Clifford bundle formulation of BF gravity generalized to the standard model

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Abstract

The structure and dynamics of the standard model and gravity are described by a Clifford valued connection and its curvature.

"Everything should be made as simple as possible, but not simpler." – A.E.

1 Introduction

The two most successful physical theories of the twentieth century were quantum mechanics, culminating in Quantum Field Theory and the standard model of particles and interactions, and General Relativity, Einstein's geometric theory of gravitation. These two theories, refined and verified to extraordinary accuracy, are beautiful mathematical descriptions of the physical universe. The fact that they have been found fundamentally incompatible stands as the greatest failure of twentieth century science, and provides the greatest challenge at the dawn of the twenty-first century.

Many attempts have been made to unify these two theories. The most popular current approaches, based on string theory, extend the methods and applications of QFT to various scenarios in a mathematically consistent but somewhat convoluted manner only tenuously connected to the standard model and GR. Although it is possible further development of string theory will lead to a coherent picture – and the development of beautiful mathematics is certainly, itself, a noble pursuit – several decades of intensive research have failed to produce a single successful experimental prediction. The one solid prediction of string theory, the existence of super-particle partners to the existing standard model particles, has so far failed to materialize. It is therefore reasonable to take a step backwards, reconsider the fundamental elements of GR and the standard model, and consider other approaches to unification following a more conservative path.

The fundamental fields of the standard model are gauge fields, spinors, and scalars over four dimensional spacetime. These elements have mathematical descriptions, respectively, as fiber bundle connections, Clifford algebra elements with anti-commuting components, and Higgs fields. QFT calculations arising from the standard model also require the introduction of BRST "ghost" fields to properly account for gauge degrees of freedom. General Relativity, in contrast, is about the geometry of spacetime itself – using a metric as well as a spin connection. But the spinor field Lagrangian of the standard model requires that the metric be alternatively described by a vierbein field, also known as a frame or tetrad, as a fundamental field. Any unified description

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of nature must employ all of these elements. Also, since the naive application of QFT methods to linearized GR fail, QFT must be generalized for a unification program to succeed.

A necessary question is how to extend QFT so it works for quantizing GR as well as reproducing existing methods. The best current attack on this problem is Loop Quantum Gravity [1] and similar approaches. The main modification to QFT suggested by LQG is that quantum transition amplitudes should not be considered between field observables at different spacetime points, but rather between boundary surfaces, with boundary surface states described by spin networks. This and other approaches to quantum gravity use a connection as the fundamental field of GR. Some very interesting recent work [2, 3, 4] revives an idea by MacDowell and Mansouri that the $so(5)$ spin connection may be broken into a $so(4)$ connection and vierbein, with the action for GR given by a restricted BF action. This restricted BF action may then be submitted to the methods of QFT, with a perturbative expansion about the purely topological BF action.

LQG and related programs of conservatively generalizing QFT to accommodate gravity seem, from an outsider's viewpoint, the most likely to succeed. In effect, they constitute the first step in a program of unification launched from the GR side rather than the particle theory side – the final goal being to extend the geometric description of General Relativity to encompass QFT and the standard model. The purpose of this article is to sketch how this unification may happen at the level of the fundamental fields, as simply as possible. In sum, the frame and spin connection 1-forms of GR may be unified in a Clifford algebra valued connection on a bundle, $\underrightarrow{\Omega} = \underrightarrow{e} + \underrightarrow{\omega}$, as in the MM method. This connection may then be incorporated in a larger Clifford fiber with the gauge and Higgs fields of the standard model, $\underrightarrow{A} = \phi\underrightarrow{e} + \underrightarrow{\omega} + \underrightarrow{Z}$. In previous work, it was shown that anti-commuting Clifford fields arise naturally from the BRST method, and the standard model fermion multiplets may be placed in a BRST extended connection, $\underrightarrow{\tilde{A}} = \phi\underrightarrow{e} + \underrightarrow{\omega} + \underrightarrow{Z} + \psi$. The resulting BRST extended curvature may then be used in a restricted BF Lagrangian, giving the standard model plus gravity from a BRST extended, Clifford algebra valued connection. Each step of this construction, building up to the full standard model plus gravity, will be described in detail using basic differential geometry so the reader may readily skim, reproduce, or absorb the material.

2 Clifford bundle connection for GR

An "n dimensional" Clifford algebra fiber, $Cl_{p,q}$, is a 2^n dimensional graded Lie algebra built from $n = p + q$ basis vector generators, γ_α, satisfying

$$\gamma_\alpha \cdot \gamma_\beta = \frac{1}{2}\left(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha\right) = \eta_{\alpha\beta} \tag{1}$$

with $\eta_{\alpha\beta}$ the generalized diagonal Minkowski metric having p positive and q negative entries. Any two unequal basis vectors anti-commute and produce a non-zero bivector, such as $\gamma_1\gamma_2 = -\gamma_2\gamma_1 = \gamma_{12}$. The Clifford algebra product, equivalent to the matrix product in a suitable representation, decomposes into symmetric and anti-symmetric parts,

$$A \cdot B = \frac{1}{2}(AB + BA)$$
$$[A, B] = A \times B = \frac{1}{2}(AB - BA)$$

with this bracket operator, equivalent to the usual commutator bracket with an added factor of $\frac{1}{2}$, extending to handle arbitrary numbers of multivectors – for example:

$$[A, B, C] = \frac{1}{3!}(ABC + BCA + CAB - ACB - CBA - BAC)$$

In this way any Clifford element may be written in terms of the basis elements as

$$
\begin{aligned}
A &= A^s + A^\alpha \gamma_\alpha + \frac{1}{2} A^{\alpha\beta} \left[\gamma_\alpha, \gamma_\beta\right] + \frac{1}{3!} A^{\alpha\beta\gamma} \left[\gamma_\alpha, \gamma_\beta, \gamma_\gamma\right] + \ldots + A^p \left[\gamma_0, \ldots, \gamma_{n-1}\right] \\
&= A^s + A^\alpha \gamma_\alpha + \frac{1}{2} A^{\alpha\beta} \gamma_{\alpha\beta} + \frac{1}{3!} A^{\alpha\beta\gamma} \gamma_{\alpha\beta\gamma} + \ldots + A^p \gamma
\end{aligned}
$$

The bivector (grade 2) part of this element is equal to $\langle A \rangle_2 = \frac{1}{2} A^{\alpha\beta} \gamma_{\alpha\beta} = A^2$, determined by the $\frac{1}{2} n(n-1)$ real coefficients, $A^{\alpha\beta} = A^{[\alpha\beta]}$, multiplying the corresponding Lie basis elements, $T_A = \frac{1}{2} \gamma_{\alpha\beta}$. The scalar (grade 0) part of a Clifford element, $\langle A \rangle = A^s$, is equivalent to the trace divided by the dimension of the matrix representation of the element; so for any Clifford elements, $\langle AB \rangle = \langle BA \rangle$. The structure constants for the algebra may be read off identities built from straightforward computations:

$$
\begin{aligned}
\gamma_\alpha \times \gamma_\beta &= \gamma_{\alpha\beta} \\
\gamma_\alpha \times \gamma_{\beta\gamma} &= \eta_{\alpha\beta} \gamma_\gamma - \eta_{\alpha\gamma} \gamma_\beta \\
\gamma_{\alpha\beta} \times \gamma_{\gamma\delta} &= \eta_{\alpha\gamma} \gamma_{\beta\delta} + \eta_{\alpha\delta} \gamma_{\beta\gamma} + \eta_{\beta\gamma} \gamma_{\alpha\delta} + \eta_{\beta\delta} \gamma_{\alpha\gamma} \\
&\vdots
\end{aligned}
$$

Equally useful identities follow from the symmetric product, such as

$$
\gamma_{\alpha\beta} \cdot \gamma_{\gamma\delta} = \left(\eta_{\alpha\delta} \eta_{\beta\gamma} - \eta_{\alpha\gamma} \eta_{\beta\delta}\right) + \gamma_{\alpha\beta\gamma\delta}
$$

giving the scalar and 4-vector parts of the symmetric product of two bivector basis elements. Taking the scalar part of two multiplied basis elements, both of grade r, gives the orthogonality relation,

$$
\left\langle \gamma_{\alpha\ldots\beta} \gamma^{\gamma\ldots\delta} \right\rangle = r! \delta^\gamma_{[\beta} \ldots \delta^\delta_{\alpha]}
$$

with indices raised by $\eta^{\alpha\beta}$. The above identities imply that the sub-algebra of anti-symmetric products of bivectors of an n dimensional Clifford algebra, spin(n), is equivalent to the algebra of anti-symmetric products of vector and bivector elements of an $n-1$ dimensional Clifford algebra. This sub-algebra of vectors and bivectors is nearly, but not, equivalent to the Poincare algebra of corresponding dimension. Also, the even graded sub-algebra of an n dimensional Clifford algebra is equal to an $n-1$ dimensional Clifford algebra.

The fundamental Clifford identity (1), and thus the Clifford algebra itself, is invariant under the Clifford group adjoint operation,

$$
\gamma'^\alpha = U \gamma U^{-1} \simeq \left(1 + \frac{1}{2} C\right) \gamma^\alpha \left(1 - \frac{1}{2} C\right) \simeq \gamma^\alpha + C \times \gamma^\alpha \tag{2}
$$

in which, for infinitesimal transformations, $U \simeq 1 + \frac{1}{2} C$ for some "small" multivector C. This operation gives the form of the transition functions acting on fiber basis elements over the base manifold.

Of special algebraic interest is the Clifford pseudo-scalar, $\gamma = \gamma_0 \gamma_1 \ldots \gamma_{n-1}$, which squares to

$$
\gamma\gamma = (-1)^q (-1)^{\frac{n(n+1)}{2}}
$$

implying $\gamma^{-1} = -\gamma$ for dimension $(1,3)$. Multiplication by γ^{-1} acts as the Clifford duality transformation, taking a Clifford r-vector to its "Clifford dual" $(n-r)$-vector,

$$
A_r \gamma^{-1} = \frac{1}{r!} A^{\alpha\ldots\beta} \gamma_{\alpha\ldots\beta} \gamma^{-1} = \frac{1}{r!\,(n-r)!} A^{\alpha\ldots\beta} \epsilon_{\alpha\ldots\beta\gamma\ldots\delta} \gamma^{\gamma\ldots\delta} \tag{3}
$$

in which $\gamma^{\gamma\ldots\delta}$ are the $(n-r)$-vector basis elements with indices raised by η and

$$\epsilon_{\alpha\ldots\delta} = n!\delta^0_{[\alpha}\ldots\delta^{n-1}_{\delta]} = \left\langle \gamma_{\alpha\ldots\delta}\gamma^{-1}\right\rangle$$

is the anti-symmetric permutation symbol. The pseudo-scalar always commutes with even graded elements, $A^e\gamma = \gamma A^e$, and anti-commutes with odd graded elements only in even dimensions, $A^o\gamma = (-1)^{n+1}\gamma A^o$. So for even n we have $A\cdot\gamma = A^e\gamma$.

The covariant derivative acting on basis elements, consistent with the transition functions, encodes how the basis vectors change as we move around on the base manifold,

$$\underrightarrow{\nabla}\gamma_\alpha = d\underrightarrow{x}^i\nabla_i\gamma_\alpha = \underrightarrow{\Omega}\times\gamma_\alpha$$

and does not necessarily preserve the grade of the basis elements when $\underrightarrow{\Omega}$ is a Clifford valued connection 1-form of arbitrary grade. This implies that for any section, C (a Clifford valued field over the base manifold), its covariant derivative is

$$\underrightarrow{\nabla}C = \underrightarrow{\partial}C + \underrightarrow{\Omega}\times C$$

with $\underrightarrow{\partial} = d\underrightarrow{x}^i\partial_i$ the exterior derivative operator. The spin connection of General Relativity, $\underrightarrow{\omega} = \frac{1}{2}d\underrightarrow{x}^i\omega_i{}^{\alpha\beta}\gamma_{\alpha\beta}$, is a bivector valued 1-form, encoding how the basis vectors rotate as we move around – it does preserve the grade of the basis elements. For example, an observer moving along a path $x(\tau)$ parameterized by τ with velocity $\vec{v} = \frac{dx^i(\tau)}{dt}\vec{\partial}_i = v^i\vec{\partial}_i$ would have the basis vectors changing over them by

$$\vec{v}\underrightarrow{\nabla}^\omega\gamma_\alpha = \vec{v}\underrightarrow{\omega}\times\gamma_\alpha = v^i\omega_i{}^\beta{}_\alpha\gamma_\beta$$

in which the vector-form contraction rule employed above is traditionally written less compactly as $\vec{\partial}_i d\underrightarrow{x}^j = \delta^j_i = \mathbf{i}_{\vec{\partial}_i}d\underrightarrow{x}^j$. It is pedagogically useful to mark all tangent bundle vectors and forms with accents indicating their grade so as to better identify their nature and commutative properties. If there is a section of the bundle, a field $C(x)$, an observer moving along a path would see this field change over them as a function of τ to $C(x(\tau))$. A section is said to be parallel transported by the spin connection over their path iff

$$0 = \vec{v}\underrightarrow{\nabla}^\omega C = \vec{v}\left(\underrightarrow{\partial}C + \underrightarrow{\omega}\times C\right) = v^i\partial_i C + v^i\omega_i\times C = \frac{d}{d\tau}C + v^i\omega_i\times C$$

evaluated along the path. Since the section and its derivatives are only evaluated along the path, this extends to describe a fiber element defined only over the path, $C(\tau)$, that is said to be parallel transported if it satisfies the same equation.

The frame, a Clifford vector valued 1-form,

$$\underrightarrow{e} = d\underrightarrow{x}^i\left(e_i\right)^\alpha\gamma_\alpha = d\underrightarrow{x}^i e_i = \underrightarrow{e}^\alpha\gamma_\alpha$$

encodes a metric on a base manifold of dimension n through contraction with vectors,

$$\left(\vec{v}\underrightarrow{e}\right)\cdot\left(\vec{w}\underrightarrow{e}\right) = v^i\left(\vec{\partial}_i d\underrightarrow{x}^j\right)\left(e_j\right)^\alpha\gamma_\alpha\cdot w^k\left(\vec{\partial}_k d\underrightarrow{x}^m\right)\left(e_m\right)^\beta\gamma_\beta = v^i w^k\left(\left(e_i\right)^\alpha\left(e_k\right)^\beta\eta_{\alpha\beta}\right) = v^i w^k g_{ik}$$

and has an inverse, $\vec{e}$, such that

$$\vec{e}\underrightarrow{e} = \gamma^\beta\left(e^{-1}_\beta\right)^j\vec{\partial}_j d\underrightarrow{x}^i\left(e_i\right)^\alpha\gamma_\alpha = \gamma^\beta\left(e^{-1}_\beta\right)^i\left(e_i\right)^\alpha\gamma_\alpha = \gamma^\beta\delta^\alpha_\beta\gamma_\alpha = n$$

It turns out to be remarkably fruitful to consider the frame and spin connection together in a Clifford connection,

$$\underrightarrow{\Omega} = \underrightarrow{e} + \underrightarrow{\omega} \tag{4}$$

of mixed grade 1 and 2. Although this is algebraicly equivalent to a bivector connection in one higher dimension, it seems natural to consider this combined connection in $n = 4$ over a four dimensional base manifold.

3 BF action for GR and equations of motion

The geometry of a fiber bundle is described by the curvature of its connection. Just as the connection arises as the description of how sections change over the paths of observers traveling on the base, the curvature continues this description to second order in path length and determines how fiber elements change when parallel transported around small loops.

3.1 Parallel transport and Curvature

Any Clifford multivector, C, is parallel transported along a parameterized curve, $x(\tau)$, with velocity $\vec{v}$ iff

$$0 = v^i \nabla_i C = \vec{v} \left(\underrightarrow{\partial} C + \underrightarrow{\Omega} \times C \right) = \frac{d}{d\tau} C + \vec{v} \underrightarrow{\Omega} \times C \tag{5}$$

for a specified Clifford connection. Parallel transport transforms the multivector via a path dependent Clifford valued adjoint operator, $U(\tau)$, as it moves along the curve,

$$C(\tau) = U(\tau)\, C(0)\, U^{-1}$$

with initial condition $U(0) = 1$. This parallel transport operator is independent of C and, from (5), must satisfy the parallel transport equation:

$$0 = \frac{d}{d\tau} U + \frac{1}{2} \vec{v} \underrightarrow{\Omega} U$$

For small distances along a path, $x(\tau) = x_0 + \varepsilon(\tau)$, this operator may be approximated to arbitrary order. To first order

$$U(\tau) = 1 - \frac{1}{2} \int_0^\tau \underrightarrow{dt}\, \frac{d\varepsilon^i}{dt} \Omega_i(x(t))\, U(t) \simeq 1 - \frac{1}{2} \varepsilon^i \Omega_i(x_0)$$

and to second order

$$\begin{aligned} U(\tau) &\simeq 1 - \frac{1}{2} \int_0^\tau \underrightarrow{dt}\, \frac{d\varepsilon^i}{dt} \left[\Omega_i + \varepsilon^j \partial_j \Omega_i \right] \left[1 - \frac{1}{2} \varepsilon^k \Omega_k \right] \\ &\simeq 1 - \frac{1}{2} \varepsilon^i \Omega_i + \frac{1}{2} \varepsilon^{ij} \left[-\partial_j \Omega_i + \frac{1}{2} \Omega_i \Omega_j \right] \end{aligned} \tag{6}$$

with the second order path dependence above defined as

$$\varepsilon^{ij} = \int_0^\tau \underrightarrow{dt}\, \frac{d\varepsilon^i}{dt} \varepsilon^j$$

Continuing the series suggests the formal expression

$$U(\tau) = \exp\left(-\frac{1}{2} \int_0^\tau \underrightarrow{dt}\, \left(\vec{v} \underrightarrow{\Omega} \right) \right) = \exp\left(-\frac{1}{2} \int_0^\tau \underrightarrow{\Omega} \right)$$

The curvature is a geometric object determining the approximate change in any multivector parallel transported around a small loop. A loop may be specified by choosing two orthonormal vectors, $\vec{u}$ and $\vec{v}$, at a point x_0 and making a square-ish path by going ε in the $\vec{u}$ direction, then ε along $\vec{v}$, ε along $-\vec{u}$, then ε along $-\vec{v}$ back to x_0. These four parameterized path segments are given by

$$\varepsilon_1^i = t u^i, \quad \varepsilon_2^i = \varepsilon u^i + t v^i, \quad \varepsilon_3^i = \varepsilon u^i + \varepsilon v^i - t u^i, \quad \varepsilon_4^i = \varepsilon v^i - t v^i$$

and produce an anti-symmetric path dependence,

$$\varepsilon^{ij} = \int_0^\varepsilon \underset{\rightarrow}{dt}\,\frac{d\varepsilon_1^i}{dt}\varepsilon_1^j + \int_0^\varepsilon \underset{\rightarrow}{dt}\,\frac{d\varepsilon_2^i}{dt}\varepsilon_2^j + \int_0^\varepsilon \underset{\rightarrow}{dt}\,\frac{d\varepsilon_3^i}{dt}\varepsilon_3^j + \int_0^\varepsilon \underset{\rightarrow}{dt}\,\frac{d\varepsilon_4^i}{dt}\varepsilon_4^j = \varepsilon^2\left(v^i u^j - v^j u^i\right)$$

implying the loop is described by a tangent 2-vector,

$$\overset{\rightrightarrows}{L} = \varepsilon^2 \vec{v}\,\vec{u} = \varepsilon^2 v^i u^j \vec{\partial}_i \vec{\partial}_j = \frac{1}{2}\varepsilon^{ij}\vec{\partial}_i\vec{\partial}_j = \frac{1}{2}L^{ij}\vec{\partial}_i\vec{\partial}_j$$

From (6), the operator for parallel transport completely around a small loop is approximately,

$$U \simeq 1 + \frac{1}{2}L^{ij}\left[-\partial_j\Omega_i + \frac{1}{2}\Omega_i\Omega_j\right] = 1 + \frac{1}{4}L^{ij}\left[\partial_i\Omega_j - \partial_j\Omega_i + \Omega_i \times \Omega_j\right] = 1 + \frac{1}{4}L^{ij}F_{ij} = 1 - \frac{1}{2}\overset{\rightrightarrows}{L}\underset{\rightrightarrows}{F}$$

with the Clifford valued curvature 2-form coefficients here emerging as

$$F_{ij} = \partial_i\Omega_j - \partial_j\Omega_i + \Omega_i \times \Omega_j$$

and the index free Clifford valued curvature 2-form written as

$$\underset{\rightrightarrows}{F} = \frac{1}{2}\underset{\rightarrow}{dx^i}\underset{\rightarrow}{dx^j}F_{ij} = \underset{\rightarrow}{\partial}\underset{\rightarrow}{\Omega} + \frac{1}{2}\underset{\rightarrow}{\Omega} \times \underset{\rightarrow}{\Omega}$$

The wedge product between forms is not written since forms always wedge, and the cross product occurs only between Clifford basis elements. Any Clifford element, C, parallel transported around a small loop is changed to

$$C \mapsto C' = UCU^{-1} \simeq C - \overset{\rightrightarrows}{L}\underset{\rightrightarrows}{F} \times C$$

to first order in loop area.

For a bivector valued spin connection, the bivector valued Riemann curvature 2-form is

$$\underset{\rightrightarrows}{R} = \frac{1}{4}\underset{\rightarrow}{dx^i}\underset{\rightarrow}{dx^j}R_{ij}{}^{\alpha\beta}\gamma_{\alpha\beta} = \underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega} + \frac{1}{2}\underset{\rightarrow}{\omega} \times \underset{\rightarrow}{\omega}$$

for which, in components,

$$\frac{1}{2}\underset{\rightarrow}{\omega} \times \underset{\rightarrow}{\omega} = \frac{1}{8}\underset{\rightarrow}{dx^i}\underset{\rightarrow}{dx^j}\omega_i{}^{\alpha\beta}\omega_j{}^{\gamma\delta}\gamma_{\alpha\beta} \times \gamma_{\gamma\delta} = \frac{1}{2}\underset{\rightarrow}{\omega}\underset{\rightarrow}{\omega} = \frac{1}{2}\underset{\rightarrow}{dx^i}\underset{\rightarrow}{dx^j}\omega_i{}^{\alpha\beta}\omega_{j\alpha}{}^{\delta}\gamma_{\beta\delta}$$

The writing of the cross between the product of any two identical 1-forms is redundant, $\underset{\rightarrow}{\Omega} \times \underset{\rightarrow}{\Omega} = \underset{\rightarrow}{\Omega}\underset{\rightarrow}{\Omega}$, since basis 1-forms anti-commute.

The Clifford curvature of a combined vector and bivector connection (4) naturally splits into Clifford vector and bivector graded parts,

$$\begin{aligned}
\underset{\rightrightarrows}{F} &= \underset{\rightarrow}{\partial}\underset{\rightarrow}{\Omega} + \frac{1}{2}\underset{\rightarrow}{\Omega}\underset{\rightarrow}{\Omega} \\
&= \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{e} + \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{e}\right) + \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega} + \frac{1}{2}\underset{\rightarrow}{\omega}\underset{\rightarrow}{\omega} + \frac{1}{2}\underset{\rightarrow}{e}\underset{\rightarrow}{e}\right) \\
&= \underset{\rightrightarrows}{T} + \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E}\right)
\end{aligned}$$

identifiable as the torsion vector valued 2-form, the Riemann curvature bivector, and the bivector area 2-form. The curvature may also be obtained by twice applying the covariant derivative,

$$\underset{\rightarrow}{\nabla}\underset{\rightarrow}{\nabla}C = \underset{\rightrightarrows}{F} \times C$$

3.2 Action

Over a four dimensional base manifold a restricted BF action equivalent to that of General Relativity may be written with Clifford $n = 4$ as

$$S = \int \underline{L} = \int \left\langle \underset{\rightrightarrows}{B} \underset{\rightrightarrows}{F} - \frac{1}{2} \underset{\rightrightarrows}{B} \underset{\rightrightarrows}{B} \gamma \right\rangle \tag{7}$$

employing the Clifford pseudo-scalar, $\gamma = (-1)^q \gamma^{-1}$, a new vector and bivector valued 2-form variable, $\underset{\rightrightarrows}{B}$, and using an under-bar to designate forms of large or arbitrary grade. Since Clifford basis orthogonality implies the scalar part of two multiplied Clifford elements splits into terms of multiplied equal graded parts, this Lagrangian splits into

$$\underline{L} = \left\langle \underset{\rightrightarrows}{B}^1 \underset{\rightrightarrows}{T} + \underset{\rightrightarrows}{B}^2 \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} - \frac{1}{2} \underset{\rightrightarrows}{B}^2 \gamma \right) \right\rangle$$

Varying the action with respect to $\underset{\rightrightarrows}{B}$, the vector part, $\underset{\rightrightarrows}{B}^1$, is a Lagrange multiplier enforcing zero torsion, $\underset{\rightrightarrows}{T} = 0$, and the bivector part gives the equation

$$\underset{\rightrightarrows}{B}^2 = \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} \right) \gamma^{-1} = \underset{\rightrightarrows}{F}^2 \gamma^{-1} \tag{8}$$

in which $\underset{\rightrightarrows}{F}^2$ is the bivector part of the curvature. Plugging this back in gives the Lagrangian purely in terms of the curvature,

$$\underline{L} = \frac{1}{2} \left\langle \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} \right) \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} \right) \gamma^{-1} \right\rangle = \frac{1}{2} \left\langle \underset{\rightrightarrows}{F} \underset{\rightrightarrows}{F} \gamma^{-1} \right\rangle \tag{9}$$

Since the Clifford pseudo-scalar commutes with even graded Clifford elements, basis 1-forms anti-commute, and the exterior derivative is nilpotent, the Riemann squared term in (9) is a Chern-Simons boundary term,

$$\left\langle \underset{\rightrightarrows}{R} \underset{\rightrightarrows}{R} \gamma^{-1} \right\rangle = \left\langle \left(\underset{\rightrightarrows}{\partial \omega} + \frac{1}{2} \underset{\rightrightarrows}{\omega \omega} \right) \left(\underset{\rightrightarrows}{\partial \omega} + \frac{1}{2} \underset{\rightrightarrows}{\omega \omega} \right) \gamma^{-1} \right\rangle = \underset{\rightarrow}{\partial} \left\langle \left(\underset{\rightarrow}{\omega} \left(\underset{\rightrightarrows}{\partial \omega} \right) + \frac{1}{3} \underset{\rightrightarrows}{\omega \omega \omega} \right) \gamma^{-1} \right\rangle$$

So the Lagrangian is equivalent, up to this boundary term, to the Lagrangian for Einstein-Cartan gravity including a cosmological constant term,

$$\underline{L}^{\text{eff}} = \frac{1}{2} \left\langle \left(\underset{\rightrightarrows}{E} \underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{R} \underset{\rightrightarrows}{E} + \underset{\rightrightarrows}{E} \underset{\rightrightarrows}{E} \right) \gamma^{-1} \right\rangle = \frac{1}{2} \left\langle \left(\underset{\rightarrow}{e} \underset{\rightarrow}{e} \underset{\rightrightarrows}{R} + \frac{1}{4} \underset{\rightarrow}{e} \underset{\rightarrow}{e} \underset{\rightarrow}{e} \underset{\rightarrow}{e} \right) \gamma^{-1} \right\rangle = \underline{e} \frac{1}{2} \langle R + 6 \rangle$$

in which $\underline{e} = \frac{1}{4!} \left\langle \underset{\rightarrow}{e} \underset{\rightarrow}{e} \underset{\rightarrow}{e} \underset{\rightarrow}{e} \gamma^{-1} \right\rangle = \underline{d^4 x} \, |e|$ is the volume 4-form and $R = \left\langle \vec{e} \vec{e} \underset{\rightrightarrows}{R} \right\rangle$ is the curvature scalar. The appearance of the cosmological constant is made explicit by the appropriate vierbein scaling – rescaling $\underset{\rightarrow}{e}$ to $\sqrt{\Lambda} \underset{\rightarrow}{e}$. The equivalence of terms written in Clifford valued form notation to those written in components is seen by expanding in basis elements and employing the orthogonality rules. Although the methods and action described so far are equivalent to those of MacDowell-Mansouri and others, the formulation here allows the action to be written naturally in terms of the Clifford dual and does not require symmetry breaking to step down from $so(5)$. Also, the Clifford algebra approach easily generalizes to additional interesting systems, such as Clifford algebras and sub-algebras of different signatures and dimensions. As in the MM approach, the main observation is that the dynamics of GR may be described purely in terms of a connection, without needing a metric on the base manifold. Viewed in this light, the scale of the vierbein should properly be interpreted as a Higgs field – an idea that will pop up again later.

3.3 Equations of motion

The first equation of motion is the vanishing of the torsion,

$$0 = \underset{\rightrightarrows}{T} = \underset{\rightarrow}{\partial}\underset{\rightarrow}{e} + \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{e} \tag{10}$$

which, for $n = 4$, may be solved explicitly for the spin connection in terms of the exterior derivative and inverse of the vierbein,

$$\underset{\rightarrow}{\omega} = -\vec{e} \times \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{e}\right) + \frac{1}{4}\left(\vec{e} \times \vec{e}\right)\left(\underset{\rightarrow}{e} \cdot \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{e}\right)\right)$$

Varying the action (7) with respect to $\underset{\rightarrow}{\Omega}$ gives

$$\delta S = \int \left\langle \underset{\rightrightarrows}{B} \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{\delta\Omega} + \frac{1}{2}\underset{\rightarrow}{\delta\Omega}\underset{\rightarrow}{\Omega} + \frac{1}{2}\underset{\rightarrow}{\Omega}\underset{\rightarrow}{\delta\Omega} \right) \right\rangle = \int \left\langle \underset{\rightarrow}{\delta\Omega} \left(\underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B} + \frac{1}{2}\underset{\rightarrow}{\Omega}\underset{\rightrightarrows}{B} - \frac{1}{2}\underset{\rightrightarrows}{B}\underset{\rightarrow}{\Omega} \right) + \underset{\rightarrow}{\partial}\left(\underset{\rightrightarrows}{B}\underset{\rightarrow}{\delta\Omega} \right) \right\rangle$$

and hence the second equation of motion,

$$0 = \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B} + \underset{\rightarrow}{\Omega} \times \underset{\rightrightarrows}{B} = \underset{\rightarrow}{\nabla}\underset{\rightrightarrows}{B}$$

which includes the odd and even graded parts,

$$\begin{aligned} 0 &= \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B}^1 + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{B}^1 + \underset{\rightarrow}{e} \times \underset{\rightrightarrows}{B}^2 \\ 0 &= \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B}^2 + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{B}^2 + \underset{\rightarrow}{e} \times \underset{\rightrightarrows}{B}^1 \end{aligned}$$

Incorporating the first equation of motion (8) and the "second" Bianchi identity,

$$\underset{\rightarrow}{\nabla}^{\omega}\underset{\rightrightarrows}{R} = \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{R} + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{R} = \underset{\rightarrow}{\partial}\left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega} + \frac{1}{2}\underset{\rightarrow}{\omega}\underset{\rightarrow}{\omega} \right) + \underset{\rightarrow}{\omega} \times \left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega} + \frac{1}{2}\underset{\rightarrow}{\omega}\underset{\rightarrow}{\omega} \right) = 0$$

these become

$$\begin{aligned} 0 &= \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B}^1 + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{B}^1 + \underset{\rightarrow}{e} \times \left(\left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} \right) \gamma^{-1} \right) \\ 0 &= \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{E}\gamma^{-1} + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{E}\gamma^{-1} + \underset{\rightarrow}{e} \times \underset{\rightrightarrows}{B}^1 \end{aligned}$$

Using the vanishing torsion, the last equation becomes $\underset{\rightrightarrows}{B}^1 = 0$ and the only remaining equation is

$$0 = \underset{\rightarrow}{e} \cdot \left(\underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} \right) \tag{11}$$

Einstein's equation, where $\underset{\rightarrow}{e}\gamma = -\gamma\underset{\rightarrow}{e}$ has been used, presuming even n.

The equations of motion may alternatively be obtained by varying $\underset{\rightarrow}{\Omega}$ in (9) to get

$$0 = \underset{\rightarrow}{\nabla}\left(\underset{\rightrightarrows}{F} \cdot \gamma^{-1} \right) \tag{12}$$

Combining with the Clifford Bianchi identity, $\underset{\rightarrow}{\nabla}\underset{\rightrightarrows}{F} = 0$, and breaking into graded parts gives vanishing torsion, Einstein's equation, and the "first" Bianchi identity, $\underset{\rightarrow}{e} \times \underset{\rightrightarrows}{R} = 0$.

3.4 Gauge symmetry

Under a gauge transformation (2), parameterized by Clifford element $U(x) \simeq 1 + \frac{1}{2}C(x)$, the connection transforms to

$$\underrightarrow{\Omega}' = U\underrightarrow{\Omega}U^{-1} + 2U\left(\underrightarrow{\partial}U^{-1}\right) \simeq \underrightarrow{\Omega} + \frac{1}{2}C\underrightarrow{\Omega} - \frac{1}{2}\underrightarrow{\Omega}C - \underrightarrow{\partial}C = \underrightarrow{\Omega} - \underrightarrow{\nabla}C$$

which may be written as

$$\delta_C \underrightarrow{\Omega} = -\underrightarrow{\nabla}C \tag{13}$$

and the curvature transforms to

$$\underrightarrow{\underrightarrow{F}}' = \underrightarrow{\partial}\underrightarrow{\Omega}' + \frac{1}{2}\underrightarrow{\Omega}'\underrightarrow{\Omega}' = U\underrightarrow{\underrightarrow{F}}U^{-1} \simeq \underrightarrow{\underrightarrow{F}} + C \times \underrightarrow{\underrightarrow{F}}$$

giving $\delta_C \underrightarrow{\underrightarrow{F}} = C \times \underrightarrow{\underrightarrow{F}}$. This produces a transformation of the Lagrangian (9) to

$$\underline{L}' = \frac{1}{2}\left\langle \underrightarrow{\underrightarrow{F}}'\underrightarrow{\underrightarrow{F}}'\gamma^{-1}\right\rangle = \frac{1}{2}\left\langle \underrightarrow{\underrightarrow{F}}\underrightarrow{\underrightarrow{F}}U^{-1}\gamma^{-1}U\right\rangle \simeq \underline{L} - \frac{1}{2}\left\langle \underrightarrow{\underrightarrow{F}}\underrightarrow{\underrightarrow{F}}\left(C \times \gamma^{-1}\right)\right\rangle = \underline{L} - \left\langle \gamma^{-1}\left(\underrightarrow{\underrightarrow{T}} \cdot \underrightarrow{\underrightarrow{F}}^2\right)C^1\right\rangle$$

in which C^1 is the Clifford vector part of $C = C^1 + C^2$. The gauge transformation decomposes into vector and bivector parts,

$$\begin{aligned}
\delta_C \underrightarrow{e} &= -\underrightarrow{\partial}C^1 - \underrightarrow{\omega} \times C^1 - \underrightarrow{e} \times C^2 \\
\delta_C \underrightarrow{\omega} &= -\underrightarrow{\partial}C^2 - \underrightarrow{\omega} \times C^2 - \underrightarrow{e} \times C^1
\end{aligned}$$

C^2 parameterizing Lorentz transformations and C^1 related to diffeomorphisms. For a transformation to be a symmetry, the action must be invariant up to a boundary term under the transformation. In the space of all possible connections, $\underrightarrow{\Omega}$, the space of solutions to the equations of motion, (12), form a subspace referred to as the "shell." An equation that holds only when the equations of motion are enforced is said to be true "on shell," and one that holds even when they are not enforced is true "off shell." Since $\delta_{C^2}\underline{L} = 0$, the gauge transformation parameterized by an arbitrary bivector, $C = C^2 = \Sigma$, is an "off shell" symmetry – giving zero variation to the Lagrangian even if the equations of motion are not enforced. The gauge transformation generated by arbitrary $C = C^1$ is an "on shell" symmetry – giving $\delta_{C^1}\underline{L} = 0$ only when $\underrightarrow{\underrightarrow{T}} = 0$. However, the gauge transformation parameterized by $C = C^1$ is an "off shell" symmetry if it is constrained such that the change in the Lagrangian is exact,

$$\delta_{C^1}\underline{L} = -\left\langle \gamma^{-1}\left(\underrightarrow{\underrightarrow{T}} \cdot \underrightarrow{\underrightarrow{F}}^2\right)C^1\right\rangle = \underrightarrow{\partial}\underline{b} \tag{14}$$

with $\underline{b}$ some scalar valued 3-form. This space of constrained gauge transformations is equivalent to the space of diffeomorphisms. If this constraint is not imposed, and arbitrary "on shell" gauge transformations are allowed, it would be possible to make the frame vanish, $\underrightarrow{e}' = \underrightarrow{e} + \delta_C \underrightarrow{e} = 0$, by making a gauge transformation via $C = x^i e_i$.

A diffeomorphism consists of moving the fields over the base manifold along an arbitrary flow field, $\vec{\xi}(x)$. The transformation is given by the Lie derivative, applying to any geometrically defined object:

$$\begin{aligned}
\delta_\xi \underrightarrow{\Omega} &= \pounds_{\vec{\xi}}\underrightarrow{\Omega} = \vec{\xi}\left(\underrightarrow{\partial}\underrightarrow{\Omega}\right) + \underrightarrow{\partial}\left(\vec{\xi}\underrightarrow{\Omega}\right) \\
\delta_\xi \underrightarrow{\underrightarrow{F}} &= \pounds_{\vec{\xi}}\underrightarrow{\underrightarrow{F}} = \vec{\xi}\left(\underrightarrow{\partial}\underrightarrow{\underrightarrow{F}}\right) + \underrightarrow{\partial}\left(\vec{\xi}\underrightarrow{\underrightarrow{F}}\right)
\end{aligned}$$

The Lagrangian changes under a diffeomorphism by an exact term since 5-forms are zero over a four dimensional base manifold,

$$\delta_\xi \underset{-}{L} = \pounds_{\underset{\rightarrow}{\vec{\xi}}} \underset{-}{L} = \underset{\rightarrow}{\partial} \left(\vec{\xi} \underset{-}{L} \right) = \underset{\rightarrow}{\partial} \underset{-}{b}^\xi \tag{15}$$

and diffeomorphisms are thus an "off shell" symmetry. The gauge transformation corresponding to a diffeomorphism may be found by solving $\delta_C \underset{\rightarrow}{\Omega} = \delta_\xi \underset{\rightarrow}{\Omega}$ for $C = C^\xi$ given any $\vec{\xi}$ – i.e. by finding the solution to

$$-\underset{\rightarrow}{\nabla} C = \pounds_{\underset{\rightarrow}{\vec{\xi}}} \underset{\rightarrow}{\Omega} = \vec{\xi} \underset{\rightrightarrows}{F} + \underset{\rightarrow}{\nabla} \left(\vec{\xi} \underset{\rightarrow}{\Omega} \right) \tag{16}$$

If the solution is split into $C^\xi = -\vec{\xi} \underset{\rightarrow}{\Omega} + C'$, this equation simplifies further to solving

$$\underset{\rightarrow}{\nabla} C' = -\vec{\xi} \underset{\rightrightarrows}{F}$$

for C' to get the correct gauge transformation corresponding to any diffeomorphism.

3.5 Hamiltonian formulation (an optional interlude)

The variational formulation and derivation of equations of motion may be recast in a canonical Hamiltonian framework. A functional derivative with respect to an arbitrary p-form, $\frac{\partial}{\partial \underset{-}{A}}$, may be defined so the chain rule for the exterior derivative works as, for example,

$$\underset{\rightarrow}{\partial} \underset{-}{G}(A, B) = \left(\underset{\rightarrow}{\partial} \underset{-}{A} \right) \left(\frac{\partial}{\partial \underset{-}{A}} \underset{-}{G} \right) + \left(\underset{\rightarrow}{\partial} \underset{-}{B} \right) \left(\frac{\partial}{\partial \underset{-}{B}} \underset{-}{G} \right)$$

for an arbitrary Clifford valued form functional, $\underset{-}{G}$, of arbitrary Clifford valued forms, $\underset{-}{A}$ and $\underset{-}{B}$. Although the above formula provides the most practical working definition for extracting an arbitrary functional derivative, we may also define the derivative with respect to a Clifford r-vector valued p-form in terms of coordinate and Clifford basis elements as

$$\frac{\partial}{\partial \underset{-}{A}} \underset{-}{G} = p! r! \vec{\partial}_j \dots \vec{\partial}_i \gamma^{\beta \dots \alpha} \frac{\partial}{\partial A_{i \dots j}{}^{\alpha \dots \beta}} \underset{-}{G}$$

For an action and scalar valued Lagrangian 4-form functional of a connection 1-form, $\underset{-}{L}(A, \partial A)$, the system may be cast in "first order" form by defining the Clifford valued momentum 2-form,

$$\underset{\rightrightarrows}{B} = \frac{\partial}{\partial \left(\underset{\rightarrow}{\partial} \underset{\rightarrow}{A} \right)} \underset{-}{L}$$

and scalar valued Hamiltonian 4-form,

$$\underset{-}{H}(A, B) = \left\langle \underset{\rightrightarrows}{B} \underset{\rightarrow}{\partial} \underset{\rightarrow}{A} \right\rangle - \underset{-}{L}$$

with $\underset{\rightarrow}{\partial} \underset{\rightarrow}{A}$ written in terms of $\underset{\rightrightarrows}{B}$. The variation of the action in terms of these variables is

$$\delta S = \delta \int \underset{-}{L} = \delta \int \left\langle \underset{\rightrightarrows}{B} \underset{\rightarrow}{\partial} \underset{\rightarrow}{A} - \underset{-}{H} \right\rangle = \int \left\langle \left(\delta \underset{\rightrightarrows}{B} \right) \left(\underset{\rightarrow}{\partial} \underset{\rightarrow}{A} - \frac{\partial}{\partial \underset{\rightrightarrows}{B}} \underset{-}{H} \right) + \left(\delta \underset{\rightarrow}{A} \right) \left(\underset{\rightarrow}{\partial} \underset{\rightrightarrows}{B} - \frac{\partial}{\partial \underset{\rightarrow}{A}} \underset{-}{H} \right) + \underset{\rightarrow}{\partial} \left(\underset{\rightrightarrows}{B} \left(\delta \underset{\rightarrow}{A} \right) \right) \right\rangle$$

The restricted BF action for gravity (7) is already in Hamiltonian form, with connection $\underset{\rightarrow}{\Omega}$ and momentum $\underset{\rightrightarrows}{B}$ the canonical variables, and the Hamiltonian

$$\underset{-}{H} = \left\langle -\frac{1}{2} \underset{\rightrightarrows}{B} \underset{\rightarrow}{\Omega} \underset{\rightarrow}{\Omega} + \frac{1}{2} \underset{\rightrightarrows}{B} \underset{\rightrightarrows}{B} \gamma \right\rangle \tag{17}$$

A restricted BF action may be used in a perturbative expansion carried out around the solutions of pure BF theory, $\underset{\rightrightarrows}{F}=0$, corresponding to Hamiltonian perturbation around $\underset{-}{H}_0=\left\langle -\frac{1}{2}\underset{\rightrightarrows}{B}\underset{\rightarrow}{\Omega}\underset{\rightarrow}{\Omega}\right\rangle$. The canonical equations of motion for BF restricted gravity are a re-assemblage of the equations of motion into

$$\begin{aligned}\underset{\rightarrow}{\partial}\underset{\rightarrow}{\Omega} &= \frac{\partial}{\partial \underset{\rightrightarrows}{B}}\underset{-}{H} = -\frac{1}{2}\underset{\rightarrow}{\Omega}\underset{\rightarrow}{\Omega}+\underset{\rightrightarrows}{B}\cdot\gamma \\ \underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B} &= \frac{\partial}{\partial \underset{\rightarrow}{\Omega}}\underset{-}{H} = -\underset{\rightarrow}{\Omega}\times\underset{\rightrightarrows}{B}\end{aligned} \tag{18}$$

This description of motion in terms of exterior derivatives is particularly well adapted to describe flows through boundary surfaces. To recover the traditional coordinate based description, we need only write out the forms in terms of coordinates including time, x^0, such as

$$\underset{\rightarrow}{\partial}\underset{\rightarrow}{e}=\left(\underset{\rightarrow}{dx}^0\partial_0+\underset{\rightarrow}{dx}^a\partial_a\right)\left(\underset{\rightarrow}{dx}^0 e_0+\underset{\rightarrow}{dx}^b e_b\right)=\underset{\rightarrow}{dx}^0\underset{\rightarrow}{dx}^a\left(\partial_0 e_a-\partial_a e_0\right)+\underset{\rightarrow}{dx}^a\underset{\rightarrow}{dx}^b\partial_a e_b$$

We may alternatively decompose the forms with respect to a level surface. For some scalar valued function over the base manifold, $t(x)$, any p-form may be split into parts parallel and perpendicular to a surface of constant t as

$$\underset{-}{A}=\underset{-}{A}^{\parallel}+\underset{-}{A}^{\perp}=\vec{t}\left(\underset{\rightarrow}{dt}\underset{-}{A}\right)+\underset{\rightarrow}{dt}\left(\vec{t}\underset{-}{A}\right)$$

with $\underset{\rightarrow}{dt}=\underset{\rightarrow}{dx}^i\partial_i t$ and its dual vector, $\vec{t}$, satisfying $\vec{t}\underset{\rightarrow}{dt}=1$. Or, as a third alternative, we could decompose the equations of motion via the Lie derivative,

$$\pounds_{\vec{t}}\underset{-}{A}=\vec{t}\left(\underset{\rightarrow}{\partial}\underset{-}{A}\right)+\underset{\rightarrow}{\partial}\left(\vec{t}\underset{-}{A}\right)$$

The canonical equations of motion may also be obtained by defining a Poisson-like bracket operator,

$$\left\{\underset{-}{F},\underset{-}{G}\right\}=\left(\frac{\partial}{\partial\underset{\rightarrow}{A}}\underset{-}{F}\right)\left(\frac{\partial}{\partial\underset{\rightrightarrows}{B}}\underset{-}{G}\right)+(-1)^{o(\underset{-}{F})}\left(\frac{\partial}{\partial\underset{\rightrightarrows}{B}}\underset{-}{F}\right)\left(\frac{\partial}{\partial\underset{\rightarrow}{A}}\underset{-}{G}\right)$$

in which $o(\underset{-}{F})$ is the form order. We may also write $\delta_{\underset{-}{F}}\underset{-}{G}=\{\underset{-}{F},\underset{-}{G}\}$ for "the canonical transformation of $\underset{-}{G}$ generated by $\underset{-}{F}$." Since $\underset{\rightarrow}{A}$ is a 1-form, this Poisson-like bracket is not necessarily anti-symmetric and should only be considered a calculational convenience, though it does satisfy $\{\underset{\rightarrow}{A},\underset{\rightrightarrows}{B}\}=-\{\underset{\rightrightarrows}{B},\underset{\rightarrow}{A}\}=1$. The Hamiltonian is the generator of the part of the exterior derivative dependent on the canonical variables. For some functional, $\underset{-}{G}(A,B,C)$, of the canonical variables and a parameterizing field, $C(x)$, the exterior derivative is

$$\underset{\rightarrow}{\partial}\underset{-}{G}=\left(\underset{\rightarrow}{\partial}C\right)\left(\frac{\partial}{\partial C}\underset{-}{G}\right)+\left(\underset{\rightarrow}{\partial}\underset{\rightrightarrows}{B}\right)\left(\frac{\partial}{\partial\underset{\rightrightarrows}{B}}\underset{-}{G}\right)+\left(\underset{\rightarrow}{\partial}\underset{\rightarrow}{A}\right)\left(\frac{\partial}{\partial\underset{\rightarrow}{A}}\underset{-}{G}\right)=\left(\underset{\rightarrow}{\partial}C\right)\left(\frac{\partial}{\partial C}\underset{-}{G}\right)+\left\{\underset{-}{H},\underset{-}{G}\right\} \tag{19}$$

when evaluated "on shell" – when the equations of motion (18) are satisfied. Canonical transformations are made by choosing a generating functional. A particularly useful class of these are scalar valued 3-form functionals, $\underset{-}{G}(A,B,C)$, parameterized by $C(x)$. Such functionals produce a canonical transformation,

$$\begin{aligned}\delta_C\underset{\rightarrow}{A} &= \left\{\underset{-}{G},\,\underset{\rightarrow}{A}\right\}=-\frac{\partial}{\partial\underset{\rightrightarrows}{B}}\underset{-}{G} \\ \delta_C\underset{\rightrightarrows}{B} &= \left\{\underset{-}{G},\,\underset{\rightrightarrows}{B}\right\}=\frac{\partial}{\partial\underset{\rightarrow}{A}}\underset{-}{G}\end{aligned}$$

and a variation in the action,

$$
\begin{aligned}
\delta_C S &= \int \left\langle \left(\delta_C \underset{\Rightarrow}{B}\right) \left(\underset{\rightarrow}{\partial} \underset{\rightarrow}{A} - \frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{H}\right) + \left(\delta_C \underset{\rightarrow}{A}\right) \left(\underset{\rightarrow}{\partial} \underset{\Rightarrow}{B} - \frac{\partial}{\partial \underset{\rightarrow}{A}} \underline{H}\right) + \underset{\rightarrow}{\partial} \left(\underset{\Rightarrow}{B} \delta_C \underset{\rightarrow}{A}\right) \right\rangle \\
&= \int \left\langle \left(\frac{\partial}{\partial \underset{\rightarrow}{A}} \underline{G}\right) \left(\underset{\rightarrow}{\partial} \underset{\rightarrow}{A} - \frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{H}\right) - \left(\frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{G}\right) \left(\underset{\rightarrow}{\partial} \underset{\Rightarrow}{B} - \frac{\partial}{\partial \underset{\rightarrow}{A}} \underline{H}\right) - \underset{\rightarrow}{\partial} \left(\underset{\Rightarrow}{B} \frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{G}\right) \right\rangle \\
&= \int \left\langle \underset{\rightarrow}{\partial} \underline{G} - \left(\underset{\rightarrow}{\partial} C\right) \left(\frac{\partial}{\partial C} \underline{G}\right) - \left\{\underline{H}, \underline{G}\right\} - \underset{\rightarrow}{\partial} \left(\underset{\Rightarrow}{B} \frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{G}\right) \right\rangle
\end{aligned}
$$

The transformation is a symmetry of the action iff the variation of the action results in a boundary term,

$$\delta_C \underline{L} = \underset{\rightarrow}{\partial} \underline{b}$$

which happens iff $\underline{G}$ satisfies

$$\left\langle \left(\underset{\rightarrow}{\partial} C\right) \left(\frac{\partial}{\partial C} \underline{G}\right) + \left\{\underline{H}, \underline{G}\right\} \right\rangle = \underset{\rightarrow}{\partial} \underline{g} \tag{20}$$

for some $\underline{g}$, in which case

$$\underline{b} = \underline{G} - \underline{g} - \left\langle \underset{\Rightarrow}{B} \frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{G} \right\rangle \tag{21}$$

Furthermore, when the equations of motion are satisfied (on shell), there is a C dependent conserved current related to the symmetry, $\underline{J} = \underline{G} - \underline{g}$, which by (19) and (20) satisfies

$$\underset{\rightarrow}{\partial} \underline{J} = \underset{\rightarrow}{\partial} \left(\underline{G} - \underline{g}\right) = \left\langle \left(\underset{\rightarrow}{\partial} C\right) \left(\frac{\partial}{\partial C} \underline{G}\right) + \left\{\underline{H}, \underline{G}\right\} \right\rangle - \underset{\rightarrow}{\partial} \underline{g} = 0$$

for all choices of C.

The generator corresponding to the gauge transformation for gravity (13) is

$$\underline{G} = \left\langle \underset{\Rightarrow}{B} \underset{\rightarrow}{\nabla} C \right\rangle = \left\langle \underset{\Rightarrow}{B} \left(\underset{\rightarrow}{\partial} C + \frac{1}{2} \underset{\rightarrow}{\Omega} C - \frac{1}{2} C \underset{\rightarrow}{\Omega}\right) \right\rangle$$

with Clifford vector and bivector valued gauge parameter field, C. This generator produces transformations by

$$
\begin{aligned}
\delta_C \underset{\rightarrow}{\Omega} &= -\frac{\partial}{\partial \underset{\Rightarrow}{B}} \underline{G} = -\underset{\rightarrow}{\nabla} C \\
\delta_C \underset{\Rightarrow}{B} &= \frac{\partial}{\partial \underset{\rightarrow}{\Omega}} \underline{G} = \frac{1}{2} C \underset{\Rightarrow}{B} - \frac{1}{2} \underset{\Rightarrow}{B} C = C \times \underset{\Rightarrow}{B}
\end{aligned}
$$

familiar as the Clifford adjoint gauge transformation. It also gives, through (21), $\underline{g} = -\underline{b}$ and satisfies (20) on shell. The generator corresponding to Lorentz transformations, parameterized by a bivector, $C = C^2 = \Sigma$, is

$$\underline{G}^{\Sigma} = \left\langle \underset{\Rightarrow}{B} \underset{\rightarrow}{\nabla} \Sigma \right\rangle$$

which generates a symmetry transformation satisfying (20) off shell and giving $\underline{g}^{\Sigma} = -\underline{b}^{\Sigma} = 0$. The related conserved current is

$$\underline{J}^{\Sigma} = \underline{G}^{\Sigma} = \left\langle \underset{\Rightarrow}{B} \underset{\rightarrow}{\nabla} \Sigma \right\rangle = \left\langle \underset{\Rightarrow}{B} \left(\underset{\rightarrow}{\partial} \Sigma + \underset{\rightarrow}{\Omega} \times \Sigma\right) \right\rangle = \left\langle \Sigma \left(\underset{\rightarrow}{\partial} \underset{\Rightarrow}{B} + \underset{\rightarrow}{\Omega} \times \underset{\Rightarrow}{B}\right) \right\rangle + \underset{\rightarrow}{\partial} \left\langle \underset{\Rightarrow}{B} \Sigma \right\rangle$$

for any Σ. When considered off shell, $\underset{\rightarrow}{\partial} \underset{-}{J}^\Sigma = 0$ implies the constraint equation associated to the Lorentz symmetry,

$$0 = \left\langle \underset{\rightarrow}{\nabla} \underset{\rightrightarrows}{B} \right\rangle_2 = \left(\underset{\rightarrow}{\partial} \underset{\rightrightarrows}{B}^2 + \underset{\rightarrow}{\omega} \times \underset{\rightrightarrows}{B}^2 + \underset{\rightarrow}{e} \times \underset{\rightrightarrows}{B}^1 \right)$$

equivalent to part of one of the equations of motion (18).

For a diffeomorphism, the gauge parameter is constrained to solve (16), so the generator, parameterized by $\vec{\xi}$, is

$$\underset{-}{G}^\xi = \left\langle \underset{\rightrightarrows}{B} \underset{\rightarrow}{\nabla} C^\xi \right\rangle = - \left\langle \underset{\rightrightarrows}{B} \pounds_{\vec{\xi}} \underset{\rightarrow}{\Omega} \right\rangle$$

giving the canonical transformations

$$\delta_\xi \underset{\rightarrow}{\Omega} = - \frac{\partial}{\partial \underset{\rightrightarrows}{B}} \underset{-}{G}^\xi = - \underset{\rightarrow}{\nabla} C^\xi = \pounds_{\vec{\xi}} \underset{\rightarrow}{\Omega}$$

$$\delta_\xi \underset{\rightrightarrows}{B} = \frac{\partial}{\partial \underset{\rightarrow}{\Omega}} \underset{-}{G}^\xi = C^\xi \times \underset{\rightrightarrows}{B} = \pounds_{\vec{\xi}} \underset{\rightrightarrows}{B}$$

and, from (15),

$$\underset{-}{g}^\xi = - \underset{-}{b}^\xi = - \vec{\xi} \underset{-}{L} = - \vec{\xi} \left\langle \underset{\rightrightarrows}{B} \underset{\rightarrow}{\partial} \underset{\rightarrow}{\Omega} - \underset{-}{H} \right\rangle$$

and the conserved current associated to diffeomorphisms,

$$\underset{-}{J}^\xi = \underset{-}{G}^\xi - \underset{-}{g}^\xi = - \left\langle \underset{\rightrightarrows}{B} \pounds_{\vec{\xi}} \underset{\rightarrow}{\Omega} \right\rangle + \vec{\xi} \left\langle \underset{\rightrightarrows}{B} \underset{\rightarrow}{\partial} \underset{\rightarrow}{\Omega} - \underset{-}{H} \right\rangle = - \vec{\xi} \underset{-}{H} + \left\langle \left(\underset{\rightarrow}{\partial} \underset{\rightrightarrows}{B} \right) \left(\vec{\xi} \underset{\rightarrow}{\Omega} \right) + \left(\vec{\xi} \underset{\rightrightarrows}{B} \right) \left(\underset{\rightarrow}{\partial} \underset{\rightarrow}{\Omega} \right) \right\rangle - \underset{\rightarrow}{\partial} \left\langle \underset{\rightrightarrows}{B} \left(\vec{\xi} \underset{\rightarrow}{\Omega} \right) \right\rangle$$

4 Cl representations

A Clifford algebra of dimension n has a faithful representation in the complex matrices, $GL(2^{[n/2]}, \mathbb{C})$, with the Clifford product isomorphic to matrix multiplication. This corresponds to the traditional use of Pauli and Dirac matrices to represent the basis vectors, γ_μ. Clifford algebra elements may also be represented as matrices of reals, complex numbers, or quaternions – depending on signature. The Clifford algebra $Cl_{0,2} = \mathbb{H}$, equivalent to the algebra of quaternions, is generated by the 2×2 complex, anti-Hermitian, Clifford grade 1 basis vectors with off-diagonal elements,

$$K = q_1 = i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad J = q_2 = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Their products generate the grade 0 Hermitian scalar and the grade 2 anti-Hermitian pseudo-scalar,

$$1 = q_0 = i\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = q_3 = i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

completing the description of the four elements of $Cl_{0,2}$ and $\mathbb{H}$ in terms of matrices. The Clifford basis elements, represented by these Pauli matrices, satisfy the commutation relations,

$$q_\pi \times q_\rho = i\sigma_\pi \times i\sigma_\rho = -\epsilon_{\pi\rho\sigma} i\sigma_\sigma = \epsilon_{\pi\rho}{}^{\sigma} q_\sigma$$

with these Greek indices running from 1 to 3. An arbitrary quaternion is encoded by four real or two complex numbers,

$$h = h^\mu q_\mu = \begin{bmatrix} h^0 + ih^3 & h^2 + ih^1 \\ -h^2 + ih^1 & h^0 - ih^3 \end{bmatrix} = \begin{bmatrix} h_\uparrow & -h_\downarrow^* \\ h_\downarrow & h_\uparrow^* \end{bmatrix} \qquad (22)$$

with a star denoting complex conjugation.

A Clifford algebra relevant to spacetime, $Cl_{1,3}$, may be generated by the four 4×4 complex Hermitian or anti-Hermitian, Clifford grade 1 basis vectors with off-diagonal blocks,

$$\gamma_0 = \begin{bmatrix} 0 & i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q_0 \\ q_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma_\pi = \begin{bmatrix} 0 & \sigma_\pi \\ -\sigma_\pi & 0 \end{bmatrix} = \begin{bmatrix} 0 & -iq_\pi \\ iq_\pi & 0 \end{bmatrix} \tag{23}$$

the "chiral" Weyl representation. These give the six bivector basis elements,

$$\gamma_{0\pi} = \begin{bmatrix} -\sigma_\pi & 0 \\ 0 & \sigma_\pi \end{bmatrix} = \begin{bmatrix} iq_\pi & 0 \\ 0 & -iq_\pi \end{bmatrix} \qquad \gamma_{\pi\rho} = \begin{bmatrix} -q_\pi q_\rho & 0 \\ 0 & -q_\pi q_\rho \end{bmatrix} = -\epsilon_{\pi\rho}{}^{\sigma} \begin{bmatrix} q_\sigma & 0 \\ 0 & q_\sigma \end{bmatrix}$$

and pseudo-scalar,

$$\gamma = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} -iq_0 & 0 \\ 0 & iq_0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -\gamma^{-1} \tag{24}$$

The catalog of sixteen basis elements for $Cl_{1,3}$ is completed by **1** and the basis trivectors,

$$-\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma^{-1} = \begin{bmatrix} 0 & -iq_0 \\ iq_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \gamma_\pi\gamma^{-1} = \begin{bmatrix} 0 & -q_\pi \\ -q_\pi & 0 \end{bmatrix}$$

As shown above, this Clifford algebra is also faithfully represented by 2×2 matrices of quaternions. All the odd graded basis elements are non-zero only in off-diagonal blocks, and all even graded elements are non-zero only in diagonal blocks. This is the sense in which a representation is "chiral" – a useful property. An arbitrary $Cl_{1,3}$ element can be written as

$$\begin{aligned} C &= C_s + C_v^0\gamma_0 + C_v^\pi\gamma_\pi + C_b^{0\pi}\gamma_{0\pi} + \frac{1}{2}C_b^{\pi\rho}\gamma_{\pi\rho} + C_t^0\gamma_0\gamma^{-1} + C_t^\pi\gamma_\pi\gamma^{-1} + C_p\gamma \\ &= \begin{bmatrix} (C_s - iC_p)\,q_0 + \left(iC_b^{0\sigma} - \frac{1}{2}C_b^{\pi\rho}\epsilon_{\pi\rho}{}^{\sigma}\right) q_\sigma & (C_v^0 - iC_t^0)\,q_0 + (-iC_v^\pi - C_t^\pi)\,q_\pi \\ (C_v^0 + iC_t^0)\,q_0 + (iC_v^\pi - C_t^\pi)\,q_\pi & (C_s + iC_p)\,q_0 + \left(-iC_b^{0\sigma} - \frac{1}{2}C_b^{\pi\rho}\epsilon_{\pi\rho}{}^{\sigma}\right) q_\sigma \end{bmatrix} \\ &= \begin{bmatrix} C_e^0 q_0 + C_e^\sigma q_\sigma & C_o^0 q_0 + C_o^\sigma q_\sigma \\ C_o^{0*} q_0 + C_o^{\sigma*} q_\sigma & C_e^{0*} q_0 + C_e^{\sigma*} q_\sigma \end{bmatrix} = \begin{bmatrix} C_{Te} + C_{Se} & C_{To} + C_{So} \\ C_{To}^\dagger - C_{So}^\dagger & C_{Te}^\dagger - C_{Se}^\dagger \end{bmatrix} \\ &= \begin{bmatrix} C_e^\mu q_\mu & C_o^\mu q_\mu \\ C_o^{\mu*} q_\mu & C_e^{\mu*} q_\mu \end{bmatrix} = \begin{bmatrix} C_e^L & C_o^R \\ C_o^L & C_e^R \end{bmatrix} \end{aligned} \tag{25}$$

a 2×2 matrix of quaternions with complex coefficients. The e/o labels stand for even and odd Clifford grade components, the T/S labels for time and space components, and the L/R labels for left and right chiralities. Each matrix element consists of complex coefficients multiplying quaternions. If charge conjugation is defined using the very useful rule,

$$-q_2 q_\mu^* q_2 = q_\mu \tag{26}$$

such that it acts as complex conjugation on the coefficients, but not within the matrices representing the quaternions, the relationship between left and right components of any Clifford element may be written as

$$\begin{aligned} \overline{C_e^L} &= \overline{(C_e^\mu q_\mu)} = -q_2\left(C_e^L\right)^* q_2 = C_e^{\mu*} q_\mu = C_e^R \\ \overline{C_o^R} &= \overline{(C_o^\mu q_\mu)} = -q_2\left(C_o^R\right)^* q_2 = C_o^{\mu*} q_\mu = C_o^L \end{aligned} \tag{27}$$

The 16 real variables in an element, C, of $Cl_{1,3}$ are thus encoded by the 4 complex variables each in C_e^L and C_o^R. Also, since $C_e^R = \overline{C_e^L}$, the scalar part of a 4×4 Clifford element is equal to the scalar part of the associated complex quaternionic (2×2) even representative.

$$\langle C\rangle = C_s = \left\langle C_e^L \right\rangle \tag{28}$$

since the scalar part operator also returns just the real part.

If we were to work in $Cl_{4,0}$, obtained by changing the vector basis representatives, $\gamma_\pi \to i\gamma_\pi$, the resulting 2×2 chiral matrix would consist of quaternions with real coefficients. There is a representation for $Cl_{1,3}$ of quaternions with real coefficients, but it's not chiral – the even and odd graded components mix between diagonal and off-diagonal blocks. Nevertheless, some Clifford algebras decompose into a quaternionic "S-sub-algebra" times a Clifford algebra of lower grade, such as the case with $Cl_{1,3} = \mathbb{H} \otimes Cl_{2,0}$. The basis vectors for this $Cl_{2,0}$ corresponding to the Weyl representation is found from

$$\gamma_0 = q_0 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma_\pi = q_\pi \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Although the choice of representation is a useful device for discerning structure, all physical Lagrangians are invariant under the global version of the Clifford adjoint (2), and therefore under a change of representation for the basis matrices.

Clifford algebra projectors may be built by combining elements having a diagonal representation, such as the chiral projector, $P^{L/R} = \frac{1}{2}(1 \pm i\gamma)$, that projects out collections of "left-acting" elements,

$$CP^L = \begin{bmatrix} C_e^L & C_o^R \\ C_o^L & C_e^R \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_e^L & 0 \\ C_o^L & 0 \end{bmatrix}$$

The explicit appearance of "i" in constructing some projectors implies the use of the corresponding complex Clifford algebra – Clifford algebras with complex coefficients. Projectors may also be used to create mixed basis elements, useful for breaking elements up into two parts such as

$$\begin{aligned}
C_b &= \begin{bmatrix} C_b^\sigma q_\sigma & \\ & C_b^{\sigma *} q_\sigma \end{bmatrix} = \left(C_b^{0\pi} \gamma_{0\pi} + \frac{1}{2} C_b^{\pi\rho} \gamma_{\pi\rho} \right) \left(P^L + P^R \right) \\
&= \left(i C_b^{0\sigma} - \frac{1}{2} C_b^{\pi\rho} \epsilon_{\pi\rho}{}^\sigma \right) \begin{bmatrix} q_\sigma & \\ & 0 \end{bmatrix} + \left(-i C_b^{0\sigma} - \frac{1}{2} C_b^{\pi\rho} \epsilon_{\pi\rho}{}^\sigma \right) \begin{bmatrix} 0 & \\ & q_\sigma \end{bmatrix} \\
&= C_b^\sigma \frac{1}{2} \left(-i\gamma_{0\sigma} - \epsilon^{\pi\rho}{}_\sigma \gamma_{\pi\rho} \right) + C_b^{\sigma *} \left(i\gamma_{0\sigma} - \epsilon^{\pi\rho}{}_\sigma \gamma_{\pi\rho} \right) \\
&= C_b^\sigma T_\sigma^L + C_b^{\sigma *} T_\sigma^R
\end{aligned}$$

a bivector broken up into its left-chiral (self-dual) and right-chiral (anti-self-dual) parts, with chiral ((anti-)self-dual) basis elements satisfying $T_\sigma^{L/R} P^{L/R} = T_\sigma^{L/R} = \pm T_\sigma^{L/R} i\gamma^{-1}$. Although it is possible to equate $C_b^\sigma T_\sigma^L$ and $C_b^L = C_b^\sigma q_\sigma$ in most computations, care should be taken since q_σ is a 2×2 matrix and T_σ^L a 4×4 matrix with non-zero elements equal to q_σ.

4.1 Chiral Gravity

The Clifford connection for $Cl_{1,3}$ in the Weyl rep is

$$\underset{\rightarrow}{\Omega} = \underset{\rightarrow}{e} + \underset{\rightarrow}{\omega} = \begin{bmatrix} \underset{\rightarrow}{\omega}^L & \underset{\rightarrow}{e}^R \\ \underset{\rightarrow}{e}^L & \underset{\rightarrow}{\omega}^R \end{bmatrix}$$

with quaternionic elements times complex coefficients, equivalent to 2×2 complex matrices, equal to

$$\begin{aligned}
\underset{\rightarrow}{e}^R &= \underset{\rightarrow}{e}^0 q_0 - i \underset{\rightarrow}{e}^\pi q_\pi \\
\underset{\rightarrow}{\omega}^L &= \left(i \underset{\rightarrow}{\omega}^{0\sigma} - \frac{1}{2} \underset{\rightarrow}{\omega}^{\pi\rho} \epsilon_{\pi\rho}{}^\sigma \right) q_\sigma
\end{aligned}$$

and their chiral partners given by (27). The curvature is

$$\underset{\rightrightarrows}{F} = \underset{\rightarrow}{\partial}\underset{\rightarrow}{\Omega} + \frac{1}{2}\underset{\rightarrow}{\Omega}\underset{\rightarrow}{\Omega} = \underset{\rightrightarrows}{T} + \underset{\rightrightarrows}{R} + \underset{\rightrightarrows}{E} = \left[\begin{array}{cc} \underset{\rightrightarrows}{R}^L + \underset{\rightrightarrows}{E}^L & \underset{\rightrightarrows}{T}^R \\ \underset{\rightrightarrows}{T}^L & \underset{\rightrightarrows}{R}^R + \underset{\rightrightarrows}{E}^R \end{array}\right]$$

with

$$\begin{aligned}
\underset{\rightrightarrows}{T}^R &= \underset{\rightarrow}{\partial}\underset{\rightarrow}{e}^R + \frac{1}{2}\underset{\rightarrow}{\omega}^L\underset{\rightarrow}{e}^R + \frac{1}{2}\underset{\rightarrow}{e}^R\underset{\rightarrow}{\omega}^R \\
\underset{\rightrightarrows}{E}^L &= \frac{1}{2}\underset{\rightarrow}{e}^R\underset{\rightarrow}{e}^L = \frac{1}{2}\left(\underset{\rightarrow}{e}^0 q_0 - i\underset{\rightarrow}{e}^\pi q_\pi\right)\left(\underset{\rightarrow}{e}^0 q_0 + i\underset{\rightarrow}{e}^\rho q_\rho\right) = \left(i\underset{\rightarrow}{e}^0\underset{\rightarrow}{e}^\sigma + \underset{\rightarrow}{e}^\pi\underset{\rightarrow}{e}^\rho \epsilon_{\pi\rho}{}^\sigma\right) q_\sigma \\
\underset{\rightrightarrows}{R}^L &= \underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega}^L + \frac{1}{2}\underset{\rightarrow}{\omega}^L\underset{\rightarrow}{\omega}^L
\end{aligned}$$

The gravitational Lagrangian (9), using (28) and (24), is

$$\begin{aligned}
\underset{-}{L} &= \frac{1}{2}\left\langle \underset{\rightrightarrows}{F}\underset{\rightrightarrows}{F}\gamma^{-1}\right\rangle = \frac{1}{2}\left\langle i\left(\underset{\rightrightarrows}{R}^L + \underset{\rightrightarrows}{E}^L\right)\left(\underset{\rightrightarrows}{R}^L + \underset{\rightrightarrows}{E}^L\right) + i\underset{\rightrightarrows}{T}^R\overline{\underset{\rightrightarrows}{T}^R}\right\rangle \\
&= \left\langle i\underset{\rightrightarrows}{E}^L\underset{\rightrightarrows}{R}^L + \frac{i}{2}\underset{\rightrightarrows}{E}^L\underset{\rightrightarrows}{E}^L\right\rangle + \underset{\rightarrow}{\partial}\underset{-}{b} \\
&= \left\langle i\underset{\rightrightarrows}{E}^L\underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega}^L - \left(-\frac{i}{2}\underset{\rightrightarrows}{E}^L\underset{\rightarrow}{\omega}^L\underset{\rightarrow}{\omega}^L - \frac{i}{2}\underset{\rightrightarrows}{E}^L\underset{\rightrightarrows}{E}^L\right)\right\rangle + \underset{\rightarrow}{\partial}\underset{-}{b}
\end{aligned}$$

the chiral (also known as “self dual”) Lagrangian for gravity plus Chern-Simons boundary term. The new dynamical variables, $i\underset{\rightrightarrows}{E}^L$ and $\underset{\rightarrow}{\omega}^L$, are spatial quaternions with complex coefficients encoding the same information as $\underset{\rightarrow}{e}$ and $\underset{\rightarrow}{\omega}$. The Ashtekar Hamiltonian for gravity with cosmological term, appearing above, is

$$\underset{-}{H} = \left\langle -\frac{1}{2}i\underset{\rightrightarrows}{E}^L\underset{\rightarrow}{\omega}^L\underset{\rightarrow}{\omega}^L + \frac{i}{2}i\underset{\rightrightarrows}{E}^L i\underset{\rightrightarrows}{E}^L\right\rangle$$

with resulting canonical equations of motion,

$$\begin{aligned}
\underset{\rightarrow}{\partial}\underset{\rightarrow}{\omega}^L &= \frac{\partial}{\partial i\underset{\rightrightarrows}{E}^L}\underset{-}{H} = -\frac{1}{2}\underset{\rightarrow}{\omega}^L\underset{\rightarrow}{\omega}^L - \underset{\rightrightarrows}{E}^L \\
\underset{\rightarrow}{\partial} i\underset{\rightrightarrows}{E}^L &= \frac{\partial}{\partial\underset{\rightarrow}{\omega}^L}\underset{-}{H} = -\underset{\rightarrow}{\omega}^L \times i\underset{\rightrightarrows}{E}^L
\end{aligned}$$

4.2 Spinors

A spinor in four dimensions is conventionally defined as a column of four complex Grassmann valued (anti-commuting) numbers, which in the chiral representation breaks into two Weyl column spinors,

$$\psi^{|} = \left[\begin{array}{c} \psi_{L\uparrow} \\ \psi_{L\downarrow} \\ \psi_{R\uparrow} \\ \psi_{R\downarrow} \end{array}\right] = \left[\begin{array}{c} \psi_L \\ \psi_R \end{array}\right]$$

The real valued Dirac Lagrangian in curved spacetime naturally splits and mixes chiral components,

$$
\begin{aligned}
\underset{-}{L}_{\text{Dirac}} &= \underset{-}{e}\left\langle \left(\psi^{|}\right)^\dagger \gamma_0 \vec{e}\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega} - i\underset{\to}{A} + \frac{im}{4}\underset{\to}{e}\right)\psi^{|}\right\rangle \\
&= \underset{-}{e}\left\langle \left[\begin{array}{cc} (\psi_L)^\dagger & (\psi_R)^\dagger \end{array}\right]\left[\begin{array}{cc} & 1 \\ 1 & \end{array}\right]\left[\begin{array}{cc} & \vec{e}_R \\ \vec{e}_L & \end{array}\right]\left[\begin{array}{cc} \underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^L - i\underset{\to}{A} & \frac{im}{4}\underset{\to}{e}^R \\ \frac{im}{4}\underset{\to}{e}^L & \underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^R - i\underset{\to}{A} \end{array}\right]\left[\begin{array}{c} \psi_L \\ \psi_R \end{array}\right]\right\rangle \\
&= \underset{-}{e}\left\langle (\psi_L)^\dagger\left[\vec{e}_L\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^L - i\underset{\to}{A}\right)\psi_L + im\psi_R\right] + (\psi_R)^\dagger\left[\vec{e}_R\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^R - i\underset{\to}{A}\right)\psi_R + im\psi_L\right]\right\rangle
\end{aligned}
\tag{29}
$$

Note the novel appearance of the mass term in the Dirac operator, (29), consistent with $\underset{\to}{\Omega} = \underset{\to}{\omega} + \frac{im}{2}\underset{\to}{e}$, as well as a $u(1)$ gauge field with generator $-i$. The Dirac Lagrangian is invariant under charge conjugation, $\psi^{|} \to \psi^{|c} = i\gamma_2(\psi^{|})^*$, with

$$
\begin{aligned}
\psi_L^c &= q_2\psi_R^* = \overline{\psi_R} \\
\psi_R^c &= -q_2\psi_L^* = \overline{\psi_L} \\
(-i\underset{\to}{A})^c &= (-i\underset{\to}{A})^*
\end{aligned}
$$

The anti-particle partner to any two-component fermion is labeled by an overline, similar to the labeling for quaternions. This invariance can be confirmed using (26) to get

$$
\begin{aligned}
-q_2\vec{e}_L^{\,*}q_2 &= \overline{\vec{e}_L} = \vec{e}_R \\
-q_2\underset{\to}{\omega}^{L*}q_2 &= \overline{\underset{\to}{\omega}^L} = \underset{\to}{\omega}^R
\end{aligned}
$$

and then

$$
\begin{aligned}
\underset{-}{L}'_{\text{Dirac}} &= \underset{-}{e}\left\langle (\psi_L^c)^\dagger\left[\vec{e}_L\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^L + (-i\underset{\to}{A})^c\right)\psi_L^c + im\psi_R^c\right] + (\psi_R^c)^\dagger\left[\vec{e}_R\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^R + (-i\underset{\to}{A})^c\right)\psi_R^c + im\psi_L^c\right]\right\rangle \\
&= \underset{-}{e}\left\langle -(\psi_R^*)^\dagger q_2\left[\vec{e}_L\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^L + i\underset{\to}{A}^*\right)q_2\psi_R^* - imq_2\psi_L^*\right] + (\psi_L^*)^\dagger q_2\left[-\vec{e}_R\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^R + i\underset{\to}{A}^*\right)q_2\psi_L^* + imq_2\psi_R^*\right]\right\rangle \\
&= \underset{-}{e}\left\langle -(\psi_R)^\dagger q_2\left[\vec{e}_L^{\,*}\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^{L*} - i\underset{\to}{A}\right)q_2\psi_R + imq_2\psi_L\right] + (\psi_L)^\dagger q_2\left[-\vec{e}_R^{\,*}\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^{R*} - i\underset{\to}{A}\right)q_2\psi_L - imq_2\psi_R\right]\right\rangle \\
&= \underset{-}{e}\left\langle (\psi_R)^\dagger\left[\vec{e}_R\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^L - i\underset{\to}{A}\right)\psi_R + im\psi_L\right] + (\psi_L)^\dagger\left[\vec{e}_L\left(\underset{\to}{\partial} + \frac{1}{2}\underset{\to}{\omega}^R - i\underset{\to}{A}\right)\psi_L + im\psi_R\right]\right\rangle = \underset{-}{L}_{\text{Dirac}}
\end{aligned}
$$

Note in the above that the anti-partner to a left chiral fermion is a right chiral fermion, and vice versa.

The Dirac Lagrangian is also invariant under chirality conjugation, $L \leftrightarrow R$. However, the weak force breaks this invariance and it is useful to construct rectangular blocks of spinors to describe its interaction. Conventionally, these blocks are single columns of two component spinors (Clifford ideals); however, it is also natural to represent multiple fermions using more than one column in a single Clifford field. For example, a coupling between a left chiral gauge field, $\underset{\to}{W}^L = -i\underset{\to}{W}^\pi\Sigma_\pi$, and a two element high block of two component spinors is conventionally written as

$$
\underset{-}{L}_{W\psi} = \underset{-}{e}\left\langle (\psi_L)^\dagger\vec{e}_L\frac{1}{2}\underset{\to}{W}^L\psi_L\right\rangle = \underset{-}{e}\left\langle \left[\begin{array}{cc} \nu_L^{e\dagger} & e_L^\dagger \end{array}\right]\left[\begin{array}{cc} -\frac{i}{2}\vec{e}_L\underset{\to}{W}^3 & -\frac{i}{2}\vec{e}_L\left(\underset{\to}{W}^1 - i\underset{\to}{W}^2\right) \\ -\frac{i}{2}\vec{e}_L\left(\underset{\to}{W}^1 + i\underset{\to}{W}^2\right) & \frac{i}{2}\vec{e}_L\underset{\to}{W}^3 \end{array}\right]\left[\begin{array}{c} \nu_L^e \\ e_L \end{array}\right]\right\rangle \tag{30}
$$

with Σ_π equal to a 4×4 block representation of the Pauli matrices. It is natural in the Clifford algebra context to replace the 4×1 block of fermion components with a square 4×4 block containing new particles as well as appropriate anti-particle partners, such as

$$
\psi_L = \left[\begin{array}{cc} \left[\begin{array}{cc} \nu_L^e & -\overline{e_R} \\ e_L & \overline{\nu_R^e} \end{array}\right] & \left[\begin{array}{cc} u_L & -\overline{d_R} \\ d_L & \overline{u_R} \end{array}\right] \end{array}\right]
$$

This gives the new particles interacting in the same way with the gauge field in (30). Each 2×2 quadrant of this block of fermions may be interpreted as a quaternion with complex coefficients.

5 BRST gauge fixing

The BRST method fixes and accounts for gauge symmetries by introducing new fields, with anti-commuting (Grassmann valued) coefficients, having dynamics and interactions with existing fields that breaks the original local gauge symmetry but includes a new global (super) symmetry involving a "rotation" between old and new fields. This method of gauge fixing is an indispensable tool in the application of path integral methods in the quantum field theory of non-abelian gauge fields, and has a natural extension to describe the existence and dynamics of fermionic spinor fields [5].

A restricted BF Lagrangian,

$$\underline{L} = \left\langle \underrightarrow{\underrightarrow{B}} \left(\underrightarrow{\underrightarrow{F}} - \underrightarrow{\underrightarrow{\Phi}}(A,B) \right) \right\rangle$$

which is invariant off shell, $\delta_{C^o} \underline{L} = 0$, under some subset of the gauge transformation (13), such as the subset formed by odd graded gauge parameter fields, $C = C^o$, is amenable to the BRST method. Note that the gravitational Lagrangian does not satisfy this condition for odd C^o, so it is a more general Lagrangian, involving a higher dimensional Clifford algebra, being considered here. The BRST method proceeds by introducing a "ghost" field, C_g^o, with Grassmann valued (anti-commuting) coefficients but otherwise equivalent to C^o, and a Grassmann valued conjugate 3-form field, $\underline{B}_g^o$, as well as a real valued partner field, $\underline{\lambda}^o$. The new system is equipped with a global BRST transformation, a "super-symmetry rotation" between real and Grassmann valued variables,

$$\begin{aligned} \delta_g \underrightarrow{A} &= -\underrightarrow{\nabla} C_g^o & \qquad \delta_g C_g^o &= -\frac{1}{2} C_g^o \times C_g^o \\ \delta_g \underrightarrow{\underrightarrow{B}} &= C_g^o \times \underrightarrow{\underrightarrow{B}} & \qquad \delta_g \underline{B}_g^o &= -i \underline{\lambda}^o \\ \delta_g \underline{\lambda}^o &= 0 \end{aligned}$$

nilpotent by design, $\delta_g \delta_g = 0$, and leaving the Lagrangian invariant. Dynamics are introduced for the ghosts by adding a "BRST exact" term to get a BRST extended Lagrangian,

$$\underline{L}' = \underline{L} + \delta_g \underline{\Psi}$$

with some BRST potential chosen, such as

$$\underline{\Psi} = \left\langle i \underline{B}_g^o \underrightarrow{A} \right\rangle$$

which gives

$$\delta_g \underline{\Psi} = \left\langle \underline{\lambda}^o \underrightarrow{A} \right\rangle + \left\langle i \underline{B}_g^o \underrightarrow{\nabla} C_g^o \right\rangle$$

The BRST partner variable, $\underline{\lambda}^o$, acts as a Lagrange multiplier constraining the connection to be even, $\underrightarrow{A} = \underrightarrow{A}^e$ – fixing the gauge freedom. The resulting effective Lagrangian is

$$\underline{L}^{\text{eff}} = \left\langle \underrightarrow{\underrightarrow{B}}^e \left(\underrightarrow{\underrightarrow{F}}^e - \underrightarrow{\underrightarrow{\Phi}}(A^e, B^e) \right) \right\rangle + \left\langle i \underline{B}_g^o \underrightarrow{\nabla}^e C_g^o \right\rangle$$

The new fields and this Lagrangian are compatible with a Poisson bracket modified to include the canonical pair of ghost fields and the Hamiltonian

$$\underline{H}^{\text{eff}} = \underline{H}^e - \left\langle i \underline{B}_g^o \left(\underrightarrow{A}^e \times C_g^o \right) \right\rangle$$

as well as a generator for the BRST transformation. The structure of this Lagrangian suggests the construction of a BRST restricted and extended connection ("super connection"), $\underrightarrow{\tilde{A}} = \underrightarrow{A}^e + C_g^o$, having curvature

$$\underrightarrow{\underrightarrow{\tilde{F}}} = \underrightarrow{\partial} \underrightarrow{\tilde{A}} + \frac{1}{2} \underrightarrow{\tilde{A}} \times \underrightarrow{\tilde{A}} = \left(\underrightarrow{\partial} \underrightarrow{A}^e + \frac{1}{2} \underrightarrow{A}^e \times \underrightarrow{A}^e \right) + \left(\underrightarrow{\partial} C_g^o + \underrightarrow{A}^e \times C_g^o \right) + \frac{1}{2} C_g^o \times C_g^o = \underrightarrow{\underrightarrow{F}}^e + \underrightarrow{\nabla}^e C_g^o + \frac{1}{2} C_g^o \times C_g^o$$

and giving the extended BF Lagrangian as

$$\underset{-}{L}^{\text{eff}} = \left\langle \left(\underset{\rightrightarrows}{B}^e + i\underset{-}{B}^o_g\right)\left(\underset{\rightrightarrows}{F}^e - \underset{\rightrightarrows}{\Phi}(A^e, B^e) + \underset{\rightarrow}{\nabla}^e C^o_g\right)\right\rangle = \left\langle \underset{\rightrightarrows}{\tilde{B}}\left(\underset{\rightrightarrows}{\tilde{F}} - \underset{\rightrightarrows}{\tilde{\Phi}}(A^e, B^e, C^o_g)\right)\right\rangle = \left\langle \underset{\rightrightarrows}{\tilde{B}}\underset{\rightarrow}{\partial}\underset{\rightarrow}{\tilde{A}}\right\rangle - \underset{-}{H}^{\text{eff}} \quad (31)$$

With a "chiral" Clifford algebra representation, splitting into even and odd quadrants, the extended connection can be written in blocks as

$$\underset{\rightarrow}{\tilde{A}} = \underset{\rightarrow}{A}^e + C^o_g = \begin{bmatrix} \underset{\rightarrow}{A}^l & C^r_g \\ C^l_g & \underset{\rightarrow}{A}^r \end{bmatrix} \quad (32)$$

with curvature

$$\underset{\rightrightarrows}{\tilde{F}} = \begin{bmatrix} \underset{\rightrightarrows}{F}^l + \frac{1}{2}C^r_g \times C^l_g & \underset{\rightarrow}{\partial}C^r_g + \frac{1}{2}\underset{\rightarrow}{A}^l C^r_g - \frac{1}{2}C^r_g\underset{\rightarrow}{A}^r \\ \underset{\rightarrow}{\partial}C^l_g + \frac{1}{2}\underset{\rightarrow}{A}^r C^l_g - \frac{1}{2}C^l_g\underset{\rightarrow}{A}^l & \underset{\rightrightarrows}{F}^r + \frac{1}{2}C^l_g \times C^r_g \end{bmatrix} \quad (33)$$

We may re-label the variable $C^r_g = \psi$ and interpret this field as a block of fermionic spinor fields, and write the conjugate BRST field as $\underset{-}{B}^l_g = -i\underset{-}{e}\chi\gamma_0\vec{e}_l$, in terms of another block of spinor fields, χ. Presuming that $\underset{\rightarrow}{e}$ is independent of the BRST transformation, the fermionic Lagrangian term is proportional to

$$\underset{-}{L}^{\text{f}} = \left\langle i\underset{-}{B}^o_g\underset{\rightarrow}{\nabla}^e C^o_g\right\rangle \sim \underset{-}{e}\left\langle \chi\gamma_0\vec{e}_l\left(\underset{\rightarrow}{\partial}\psi + \frac{1}{2}\underset{\rightarrow}{A}^l\psi - \frac{1}{2}\psi\underset{\rightarrow}{A}^r\right)\right\rangle$$

with one block of gauge fields operating on the fermion block from the left, the "left-acting" $\underset{\rightarrow}{A}^l$, and one from the right, the "right-acting" $\underset{\rightarrow}{A}^r$. In this manner the BRST method produces blocks of odd graded fermions with left and right acting blocks of gauge fields.

6 The standard model and gravity

The standard model of gauge forces and fermions is constructed by considering Clifford algebra fibers of higher dimension, still over a four dimensional base manifold. One Clifford algebra in particular, $Cl_{1,7} = \mathbb{H} \otimes Cl_{0,6}$, has a quaternionic decomposition as well as an appealing bivector sub-algebra, $spin(6) = su(4)$. Applying the methods and tools established so far leads to a concise description of the standard model compatible with the elegant description put forth by Greg Trayling [6], with a few cosmetic modifications and the natural inclusion of gravity.

6.1 Basis vectors

The $Cl_{1,7}$ algebra has representations as real 16×16 matrices, restricted complex 16×16 matrices, 8×8 matrices of quaternions, or as restricted 8×8 matrices of quaternions with complex coefficients – similar to (23). Inspired by the Weyl representation, the eight Hermitian and anti-Hermitian Clifford basis vectors, Γ_α, are chosen to be

$$\Gamma_0 + \Gamma_\pi + z\Gamma_4 + a\Gamma_5 + b\Gamma_6 + c\Gamma_7 = \begin{bmatrix} &&&& 1-iq_\pi && -c+iz & -a+ib \\ &&&&& 1-iq_\pi & -a-ib & c+iz \\ &&&& -c-iz & -a+ib & 1+iq_\pi & \\ &&&& -a-ib & c-iz && 1+iq_\pi \\ 1+iq_\pi && c-iz & a-ib &&&& \\ & 1+iq_\pi & a+ib & -c-iz &&&& \\ c+iz & a-ib & 1-iq_\pi &&&&& \\ a+ib & -c+iz && 1-iq_\pi &&&& \end{bmatrix}$$

in which the higher Clifford basis vector elements, $\Gamma_{\phi=4,5,6,7}$, are directly related to the 4×4 block Pauli sigma matrices, $\Sigma_{\phi-4}$, similar to the way the lower basis vectors, $\Gamma_{\mu=0,1,2,3}$, are related to the corresponding 2×2

Pauli matrices or quaternions. The resulting pseudo-scalar is

$$\Gamma = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = \begin{bmatrix} -i & \\ & i \end{bmatrix}$$

used to build the fundamental left/right projector, $P^{l/r} = \frac{1}{2}(1 \pm i\Gamma)$. This projector necessarily contains i, and its use implies consideration of the corresponding complex Clifford algebra, $\mathbb{C}l_8$. The Left/Right "chiral" projector, $P^{L/R} = \frac{1}{2}(1 \pm \gamma_5)$, comes from

$$\gamma_5 = \Gamma_4\Gamma_5\Gamma_6\Gamma_7 = i\gamma = -\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

6.2 Trayling's model plus gravity

Following Trayling, a set of Clifford basis elements are chosen that reproduce the familiar anti-Hermitian $su(2)_{L/R}$ and $su(3)$ generators as well as those for gravity. Remarkably, the necessary generators can all be constructed from the 28 bivectors of $Cl_{1,7}$ projected into left and right acting blocks via $P^{l/r}$ – but therefore implying use of $\mathbb{C}l_8$. Almost all of the 28 bivectors are used to construct the left acting generators, while a subgroup of 8 right acting generators are picked out corresponding to the complex conjugates of the eight desired Gell-Mann matrices, as well as a final generator having non-zero elements in both left and right blocks:

$$\begin{array}{rclcrcl}
T_{\mu\phi} &=& \Gamma_{\mu\phi}P^l & \quad & T'_1 &=& \frac{1}{2}(\Gamma_{16} - \Gamma_{25})P^r \\
T_{\mu\nu} &=& \Gamma_{\mu\nu}P^l & & T'_2 &=& \frac{1}{2}(-\Gamma_{15} - \Gamma_{26})P^r \\
T^L_1 &=& \frac{1}{2}(-\Gamma_{45} + \Gamma_{67})P^l & & T'_3 &=& \frac{1}{2}(\Gamma_{12} - \Gamma_{56})P^r \\
T^L_2 &=& \frac{1}{2}(-\Gamma_{46} - \Gamma_{57})P^l & & T'_4 &=& \frac{1}{2}(-\Gamma_{14} + \Gamma_{27})P^r \\
T^L_3 &=& \frac{1}{2}(-\Gamma_{47} + \Gamma_{56})P^l & & T'_5 &=& \frac{1}{2}(\Gamma_{17} + \Gamma_{24})P^r \\
T^R_1 &=& \frac{1}{2}(\Gamma_{45} + \Gamma_{67})P^l & & T'_6 &=& \frac{1}{2}(\Gamma_{45} + \Gamma_{67})P^r \\
T^R_2 &=& \frac{1}{2}(\Gamma_{46} - \Gamma_{57})P^l & & T'_7 &=& \frac{1}{2}(-\Gamma_{46} + \Gamma_{57})P^r \\
T_0 &=& \frac{1}{2}(\Gamma_{47} + \Gamma_{56})P^l + \frac{1}{3}(\Gamma_{12} - \Gamma_{47} + \Gamma_{56})P^r & & T'_8 &=& \frac{1}{2\sqrt{3}}(\Gamma_{12} + 2\Gamma_{47} + \Gamma_{56})P^r
\end{array} \tag{34}$$

The action for the standard model is presumed to allow odd graded connection elements to be supplanted by the use of an odd graded BRST field (32). The resulting BRST extended connection, built of selected projected bivector generators from (34) and an odd graded block of fermions, with one quadrant left undetermined because it's not clear what should go in it, takes the form

$$\begin{aligned}
\underrightarrow{\tilde{A}} &= \phi\underrightarrow{e} + \underrightarrow{\omega} + \underrightarrow{W} + \underrightarrow{B} + \underrightarrow{G} + \psi \qquad (35) \\
&= \phi^\psi \underrightarrow{e}^\mu T_{\mu\psi} + \frac{1}{2}\underrightarrow{\omega}^{\mu\nu}T_{\mu\nu} + \underrightarrow{W}^\pi T^L_\pi + \underrightarrow{B}T_0 + \underrightarrow{G}^A T'_A + \nu^e + e + u + d + \overline{\nu^e} + \overline{e} + \overline{u} + \overline{d} \\
&= \begin{bmatrix}
\underrightarrow{\omega}^L - i\underrightarrow{W}^3 & -i\underrightarrow{W}^1 - \underrightarrow{W}^2 & \phi^0_2\underrightarrow{e}^R & \phi^+_1\underrightarrow{e}^R & \nu^e_L & u^r_L & u^g_L & u^b_L \\
-i\underrightarrow{W}^1 + \underrightarrow{W}^2 & \underrightarrow{\omega}^L + i\underrightarrow{W}^3 & \phi^+_2\underrightarrow{e}^R & \phi^0_1\underrightarrow{e}^R & e_L & d^r_L & d^g_L & d^b_L \\
-\phi^0_1\underrightarrow{e}^L & \phi^+_1\underrightarrow{e}^L & \underrightarrow{\omega}^R - i\underrightarrow{B} & & \nu^e_R & u^r_R & u^g_R & u^b_R \\
\phi^+_2\underrightarrow{e}^L & -\phi^0_2\underrightarrow{e}^L & & \underrightarrow{\omega}^R + i\underrightarrow{B} & e_R & d^r_R & d^g_R & d^b_R \\
 & & & & -i\underrightarrow{B} & & & \\
 & & & & & \frac{i}{3}\underrightarrow{B} - i\underrightarrow{G}^3 - \frac{i}{\sqrt{3}}\underrightarrow{G}^8 & -i\underrightarrow{G}^1 + \underrightarrow{G}^2 & -i\underrightarrow{G}^4 + \underrightarrow{G}^5 \\
 & & & & & -i\underrightarrow{G}^1 - \underrightarrow{G}^2 & \frac{i}{3}\underrightarrow{B} + i\underrightarrow{G}^3 - \frac{i}{\sqrt{3}}\underrightarrow{G}^8 & -iG^6 + \underrightarrow{G}^7 \\
 & & & & & -i\underrightarrow{G}^4 - \underrightarrow{G}^5 & -i\underrightarrow{G}^6 - \underrightarrow{G}^7 & \frac{i}{3}\underrightarrow{B} + \frac{2i}{\sqrt{3}}\underrightarrow{G}^8
\end{bmatrix}
\end{aligned}$$

This is the extended connection for the standard model and gravity. Each 2×2 fermionic block (a complex quaternion) includes an anti-particle that isn't shown, such as

$$e_L \leftrightarrow \left[\begin{array}{cc} e_L & \overline{u_R^r} \end{array} \right]$$

Some right-chiral acting gauge fields, $\underrightarrow{X}^1 T_1^R + \underrightarrow{X}^2 T_2^R$, are suggested for completeness, but left out as they're not part of the standard model. The matrix of neutral and charged Higgs coefficients is

$$\left[\begin{array}{cc} \phi_2^0 & \phi_1^+ \\ \phi_2^+ & \phi_1^0 \end{array} \right] = \left[\begin{array}{cc} (-i\phi^4 + \phi^7) & (\phi^5 - i\phi^6) \\ (\phi^5 + i\phi^6) & (-i\phi^4 - \phi^7) \end{array} \right]$$

which comes from the definition of the Higgs vector field as

$$\phi = -\phi^\psi \Gamma_\psi$$

and the spacetime vierbein as

$$\underrightarrow{e} = \underrightarrow{e}^\mu \Gamma_\mu P^l$$

This definition of the Higgs and vierbein is compatible with $\phi \underrightarrow{e} = \phi^\psi \underrightarrow{e}^\mu \Gamma_{\mu\psi} P^l$, but doesn't use all 16 degrees of freedom corresponding to the $\Gamma_{\mu\psi} P^l$ generators since there are only 4 corresponding to $\underrightarrow{e}^\mu$ and 4 to ϕ^ψ, or 8 if ϕ^ψ is allowed to be complex. Also, since ϕ and $\underrightarrow{e}$ multiply, one or the other should be normalized – letting the vierbein be free, the Higgs vector is restricted to satisfy

$$\phi \cdot \phi = \phi^\psi \phi^\chi \eta_{\psi\chi} = -M^2$$

The curvature of this extended connection (the curvature of the standard model and gravity) is

$$\begin{aligned} \underrightarrow{\underrightarrow{\tilde{F}}} =& \underrightarrow{\partial} \underrightarrow{\tilde{A}} + \frac{1}{2} \underrightarrow{\tilde{A}} \underrightarrow{\tilde{A}} \\ =& \left(\underrightarrow{\partial} \left(\phi \underrightarrow{e} \right) + \left(\underrightarrow{\omega} + \underrightarrow{W} + \underrightarrow{B} \right) \times \left(\phi \underrightarrow{e} \right) \right) \\ &+ \left(\underrightarrow{\partial} \underrightarrow{\omega} + \frac{1}{2} \underrightarrow{\omega} \underrightarrow{\omega} + \frac{1}{2} \left(\phi \underrightarrow{e} \right) \times \left(\phi \underrightarrow{e} \right) \right) \\ &+ \left(\underrightarrow{\partial} \underrightarrow{W} + \frac{1}{2} \underrightarrow{W} \underrightarrow{W} \right) + \left(\underrightarrow{\partial} \underrightarrow{B} \right) + \left(\underrightarrow{\partial} \underrightarrow{G} + \frac{1}{2} \underrightarrow{G} \underrightarrow{G} \right) \\ &+ \left(\underrightarrow{\partial} \psi + \frac{1}{2} \left(\phi \underrightarrow{e} + \underrightarrow{\omega} + \underrightarrow{W} \right) \psi + \underrightarrow{B} \times \psi - \frac{1}{2} \psi \underrightarrow{G} \right) + \frac{1}{2} \psi \times \psi \end{aligned}$$

Most of these terms are familiar, but the Higgs terms require special attention. The cosmological term is

$$\frac{1}{2} \left(\phi \underrightarrow{e} \right) \times \left(\phi \underrightarrow{e} \right) = \frac{1}{2} \phi \underrightarrow{e} \phi \underrightarrow{e} = -\frac{1}{2} \phi \phi \underrightarrow{e} \underrightarrow{e} = M^2 \frac{1}{2} \underrightarrow{e} \underrightarrow{e}$$

with cosmological constant equal to the normalization constant for the Higgs, $\Lambda = M^2$. The first term, the Higgs extended torsion, may be simplified since many of the objects commute, $\underrightarrow{\omega} \times \phi = 0$ and $\left(\underrightarrow{W} + \underrightarrow{B} \right) \times \underrightarrow{e} = 0$. It breaks up into

$$\underrightarrow{\underrightarrow{T}}' = \left(\underrightarrow{\partial} \phi + \left(\underrightarrow{W} + \underrightarrow{B} \right) \times \phi \right) \underrightarrow{e} + \phi \left(\underrightarrow{\partial} \underrightarrow{e} + \underrightarrow{\omega} \times \underrightarrow{e} \right)$$

which includes the gravitational torsion, $\underrightarrow{\underrightarrow{T}} = \underrightarrow{\partial} \underrightarrow{e} + \underrightarrow{\omega} \times \underrightarrow{e}$, and the gauge covariant derivative of the Higgs multiplet. This is rather unusual, as it relates the gravitational torsion to the weak neutral gauge field, defined as $\underrightarrow{Z} = \frac{1}{2}\left(\underrightarrow{W}^3 - \underrightarrow{B}\right)$. In terms of the representative matrices, and after multiplying by the inverse vierbein, this looks like

$$\vec{e}\underset{\rightrightarrows}{T}' \sim \left[\begin{array}{cc} \underrightarrow{\partial} + \frac{1}{4}\left(\vec{e}\underset{\rightrightarrows}{T}\right) - \frac{i}{2}\underrightarrow{W}^3 & -\frac{i}{2}\underrightarrow{W}^1 - \frac{1}{2}\underrightarrow{W}^2 \\ -\frac{i}{2}\underrightarrow{W}^1 + \frac{1}{2}\underrightarrow{W}^2 & \underrightarrow{\partial} + \frac{1}{4}\left(\vec{e}\underset{\rightrightarrows}{T}\right) + \frac{i}{2}\underrightarrow{W}^3 \end{array}\right] \left[\begin{array}{cc} \phi_2^0 & \phi_1^+ \\ \phi_2^+ & \phi_1^0 \end{array}\right] - \left[\begin{array}{cc} \phi_2^0 & \phi_1^+ \\ \phi_2^+ & \phi_1^0 \end{array}\right] \left[\begin{array}{cc} -\frac{i}{2}\underrightarrow{B} & 0 \\ 0 & \frac{i}{2}\underrightarrow{B} \end{array}\right]$$

Since ϕ is normalized to M, it is reasonable to expand around a constant value for the Higgs. If the Higgs coefficients, ϕ^ψ, are allowed to be complex, this value may be presumed to be

$$\left[\begin{array}{cc} \phi_2^0 & \phi_1^+ \\ \phi_2^+ & \phi_1^0 \end{array}\right] = \left[\begin{array}{cc} im_2 & 0 \\ 0 & im_1 \end{array}\right]$$

with $m_1^2 + m_2^2 = M^2$. This then works as per the standard Higgs mechanism to provide masses for the weak $\underrightarrow{W}$ and $\underrightarrow{Z}$ fields as well as Dirac masses for the fermions, and spawns the massless photon field, $\underrightarrow{A} = \frac{1}{2}\left(\underrightarrow{W}^3 + \underrightarrow{B}\right)$.

The various charges can be read directly from the matrix representation of the extended connection (35) by reading the coefficients of $-i\underrightarrow{W}^3$ and $-i\underrightarrow{B}$ to the left of any matrix element and the coefficient of $+i\underrightarrow{B}$ below, since these act on the fermions from the left and right respectively (33). This produces the familiar table of charges:

	ν_L^e	e_L	ν_R^e	e_R	u_L	d_L	u_R	d_R	ϕ^0	ϕ^+	W^1	X^1
W^3	1	-1	0	0	1	-1	0	0	-1	1	2	0
B^l	0	0	1	-1	0	0	1	-1	0	0	0	1
B^r	1	1	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	-1	0	-1
$B = B^l - B^r$	-1	-1	0	-2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	1	1	0	2
$A = \frac{1}{2}(W^3 + B)$	0	-1	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	1	1

The Lagrangian for the standard model plus gravity may be written in restricted BF form (31) with a nonlinear

$$\underset{\rightrightarrows}{\Phi} = -\frac{1}{2}\underset{\rightrightarrows}{B}^{\phi e+\omega}\gamma - \frac{1}{2}{*}\underset{\rightrightarrows}{B}^{W+B+G}$$

which includes the first term for gravity and a vierbein dependent Hodge dual term for the Yang-Mills fields. As an alternative, it may be interesting to consider a more unified Lagrangian with $\underset{\rightrightarrows}{\Phi} = -\frac{1}{2}{*}\underset{\rightrightarrows}{B}$, which would imply the existence of an additional SKY Lagrangian term, $\underline{L} \sim \frac{1}{2}\left\langle \underset{\rightrightarrows}{R}{*}\underset{\rightrightarrows}{R} \right\rangle$.

6.3 Bivector $u(4)$ GUT

Although that pretty much wraps it up, there are several significant deficiencies worth mentioning. There is only one generation of fermions represented, and since each complex quaternionic representative also contains an anti-particle, these fermions are represented redundantly. Also, the subgroup of $su(3)$ bivectors was picked out by hand with the exclusion of other right handed generators. Furthermore, the massive fermions all attain the same bare mass up to a single phase – it doesn't appear natural in this model to introduce separate Yukawa couplings. Lastly, it seems somewhat ad-hoc to use the left/right projector, $P^{l/r}$, in constructing the generators. These problems may all be solved by stepping away from the familiar representations of the standard model groups and considering what would happen if all the generators were unprojected $\mathbb{C}l_8$ bivectors – implying a potentially novel form of grand unification based on the group $u(4)$. If the projectors are not used and the weak $su(2)$ generators are "stretched out," so they are "lopsided" 8×8 instead of 4×4, and a set of bivector $su(3)$

generators are folded in, with a $u(1)$ gauge field added, the resulting upper left and lower right quadrants of the representative matrix could look something like:

$$\begin{bmatrix} \underrightarrow{\omega}^L - 3i\underrightarrow{W}^3 - 3i\underrightarrow{B} & -i\underrightarrow{W}^1 - \underrightarrow{W}^2 & -i\underrightarrow{W}^1 - \underrightarrow{W}^2 + \phi_4 \underrightarrow{e}^R & -i\underrightarrow{W}^1 - \underrightarrow{W}^2 + \phi_2 \underrightarrow{e}^R \\ -i\underrightarrow{W}^1 + \underrightarrow{W}^2 & \underrightarrow{\omega}^L + i\underrightarrow{W}^3 - i\underrightarrow{B} - i\underrightarrow{G}^3 - \frac{i}{\sqrt{3}}\underrightarrow{G}^8 & -i\underrightarrow{G}^1 - \underrightarrow{G}^2 + \phi_3 \underrightarrow{e}^R & -i\underrightarrow{G}^4 - \underrightarrow{G}^5 + \phi_1 \underrightarrow{e}^R \\ -i\underrightarrow{W}^1 + \underrightarrow{W}^2 - \phi_1 \underrightarrow{e}^L & -i\underrightarrow{G}^1 + \underrightarrow{G}^2 + \phi_2 \underrightarrow{e}^L & \underrightarrow{\omega}^R + i\underrightarrow{W}^3 - i\underrightarrow{B} + i\underrightarrow{G}^3 - \frac{i}{\sqrt{3}}\underrightarrow{G}^8 & -i\underrightarrow{G}^6 - \underrightarrow{G}^7 \\ -i\underrightarrow{W}^1 + \underrightarrow{W}^2 + \phi_3 \underrightarrow{e}^L & -i\underrightarrow{G}^4 + \underrightarrow{G}^5 - \phi_4 \underrightarrow{e}^L & -i\underrightarrow{G}^6 + \underrightarrow{G}^7 & \underrightarrow{\omega}^R + i\underrightarrow{W}^3 - i\underrightarrow{B} + \frac{2i}{\sqrt{3}}\underrightarrow{G}^8 \end{bmatrix}$$

This GUT model has likely been ruled out previously because these $su(2)$ and $su(3)$ bivector generator subgroups of $su(4)$ do not form proper subgroups. However, with this representation the offending cross terms, $\underrightarrow{W} \times \underrightarrow{G}$, do not fall in $\underrightarrow{W}$ or $\underrightarrow{G}$ but may stand a chance of somehow being "absorbed" by $\phi \underrightarrow{e}$. This model is very similar to, but slightly less extreme than, Tony Smith's insightful model [7]. The generators (except possibly for the $u(1)$) are all bivectors – in fact exhausting the complete set of 28. With this representation for the standard model gauge groups and gravity, the upper right quadrant of three generations of fermions might look something like:

$$\begin{bmatrix} (\nu_L^e + \nu_L^\mu + \nu_L^\tau) & (u_L^r + u_L^b + u_L^g) & (s_L^r + s_L^b + s_L^g) & (b_L^r + b_L^b + b_L^g) \\ e_L & d_L^r & d_L^b & d_L^g \\ \mu_R & c_R^r & c_R^b & c_R^g \\ \tau_R & t_R^r & t_R^b & t_R^g \end{bmatrix}$$

However, the exact form would come from calculating the eigenvalues (charges) and eigenvectors (fermions) corresponding to the standard model gauge bivectors and labeling them accordingly. The mass eigenstates would have to be independently calculated based on the Higgs vev's and, through the power of wishful thinking, the CKM mixing matrices established. One inevitable component, which will either make or break this proposal, is a gravitational connection acting on spinor blocks from the left and from the right. This idea is currently wild conjecture, but provides a possible approach towards getting particle masses from the structure of $\mathbb{C}l_8$– although the true model describing nature is likely to be a bit more complicated.

7 Discussion

This paper has progressed in small steps to construct a complete picture of gravity and the standard model from the bottom up using basic elements with as few mathematical abstractions as possible. It began and ended with the description of a Clifford algebra as a graded Lie algebra, which became the fiber over a four dimensional base manifold. The connection and curvature of this bundle, along with an appropriately restricted BF action, provided a complete description of General Relativity in terms of Lie algebra valued differential forms, without use of a metric. This "trick" is equivalent to the MacDowell-Mansouri method of getting GR from an $so(5)$ valued connection. Hamiltonian dynamics were discussed, providing a possible connecting point with the canonical approach to quantum gravity. Further tools and mathematical elements were described just before they were needed. The matrix representation of Clifford algebras was developed, as well as how spinor fields fit in with these representations. The relevant BRST method produced spinor fields with gauge operators acting on the left and right. These pieces all came together, forming a complete picture of gravity and the standard model as a single BRST extended connection. If this final picture seems very simple, it has succeeded.

As a coherent picture, this work does have weaknesses. Everything takes place purely on the level of "classical" fields – but with an eye towards their use in a QFT via the methods of quantum gravity, which must be applied in a truly complete model. The BRST approach to deriving fermions from gauge symmetries, although a straightforward application of standard techniques, may be hard to swallow. If this method is unpalatable, it is perfectly acceptable to begin instead with the picture of a fundamental fermionic field as a

Clifford element with gauge fields acting from the left and right in an appropriate action. The model conjectured at the very end, based on the related $u(4)$ GUT, is yet untested and should be treated with great skepticism until further investigated. In a somewhat ironic twist, after arguing in the beginning for the more natural description of the MM bivector $so(5)$ model in terms of mixed grade $Cl_{1,3}$ vectors and bivectors, this conjectured model is composed purely of bivector gauge fields.

Although the model stands on its own as a straightforward $\mathbb{C}l_8$ fiber bundle construction over four dimensional base, there are many other compatible geometric descriptions. One alternative is to interpret $\underset{\rightarrow}{\tilde{A}}$ as the connection for a Cartan geometry with Lie group G – with a Lie subgroup, H, formed by the generators of elements other than $\underset{\rightarrow}{e}$, and the spacetime "base" formed by G/H. Another particularly appealing interpretation is the Kaluza-Klein construction, with four compact dimensions implied by the Higgs vector, $\phi = -\phi^\psi \Gamma_\psi$, and a corresponding translation of the components of $\underset{\rightarrow}{\tilde{A}}$ into parts of a vielbein including this higher dimensional space. The model may also be extended to encompass more traditional unification schemes, such as using a ten dimensional Clifford algebra in a $so(10)$ GUT. All of these geometric ideas should be developed further in the context of the model described here, as they may provide valuable insights.

In conclusion, and in defense of its existence, this work has concentrated on producing a clear and coherent unified picture rather than introducing novel ideas in particular areas. The answer to the question of what here is really "new" is: "as little as possible." Rather, several standard and non-standard pieces have been brought together to form a unified whole describing the conventional standard model and gravity as simply as possible.

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