ 7,\ 8 \\ -\frac{1}{2}(e_3 \otimes e_6 + e_4 \otimes e_5 + e_5 \otimes e_4 + e_6 \otimes e_3), & i = 1 \\ \frac{1}{2}(e_1 \otimes e_6 + e_6 \otimes e_1), & i = 3 \\ \frac{1}{2}(e_1 \otimes e_4 + e_4 \otimes e_1), & i = 5. \end{cases}$$

It follows that

$$Ric_{ii} = \begin{cases} 0, & i = 2,\ 4,\ 6,\ 7,\ 8 \\ -\frac{1}{2}, & i = 1 \\ -\frac{1}{4}, & i = 3,\ 5 \end{cases}$$

so (M, g) is of negative scalar curvature, but not Einstein, that is, Ric has a non–trivial $\mathbf{1}$– and $[1,1,2]$–component.

Summarising, we obtain our final proposition.

Proposition 7.1. *Non–trivial compact harmonic $Sp(1) \cdot Sp(2)$– and $PSU(3)$–structures do exist. For a generic harmonic structure, the Ricci tensor takes values in $\mathbf{1} \oplus [2,2]$ and $\mathbf{1} \oplus [1,1,2]$ and in particular, we cannot improve Proposition 6.5.*

A. Appendix: A matrix representation of *Cliff*$(\mathbb{R}^8, g)$

For a fixed orthonormal basis $e_1, \ldots, e_8$ of $(\Lambda^1, g) \cong (\mathbb{R}^8, g_0)$ let $E_{ij} = e_i \wedge e_j$ denote the basis of Λ^2 which we identify with skew–symmetric matrices via

$$E_{ij} = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 1 & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \begin{matrix} \\ \ldots i \\ \\ \ldots j \\ \\ \end{matrix}$$
$$\qquad\qquad \vdots \qquad \vdots$$
$$\qquad\qquad i \qquad j$$

Then the matrix representation $\kappa : \mathit{Cliff}(\mathbb{R}^8, g_0) \to \mathrm{End}(\Delta_+ \oplus \Delta_-)$ computed from (2.1) with respect to the standard basis $e_1 = 1, e_2 = i, \ldots, e_8 = e \cdot k$ of $(\mathbb{O}, \|\cdot\|)$ is given by

$$\begin{aligned} \kappa(e_1) &= -E_{1,9} - E_{2,10} - E_{3,11} - E_{4,12} - E_{5,13} - E_{6,14} - E_{7,15} - E_{8,16}, \\ \kappa(e_2) &= E_{1,10} - E_{2,9} - E_{3,12} + E_{4,11} - E_{5,14} + E_{6,13} + E_{7,16} - E_{8,15}, \\ \kappa(e_3) &= E_{1,11} + E_{2,12} - E_{3,9} - E_{4,10} - E_{5,15} - E_{6,16} + E_{7,13} + E_{8,14}, \\ \kappa(e_4) &= E_{1,12} - E_{2,11} + E_{3,10} - E_{4,9} - E_{5,16} + E_{6,15} - E_{7,14} + E_{8,13}, \\ \kappa(e_5) &= E_{1,13} + E_{2,14} + E_{3,15} + E_{4,16} - E_{5,9} - E_{6,10} - E_{7,11} - E_{8,12}, \\ \kappa(e_6) &= E_{1,14} - E_{2,13} + E_{3,16} - E_{4,15} + E_{5,10} - E_{6,9} + E_{7,12} - E_{8,11}, \\ \kappa(e_7) &= E_{1,15} - E_{2,16} - E_{3,13} + E_{4,14} + E_{5,11} - E_{6,12} - E_{7,9} + E_{8,10}, \\ \kappa(e_8) &= E_{1,16} + E_{2,15} - E_{3,14} - E_{4,13} + E_{5,12} + E_{6,11} - E_{7,10} - E_{8,9}. \end{aligned} \tag{A.1}$$

References

[1] J. Armstrong. An ansatz for almost-Kähler, Einstein 4-manifolds. *J. Reine Angew. Math.*, **542**:53–84, 2002.

[2] S. Avis and C. Isham. Quantum field theory and fiber bundles in a general space-time. In M. Levy and S. Deser (eds.), *Recent developments in gravitation*, **44** of *NATO Advanced Study Institute Series B: Physics*, pp. 347–401. Plenum Press, 1979.

[3] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces i. *Amer. J. Math.*, **80**:458–538, 1958.

[4] R. Bryant. Metrics with exceptional holonomy. *Ann. Math.*, **126**:525–576, 1987.

[5] M. Gell-Mann and Y. Ne'eman. *The eightfold way.* W.A. Benjamin, New York, 1964.

[6] G. Gibbons and S. Hawking. Gravitational multi - instantons. *Phys. Lett.*, **B78**:430, 1978.

[7] N. Hitchin. Stable forms and special metrics. *Contemp. Math.*, **288**:70–89, 2001 [math.dg/0107101].

[8] N. Hitchin. Special holonomy and beyond. In *Strings and geometry*, **3** of *Clay Math. Proc.*, pp. 159–175. Amer. Math. Soc., 2004.

[9] H. Lawson and M.-L. Michelsohn. *Spin geometry*, **38** of *Princeton Mathematical Series.* Princeton University Press, Princeton, 1989.

[10] A. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, **39**, 1951.

[11] A. Onishchik and E. Vinberg. *Lie groups and Lie algebras III*, **41** of *Encyclopaedia of Mathematical Sciences.* Springer, Berlin, 1994.

[12] S. Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, **67**:143–171, 1982.

[13] S. Salamon. *Riemannian geometry and holonomy groups*, **201** of *Pitman Research Notes in Mathematics Series.* Longmann Scientific & Technical, Harlow, 1989.

[14] S. Salamon. Almost parallel structures. *Contemp. Math.*, **288**:162–181, 2001 [math.dg/0107146].

[15] H. Samelson. On curvature and characteristic of homogeneous spaces. *Michigan Math. J.*, **5**:13–18, 1958.

[16] A. Swann. Aspects symplectiques de la gé omé trie quaternionique. *C. R. Acad. Sci. Paris Sér. I Math.*, **308**(7):225–228, 1989.

[17] E. Thomas. Vector fields on manifolds. *Bull. Amer. Math. Soc.*, **75**:643–683, 1969.

[18] M. Čadek and J. Vanžura. Almost quaternionic structures on eight-manifolds. *Osaka J. Math.*, **35**(1):165–190, 1998.

[19] M. Čadek and J. Vanžura. On 4-fields and 4-distributions in 8-dimensional vector bundles over 8-complexes. *Colloq. Math.*, **76**(2):213–228, 1998.

[20] M. Y. Wang. Preserving parallel spinors under metric deformations. *Indiana Univ. Math. J.*, **40**(3):815–844, 1991.

[21] F. Witt. Special metric structures and closed forms, 2005, University of Oxford, math.DG/0502443.

[22] L. Woodward. The classification of orientable vector bundles over CW-complexes of small dimension. *Proc. Roy. Soc. Edinburgh Sect. A*, **92**(3-4):175–179, 1982.

[23] L. Woodward. The classification of principal $PU(n)$-bundles over a 4-complex. *J. London Math. Soc. (2)*, **25**(3):513–524, 1982.