TASI 2003 LECTURES ON ANOMALIES

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These lecture notes review the structure of anomalies and present some of their applications in field theory, string theory and M theory. They expand on material presented at the TASI 2003 summer school and the 2005 International Spring School on String Theory in Hangzhou, China.

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0 Introduction

The study of anomalies has many applications in field theory and string theory. These range from phenomenological applications such as the calculation of the decay rate for neutral pions into two photons, the computation of quantum numbers in the Skyrme model of hadrons, and mechanisms for baryogenesis in the Standard Model; to more abstract applications such as the study of dualities in gauge theory, the computation of anomalous couplings in the effective theory of D-branes, and the analysis of Black Hole entropy.

Anomalies are often a useful first line of attack in trying to understand new systems. This is because the presence of anomalies, or the way they are canceled, can often be studied without knowing the detailed dynamics of the theory. They are in a way topological properties of the theory and thus can be studied by approximate methods.

The basic definition of an anomaly is the following. Consider a quantum theory which has a symmetry group G which leaves the classical action invariant ($\delta S_{cl} = 0$). We say that G is anomalous if G is violated in the full quantum theory. Thus anomalous symmetries are symmetries of classical theories which do not survive the transition to quantum mechanics. The method of computation and physical implications of anomalies depend on the structure of G. In particular, G can be either discrete or continuous, and it can be either a global symmetry or a gauge (local) symmetry.

If the symmetry G is a global symmetry then anomalies in G do not indicate any inconsistency of the theory, but they often have interesting physical consequences. Historically the most important example of this type is the anomaly in the axial current which is important in understanding the decay rate for $\pi^0 \to \gamma\gamma$. This is the subject of lecture 1.

Classically, invariance under a continuous global symmetry group G implies the existence of conserved currents j^a_μ with a labelling the generators of G: $\partial^\mu j^a_\mu = 0$. If the symmetry is anomalous then there are quantum corrections which make the divergence of j^a_μ non-zero, $\partial^\mu j^a_\mu = \mathcal{A}^a(\phi, \partial\phi)$. Thus the variation of the quantum effective action under a symmetry transformation labelled by v^a is

$$\delta_v S_{\rm eff} = \int v^a \partial^\mu j^a_\mu = \int v^a \mathcal{A}^a \tag{1}$$

and the quantity $\mathcal{A}^a$ is the anomaly. Anomalies can be more precisely phrased as violations of the Ward identities following from G invariance, a point of view which will be developed further in sec 3.1.

On the other hand, if G is a gauge symmetry then anomalies indicate a fundamental inconsistency of the theory and must vanish. Recall that gauge symmetries are not symmetries in the conventional sense of symmetries that act on the configuration space and lead to identical physics. Rather, they are redundancies in our description of the physics when we work in the space of gauge fields rather than its quotient by gauge transformations. Anomalies in a redundancy would not be a good thing. More concretely, when we work in the space of gauge field configurations, we need gauge invariance to remove negative norm states from the spectrum and a lack of gauge invariance due to anomalies would lead to fundamental inconsistencies [9].

We can also distinguish between several possible types of gauge anomaly. Continuous gauge transformations can either be local, here meaning that they can be continuously connected to the identity transformation, or global, meaning that they cannot be so connected. It is possible for a gauge theory to have anomalies in global but not local gauge transformations. A famous example is $SU(2)$ gauge theory in $D = 4$ spacetime dimensions with a single Weyl fermion in the two-dimensional representation of $SU(2)$. Working in Euclidean space, the fact that $g(x) \to 1$ as $|x| \to \infty$ means that we can identify the boundary of R^4 with a single point, so that spacetime can be effectively viewed as S^4. Then gauge transformations are maps from S^4 to $SU(2)$, and such maps are classified by $\pi_4(SU(2)) = Z_2$. Thus there are gauge transformations which are not connected to the identity. It was shown in [10] that the quantum effective action is not invariant under global $SU(2)$ transformations, that is, there is an anomaly in global $SU(2)$ gauge transformations. This inconsistency of the theory came as a relief, since otherwise one would have had to make sense of other rather peculiar properties of this theory such as the odd number of fermion zero modes in an instanton background.

One can also consider theories where gauge symmetries are discrete and ask if these discrete transformations are symmetries of the quantum theory. See for example [11,12,13,14,15,16].

Gravity is also a gauge theory of a sort. When fermions are incorporated into gravitational theories one requires both diffeomorphism symmetry and local Lorentz symmetry. These symmetries can also be anomalous, although they are somewhat more exotic than gauge anomalies in that gravitational anomalies only appear in spacetime dimensions $D = 2 + 4k$ with k integer. A seemingly even more exotic possibility is that a theory could have global gravitational anomalies when the group of diffeomorphisms has components not connected to the identity [17]. A simple and important example of this occurs in string theory when one studies one-loop diagrams in string perturbation theory. The string world-sheet is then a two-torus, T^2, and the group of global diffeomorphisms is $SL(2, \mathbb{Z})$. Invariance under $SL(2, \mathbb{Z})$ is also known as modular invariance and provides an important constraint on the structure of chiral string theories [17,18].

A final class of anomalies of central importance in particle physics are the trace, scale, or conformal anomalies. These occur in theories which are classically scale and/or conformally invariant, but where the invariance is broken by quantum effects. Unlike the other anomalies discussed here, which only occur in certain specific theories, anomalies in scale invariance are generic due to the non-trivial renormalization group flow of interacting quantum field theories. It is only in very special theories like $N = 4$ Super Yang-Mills or special theories with $N = 2$ or $N = 1$ supersymmetry in four dimensions that they are absent.

In these lectures I will first introduce some of the basic ideas in simple systems. I will then discuss some four-dimensional, "real world" applications, present some of the mathematical tools needed to do computations with anomalies in higher-dimensional theories, and then end with some applications of anomalies to branes in string theory and M-theory. Although I will cover a number of topics in anomalies, the central topic will be anomaly inflow and its applications. There a number of good reviews of anomalies which focus on other topics. For a brief overview of anomalies see the review by Adler [1]. The review of Alvarez-Gaume and Ginsparg [2] has a comprehensive discussion of the topological interpretation of anomalies as well as many useful formulae. Scrucca and Serone [3] focus on anomalies in theories with extra dimensions and have an extensive treatment of anomalies in orbifold theories. The Bilal and Metzger review [4] discusses all aspects of anomaly cancellation in M theory, including fivebrane anomalies and anomaly cancellation in heterotic M theory. The Green-Schwarz mechanism of anomaly cancellation in superstring theory is discussed in the textbooks [5] and [6]. In preparing these lectures I found the treatment of anomalies in the textbooks by Peskin and Schroeder [7] and by Weinberg [8] to be particularly useful. This review will not discuss anomalies in discrete symmetries, whether global or gauged. Nor will it discuss scale anomalies or anomalies in supersymmetric theories, except in passing. These are all interesting topics but would take us too far afield.

0.1 Conventions

We will sometimes write gauge fields in components as A^i_μ with μ a spacetime vector index and i a gauge index. In later sections we write gauge fields as 1-forms taking values in the Lie algebra $\mathcal{G}$ of a compact Lie group G. In the earlier "phenomenological" sections we use Hermitian generators of G, while in more mathematical sections we choose anti-Hermitian matrices $(\lambda^i)^a_b$ which span the adjoint representation of $\mathcal{G}$ and write $A = A^i_\mu \lambda^i dx^\mu$. The representation matrices are normalized so that

$$\mathrm{Tr}\lambda^i\lambda^j = \frac{1}{2}\delta^{ij}.$$

The covariant derivative is $D = d + [A, \ \ ]$ and the curvature is $F = dA + A^2$. Since we wish to couple fermions to gravity we will decompose the metric in terms of vielbeins, $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$ with Greek indices used for coordinate frame indices and Latin letters for tangent space indices.

The gamma matrices γ^a obey $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. In $2n$ spacetime dimensions, spinors can be divided into positive and negative chirality components according to whether they have eigenvalue ± 1 with respect to the generalization of γ^5 in four dimensions to $2n$ dimensions:

$$\overline{\gamma} = \eta \prod_{a=1}^{2n} \gamma^a$$

where η is a phase, equal to i^n in Euclidean space and i^{n-1} in Minkowski space. Our metric convention is $\eta^{ab} = \mathrm{diag}(1, -1, \cdots - 1)$.

We will often encounter differential forms which represent characteristic classes and a set of differential forms related to these by the "descent procedure" which will be discussed in these lectures. For these forms we use a notation where subscripts indicate the degree of the form and superscripts in parentheses indicate the order of the form in the parameter of the gauge variation. Thus $\omega_2^{(1)}$ denotes a 2-form which is linear in the parameter of the gauge variation. We will also encounter formal sums of differential forms of different degrees. If α is such a sum, we will indicate the n-form part of α by $\alpha|_n$.

Finally, while I have attempted to get factors of 2 and π correct, no serious attempt has been made in these notes to check signs. For anomaly cancellation in M-theory these have been worked out carefully in the review [4].

0.2 Exercises

Each lecture is followed by a few exercises which should be attempted by students wanting to have a full grasp of the material.

1 Lecture 1: The Chiral anomaly

1.1 $\pi^0 \to \gamma\gamma$

After these generalities let me go back to the beginning. The story of anomalies could be said to originate in a computation of the rate for the decay

$$\pi^0 \to \gamma\gamma. \tag{2}$$

Since the π^0 is electrically neutral, it doesn't couple directly to electromagnetism. There can however be a coupling induced at the one-loop level. The lowest dimension parity invariant operator one can write down which would lead to such a decay process is

$$\mathcal{L}_{\pi\gamma\gamma} = A\pi^0 \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}. \tag{3}$$

A pre-QCD computation (in 1949!) by Steinberger [19] (see also [20]) used the coupling of pions to the nucleon doublet N of the form

$$G_{\pi N} \vec{\pi} \cdot \overline{N} \gamma^5 \vec{\sigma} N \tag{4}$$

to compute the one-loop diagram with virtual nucleons running in the loop and one external pion and two external photons and obtained a result which is equivalent to (3) with

$$A = e^2 G_{\pi N} / 32\pi^2 m_N \tag{5}$$

with m_N the nucleon mass. This leads to a decay rate which agrees to within factors of a few with the experimental value $\Gamma \sim 10^{16} \mathrm{sec}^{-1}$.

However it was later realized by Nambu that the pion should be thought of as a Nambu-Goldstone boson resulting from the spontaneous breaking of chiral symmetry by the QCD vacuum, $SU(2)_L \times SU(2)_R \to SU(2)_V$. This implies that pions only have derivative couplings up to terms suppressed by powers of m_π^2/m_N^2. But in this case, the coefficient A should be suppressed by m_π^2/m_N^2 relative to the "naive" value (5) (the Sutherland-Veltman theorem uses PCAC to show that the matrix element for $\pi^0 \to \gamma\gamma$ vanishes in the soft pion limit [21]). This however is inconsistent with experiment. Thus there must be something "anomalous" going on that invalidates this reasoning. The anomalous behavior was understood in 1969 due to the work of Adler [22] and Bell and Jackiw [23]. What they found is that there is a quantum violation of part of the $SU(2)_L \times SU(2)_R$ symmetry in the presence of electromagnetism which is independent of the quark masses. We will discuss their result (in modern language) in a following section, but first we turn to a simpler system where the anomaly in a chiral symmetry can be computed in a particularly straightforward manner.

1.2 The axial current anomaly in $1+1$ dimensions

For our first example we consider QED in two spacetime dimensions described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\overline{\psi}\gamma^\mu D_\mu \psi \tag{6}$$

where

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{7}$$

is a two-component Dirac spinor and we choose a basis of gamma matrices so that

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \overline{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Here $\overline{\gamma} = \gamma^0\gamma^1$ is the analog of γ^5 in four dimensions, that is $(1 \pm \overline{\gamma})/2$ are the projection operators onto chiral representations of the Lie algebra of the Lorentz group.

Classically this theory is invariant under the vector transformation $\psi \to e^{i\alpha}\psi$ and the chiral transformation $\psi \to e^{i\beta\overline{\gamma}}\psi$, leading to conservation of the vector and axial currents

$$j^V_\mu = \overline{\psi}\gamma_\mu\psi \tag{9a}$$

$$j^A_\mu = \overline{\psi}\gamma_\mu\overline{\gamma}\psi \tag{9b}$$

Note that $\gamma^\mu\overline{\gamma} = -\epsilon^{\mu\nu}\gamma_\nu$ so that the two currents are related by $j^A_\mu = -\epsilon^{\mu\nu} j^V_\nu$.

We can discover an anomaly in the conservation of j^A_μ by studying the one-loop correction to the vector current in the presence of a background gauge field A_μ. In momentum space this is given by the one-loop diagram with one external current insertion and one insertion of the background gauge field:

$$\langle j^{V\mu}(q)\rangle_A = \Pi^{\mu\nu} A_\nu(q) \tag{10}$$

where $\Pi^{\mu\nu}$ is the usual one-loop vacuum polarization diagram. Computing this diagram so as to maintain gauge invariance (e.g. by dimensional regularization) leads to

$$i\Pi^{\mu\nu}(q) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \tag{11}$$

with

$$\Pi(q^2) = \frac{1}{\pi q^2}. \tag{12}$$

The pole in Π at $q^2 = 0$ implies that the photon has acquired a mass $m^2_\gamma = e^2/\pi$ as can be seen by summing the geometric series for the photon propagator. In spite of this, the vector gauge current is still conserved, $q_\mu\langle j^{V\mu}(q)\rangle_A = 0$, but computing the divergence of the axial current gives

$$q_\mu\langle j^{A\mu}(q)\rangle_A = -q_\mu\epsilon^{\mu\nu}\langle j^V_\nu\rangle_A = \frac{1}{\pi}\epsilon^{\mu\nu} q_\mu A_\nu, \tag{13}$$

which in position space implies

$$\partial_\mu j^{A\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{14}$$

So, we have found an anomaly, but it is natural to ask whether the anomaly we have found depends on the choice of regulator. What would have happened if we had used Pauli-Villars, or a momentum space cutoff, or some other method to regularize the theory? Following the discussion in [7] we can reason as follows. Whatever choice of regulator we make, dimensional analysis and Lorentz invariance tells us that the vacuum polarization will take the form

$$i\Pi^{\mu\nu}(q) = i(A(q^2)\eta^{\mu\nu} - B(q^2)\frac{q^\mu q^\nu}{q^2}). \tag{15}$$

Computing the divergence of the vector and axial currents we find

$$q_\mu\langle j^{\mu V}(q)\rangle_A = -q^\mu A_\mu(q)(A(q^2) - B(q^2)) \tag{16}$$

and

$$q_\mu\langle j^{\mu A}(q)\rangle_A = A(q^2)\epsilon^{\mu\nu}q_\mu A_\nu(q). \tag{17}$$

Now, a little thought or calculation shows that $A(q^2)$ is logarithmically divergent, so its value certainly depends on the choice of regulator. $B(q^2)$ on the other hand, is finite, independent of the choice of regulator, and in fact determined entirely by the infrared behavior of the theory since it is the residue of the pole at $q^2 = 0$. Now we could certainly regularize the theory so that $A(q^2) = 0$, but then (19) shows that the vector current would have a non-zero divergence which would violate gauge invariance. Gauge invariance requires that we regularize the theory so that $A(q^2) = B(q^2)$, and since $B(q^2)$ is non-zero, we are then forced into having an anomaly in the divergence of the axial current, no matter what choice or regulator we use. This example shows the intricate interplay between UV divergences, IR behavior, and gauge invariance which is characteristic of anomalies.

1.3 Fujikawa analysis of chiral anomalies

The calculation in the previous section of the chiral anomaly in 1+1 dimensions can be generalized to Dirac fermions in $2n$ spacetime dimensions. There are many equivalent ways to compute the anomaly. These include:

- A direct calculation of the $(n+1)$-gon diagram with one insertion of j^A_μ and n insertions of the background gauge field using a regulator which maintains gauge invariance.

- Point Splitting: The current involves the product of field operators and so is potentially divergent. To regulate this split the fermion fields apart and insert a Wilson line to maintain gauge invariance. Thus one defines the current as

$$j^A_\mu = \lim_{\epsilon \to 0} \overline{\psi}(x + \epsilon/2)\gamma_\mu \overline{\gamma} e^{-ie\int_{x-\epsilon/2}^{x+\epsilon/2} A} \psi(x - \epsilon/2) \tag{18}$$

and computes the divergence as ϵ is taken to zero.

- The Fujikawa method [24]. A careful definition of the measure $D\overline{\psi}D\psi$ in the path integral is given in terms of the spectrum of the Dirac operator and then one finds that the measure is not invariant under chiral transformations.

All these methods lead to the same conclusion, in $2n$ spacetime dimensions, there is a one-loop anomaly in the divergence of the axial current given by

$$\partial_\mu j^{A\mu} = \frac{2(-1)^{n+1}}{n!(4\pi)^n} \epsilon^{\mu_1 \cdots \mu_{2n}} F_{\mu_1\mu_2} \cdots F_{\mu_{2n-1}\mu_{2n}} \tag{19}$$

Here I will follow the last approach pioneered by Fujikawa. His approach is conceptually attractive and computationally powerful. To explain the method I will first consider a simple example in detail and then summarize some generalizations. We start with a charged massless fermion ψ coupled to electromagnetism in $3+1$ dimensions. The partition function is given by

$$Z = \int DA_\mu D\psi D\overline{\psi} e^{iS[A,\psi,\overline{\psi}]} \tag{20}$$

with classical action

$$S = \int d^4x \left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \overline{\psi} i \gamma^\mu D_\mu \psi \right). \tag{21}$$

This theory has a chiral symmetry

$$\psi \to e^{i\alpha\overline{\gamma}}\psi = \psi + i\alpha\overline{\gamma}\psi + \cdots \tag{22}$$

with a corresponding Noether current $j^A_\mu = \overline{\psi}\gamma_\mu\overline{\gamma}\psi$.

The classical conservation law $\partial^\mu j^A_\mu = 0$ is in the quantum theory replaced by Ward identities. The standard path integral derivation of Ward identities goes as follows. We consider the change of variables in the path integral

$$\psi(x) \quad \to \quad \psi'(x) = \psi(x) + \epsilon(x), \tag{23a}$$
$$\overline{\psi}(x) \quad \to \quad \overline{\psi}'(x) = \overline{\psi}(x) + \overline{\epsilon}(x). \tag{23b}$$

According to the standard rules of path integration, this change of variables should leave the path integral unchanged so that

$$\int D\psi D\overline{\psi} e^{i\int d^4x \mathcal{L}[\psi,\overline{\psi}]} = \int D\psi' D\overline{\psi}' e^{i\int d^4x \mathcal{L}[\psi',\overline{\psi}']}. \tag{24}$$

We now apply this with $\epsilon(x) = i\alpha(x)\overline{\gamma}\psi(x)$ and $\overline{\epsilon}(x) = \overline{\psi} i\alpha(x)\overline{\gamma}$. This corresponds to an infinitesimal chiral transformation, but with a spacetime dependent parameter $\alpha(x)$.

For α constant the Lagrangian $\mathcal{L}$ is invariant, so the first order change in $\mathcal{L}$ must be proportional to $\partial_\mu \alpha$ and indeed one easily finds that

$$\int d^4x \left(\overline{\psi}' i\gamma^\mu D_\mu \psi' \right) = \int d^4x \left(\overline{\psi} i\gamma^\mu D_\mu \psi - \partial_\mu \alpha \overline{\psi} \gamma^\mu \overline{\gamma} \psi \right). \tag{25}$$

If we then assume that the measure is invariant, $D\psi' D\overline{\psi}' = D\psi D\overline{\psi}$, (an assumption which we will soon see is invalid) then integrating by parts and varying with respect to α gives the Ward identity

$$\partial_\mu \langle \overline{\psi} \gamma^\mu \overline{\gamma} \psi \rangle = 0. \tag{26}$$

Fujikawa pointed out that the assumption that the measure is invariant is not necessarily valid. This provides a very nice way of understanding why a classical symmetry might fail to be a symmetry of the quantum theory. To go from a classical theory to its quantum counterpart in the path integral formalism we need not only the classical action, but also a measure in the path integral. If the measure is not invariant then the quantum theory will not inherit the classical symmetries of the action. We can check this idea in our specific example by giving a more precise definition of the measure and then checking invariance under chiral transformations.

To do this we will expand ψ in terms of orthonormal eigenstates of $i\gamma^\mu D_\mu$:

$$i\gamma^\mu D_\mu \phi_m = \lambda_m \phi_m, \qquad \overline{\phi}_m i\overleftarrow{\gamma^\mu D_\mu} = \lambda_m \overline{\phi}_m, \tag{27}$$

and expand

$$\psi(x) = \sum_m a_m \phi_m(x), \qquad \overline{\psi}(x) = \sum_m \overline{a}_m \overline{\phi}_m(x), \tag{28}$$

where $a_m, \overline{a}_m$ are Grassmann variables multiplying the c-number eigenfunctions. We then define the measure by

$$D\psi D\overline{\psi} = \prod_m da_m d\overline{a}_m. \tag{29}$$

We now make the change of variables as before and find that this induces a change in the coefficients a_m:

$$a'_m = \int d^4x \phi_m(x)^\dagger \sum_n (1 + i\alpha(x)\overline{\gamma})\phi_n(x)a_n \tag{30}$$

which we will write in short-hand notation as

$$a'_m = \sum_n (\delta_{mn} + C_{mn})a_n \tag{31}$$

with

$$C_{mn} = i\int d^4x \phi_m^\dagger \alpha \overline{\gamma} \phi_n. \tag{32}$$

We can then compute the change in the measure from the Jacobian of this transformation, and taking into account the Grassmann property of the a_m find

$$D\psi' D\overline{\psi}' = (\det(1 + C))^{-2} D\psi D\overline{\psi}. \tag{33}$$

Thus to compute the change in the measure we need to compute $\det(1+C)$. Working to first order in α and hence to first order in C we have

$$\det(1 + C) = e^{\mathrm{Tr}\ln(1+\mathrm{C})} = e^{\mathrm{TrC}+\cdots} \tag{34}$$

so that to this order

$$\det(1 + C)^{-2} = e^{-2i\int d^4x\alpha(x)\sum_n \phi_n^\dagger(x)\overline{\gamma}\phi_n(x)} \tag{35}$$

Formally, the coefficient of $\alpha(x)$ in the exponent of (35) is $\mathrm{Tr}\overline{\gamma}$. This trace includes a trace over the Lorentz indices of $\overline{\gamma}$, which of course gives zero, but also a trace over the infinite number of eigenstates of $i\gamma^\mu D_\mu$, which gives infinity. In other words, (35) is not defined without some regularization scheme. To regulate (35) we will define momentum integrals by continuation to Euclidean space and regularize the sum by cutting off the sum at large eigenvalues via

$$\sum_n \phi_n^\dagger(x)\overline{\gamma}\phi_n(x) \equiv \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\overline{\gamma}\phi_n(x)e^{\lambda_n^2/M^2}. \tag{36}$$

The sign of the last exponential in (36) may look wrong, but we will see in a moment that it is what we need to regulate the sum when we continue to Euclidean space.

Since the ϕ_n are eigenfunctions of $i\gamma^\mu D_\mu$ we can also write (36) as

$$\lim_{M\to\infty}\sum_n \phi_n^\dagger \overline{\gamma} e^{(i\gamma^\mu D_\mu)^2/M^2}\phi_n = \lim_{M\to\infty}\langle x|tr[\overline{\gamma}e^{(i\gamma^\mu D_\mu)^2/M^2}]|x\rangle. \qquad (37)$$

Now $(i\gamma^\mu D_\mu)^2 = -D^2 + (1/2)\sigma^{\mu\nu}F_{\mu\nu}$ with $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$ so we are left with the evaluation of

$$\lim_{M\to\infty}\langle x|tr[\overline{\gamma}e^{(-D^2+(1/2)\sigma^{\mu\nu}F_{\mu\nu})/M^2}]|x\rangle. \qquad (38)$$

We now need to figure out what terms contribute in the limit that M goes to infinity. We can expand in powers of the background electromagnetic field, writing $-D^2 = -\partial^2 + \cdots$. Then the term with no powers of the background field involves the integral (after continuing to Euclidean space)

$$\langle x|e^{-\partial^2/M^2}|x\rangle = i\int \frac{d^4k_E}{(2\pi)^4}e^{-k_E^2/M^2} = \frac{iM^4}{16\pi^2}. \qquad (39)$$

However, the trace of $\overline{\gamma}$ vanishes so there is no contribution which is independent of the background field. Bringing down one power of the background field also vanishes since $\mathrm{Tr}\overline{\gamma}\sigma^{\mu\nu} = 0$. Terms with more than two powers of the background field vanish in the limit $M \to \infty$. Thus we are left with a single term which is finite and non-zero in the limit $M \to \infty$ which results from expanding to second order in the background gauge field:

$$\lim_{M\to\infty}\mathrm{Tr}\left[\overline{\gamma}\frac{1}{2}\left(\frac{1}{2\mathrm{M}^2}\sigma^{\mu\nu}\mathrm{F}_{\mu\nu}\right)^2\right]\langle \mathrm{x}|\mathrm{e}^{-\partial^2/\mathrm{M}^2}|\mathrm{x}\rangle = -\frac{1}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}\mathrm{F}_{\alpha\beta}\mathrm{F}_{\mu\nu}. \qquad (40)$$

We thus have

$$\det(1+C)^{-2} = e^{i\int d^4x\alpha(x)\left(\frac{1}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right)} \qquad (41)$$

and the change of variables we used in the proof of the Ward identity thus leads to the partition function

$$Z[A] = \int D\psi D\overline{\psi}e^{i\int d^4x\left(\overline{\psi}i\gamma^\mu D_\mu\psi + \alpha(x)\left(\partial^\mu j_\mu^A + (1/16\pi^2)\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right)\right)} \qquad (42)$$

which after varying with respect to α gives the Adler-Bell-Jackiw anomaly

$$\partial^\mu j_\mu^A = -\frac{1}{8\pi^2}\tilde{F}^{\mu\nu}F_{\mu\nu} \qquad (43)$$

where we have defined

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \qquad (44)$$

and have removed a factor of i in the final answer to express the answer in Minkowski space.

1.4 Gravitational contribution to the chiral anomaly

Fujikawa's method can also be applied to compute the anomaly in the chiral current when fermions are coupled to gravity [24]. The derivation proceeds along the same line as for the gauge anomaly with some minor technical complications and leads to a gravitational contribution to the divergence of the axial current given by

$$D^{\mu} j^{A}_{\mu} = -\frac{1}{384\pi^2}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\sigma\tau} R_{\alpha\beta}{}^{\sigma\tau}. \tag{45}$$

1.5 Why anomalies are one-loop exact

We have so far seen two calculations of anomalies. One involving a one-loop calculation in two dimensions, and the other the calculation of Fujikawa which involves regularizing the determinant from a change of variables in the fermionic path integral in four dimensions. This is essentially a one-loop calculation as well given the well-known diagrammatic interpration of fermion determinants. There are many qualitative reasons why we would expect that these one-loop calculations give the anomaly exactly, without any corrections from higher orders in perturbation theory. As will see in later sections, there are several topological interpretations of the anomaly, one involving the topology of gauge configuration space and another involving the index of the Dirac operator. This argues against perturbative corrections since topological quantities cannot change continuously.

The detailed diagrammatic proof of the the absence of higher-order corrections to the anomaly was given by Adler and Bardeen [25]. The essential idea in their proof was to note that higher-order corrections necessarily involve internal boson propagators, and then to show that these can always be regularized in such a way that the usual Ward identities are satisfied.

1.6 Why anomalies are an infrared effect

The calculations we have given of the anomaly so far seem to focus on the issue of regularization. The anomaly reflects the fact that the UV regulator does not respect the chiral symmetry of the theory, hence the lack of conservation of the chiral current. This suggests that the anomaly is an ultraviolet effect. However, it later came to be appreciated that anomalies are more accurately understood as a statement about the infrared behavior of the theory [26,27,28]. First of all, only massless particles for which no mass term is allowed that is consistent with the potentially anomalous symmetry can contribute to the anomaly. If a mass term were allowed, then the theory could be regulated in a way that respects the symmetry using the Pauli-Villars method, and so would not be anomalous.

More importantly, the anomaly can be understood as a statement about the analytic structure of current correlation functions. The anomaly equation implies the existence of discontinuities in current correlation functions at zero momentum. Only massless particles can contribute to these discontinuities, so the anomaly of a theory can always be understood purely in terms of the spectra and interactions of the massless states. We saw a simple example of this in our analysis of the chiral anomaly in $1 + 1$ dimensions. There, the value of the coefficient B in (16) was determined by the massless fields and this non-zero value, along with gauge invariance, required an anomaly in the divergence of the axial current. For more details on this point of view see [26,27,28].

1.7 Exercises for Lecture 1

- In 1+1 dimensions a $U(1)$ chiral current $j(z)$ obeys the Operator Product Expansion
$$j(z)j(0) \sim \frac{k}{z^2} \tag{46}$$
Compute the anomalous divergence of the current when coupled to a background gauge field in terms of the constant k. For help see sec. 12.2 of [6].

- Compute the vacuum polarization diagram in $1 + 1$ dimensions using a momentum-space cutoff and show that the coefficient A in (15) is logarithmically divergent and that the value you get for the coefficient B in (15) is the same as found using dimensional regularization.

2 Lecture 2: Applications of anomalies in $D = 4$

The results of the previous lecture on the evaluation of the chiral anomaly in four dimensions can be generalized in several different ways. In this lecture we consider these generalizations and some of their "real-world" applications to four-dimensional physics.

First of all, we can replace electromagnetism by a general gauge theory in the previous analysis. Redoing the Fujikawa analysis for fermions coupled to a non-Abelian gauge theory leads to a simple change in the final answer: there is a trace over gauge indices so that $F_{\mu\nu}F_{\alpha\beta}$ is replaced by $2\mathrm{Tr}(\mathrm{F}_{\mu\nu}\mathrm{F}_{\alpha\beta})$ assuming the standard normalization $\mathrm{Tr}\lambda^a\lambda^b = \frac{1}{2}\delta^{ab}$ for the generators of the gauge symmetry.

We can also generalize the Abelian chiral current to a set of non-Abelian chiral currents

$$j_\mu^{Ai} = \overline{\psi}\gamma^\mu\overline{\gamma}T^i\psi \tag{47}$$

with T^i the generators of the chiral symmetry. The divergence of j_μ^{Ai} then involves an additional trace over the generator T^i. Combining these two results, the anomalous divergence of a non-Abelian axial current coupled to non-Abelian gauge fields is

$$\partial^\mu j_\mu^{Ai} = -\frac{1}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}\mathrm{Tr\ T^iF_{\alpha\beta}F_{\mu\nu}}. \tag{48}$$

2.1 $\pi^0 \to \gamma\gamma$ revisited

As an example of these generalizations of phenomenological importance we will revisit the problem of π^0 decay[22,23] discussed in sec. 1.1 and redo the analysis in QCD. Thus, we consider QCD with two flavors in the limit $m_u = m_d = 0$ and write the quark doublet as

$$Q = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{49}$$

This theory has an $SU(2)_L \times SU(2)_R$ chiral symmetry with a triplet of axial currents given by

$$j_\mu^{Ai} = \overline{Q}\gamma_\mu\overline{\gamma}\tau^i Q \tag{50}$$

with τ^i the generators of $SU(2)$. The divergence of j_μ^{Ai} has a contribution from external gluons which is proportional to $\mathrm{Tr}\tau^{\mathrm{i}}\lambda^{\mathrm{c}}\lambda^{\mathrm{d}} = 0$ with λ^a the generators of the $SU(3)$ gauge symmetry of QCD. This vanishes since the τ^i are traceless, hence there is no QCD contribution to the anomaly. On the other hand, the electromagnetic contribution is given by

$$\partial^\mu j_\mu^{Ai}|_{QED} = -\frac{1}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\mathrm{Tr}(\tau^{\mathrm{i}}\mathrm{Q_{el}^2}) \tag{51}$$

where

$$Q_{el} = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \tag{52}$$

is the electric charge matrix acting on the quark doublet Q. This trace is non-zero only for $i = 3$, and this component of the axial current has a divergence

$$\partial^\mu j_\mu^{A3}|_{QED} = -\frac{1}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{53}$$

Using the fact that j_μ^{A3} creates a π^0, this anomaly implies an effective coupling of the π^0 to electromagnetism of the form (3) with coefficient

$$A = \frac{e^2}{16\pi^2 f_\pi}. \tag{54}$$

where $f_\pi = 93$ MeV is the pion decay constant. This gives a lifetime for $\pi^0 \to \gamma\gamma$ in excellent agreement with experiment.

2.2 Cancellation of gauge anomalies

So far we have considered situations where currents of global axial symmetries are anomalous in the presence of gauge fields coupled to vectorial currents. In the standard model and its extensions to grand unified theories and string theory we are interested in situations where the gauge currents are themselves chiral. We will not yet delve into the details of such gauge anomalies, putting this off for later when we have developed more of the necessary formalism. However, we do have the tools to see when such gauge anomalies vanish.

We focus here on the situation in $D = 4$. In this case we can always write the Lagrangian purely in terms of left-handed fields since given a right-handed fermion field $\psi_{R,\alpha}$, the field $\tilde{\psi}_{L,\alpha} = \epsilon_{\alpha\beta}\psi^*_{R\beta}$ transforms as a left-handed fermion under Lorentz transformations. The action for fermion fields in the representation $\mathbf{r}$ of the gauge group has the form

$$\mathcal{L}_f = \psi^\dagger_{Li} i\overline{\sigma}_\mu D^\mu_{ij}\psi_{Lj} + \cdots \tag{55}$$

where spinors indices have been suppressed, $i = 1 \cdots \dim \mathbf{r}$ and the covariant derivative is given by

$$D^\mu_{ij} = \partial^\mu \delta_{ij} - igA^{\mu a}(\lambda^a_{\mathbf{r}})_{ij} \tag{56}$$

with $\lambda^a_{\mathbf{r}}$ the generators of the Lie algebra of the gauge group.

We then need to compute the divergence of the gauge current, which is given by a triangle diagram with the current at one vertex and gauge fields on the other two vertices. Label the vertex with the current by μ, a. Bose symmetry demands that the diagram is invariant under the exchange of the two external gauge fields. The divergence of the current is thus proportional to $\mathrm{Tr}(\lambda^{\mathrm{a}}_{\mathbf{r}}\{\lambda^{\mathrm{b}}_{\mathbf{r}}, \lambda^{\mathrm{c}}_{\mathbf{r}}\})$ and vanishes if and only if

$$d^{abc} \equiv \mathrm{Tr}(\lambda^{\mathrm{a}}_{\mathbf{r}}\{\lambda^{\mathrm{b}}_{\mathbf{r}}, \lambda^{\mathrm{c}}_{\mathbf{r}}\}) = 0. \tag{57}$$

This quantity vanishes if the fermion representation $\mathbf{r}$ is real (or pseudo-real). There are several ways to understand this. Physically, if the fermions are in a real representation then it is possible to add a gauge invariant mass term to the Lagrangian. We can thus regulate the theory in a gauge invariant way using Pauli-Villars regularization and hence the anomaly must vanish. We can also show this directly.

A field $\psi_{\mathbf{r}}$ in the representation $\mathbf{r}$ transforms under infinitesimal gauge transformations by

$$\psi_{\mathbf{r}} \to (1 + i\alpha^a \lambda^a_{\mathbf{r}})\psi_{\mathbf{r}}. \tag{58}$$

Therefore the complex conjugate field transforms as

$$(\psi_{\mathbf{r}})^* \to (1 - i\alpha^a (\lambda^a_{\mathbf{r}})^*)(\psi_{\mathbf{r}})^* \tag{59}$$

which shows that the matrices which represent the conjugate representation $\overline{\mathbf{r}}$ are $\lambda^a_{\overline{\mathbf{r}}} = -(\lambda^a_{\mathbf{r}})^* = -(\lambda^a_{\mathbf{r}})^T$ where in the last step we have used the fact that we can choose the $\lambda^a_{\mathbf{r}}$ to be Hermitian.

Now if the representation $\mathbf{r}$ is real or pseudo-real then we can find a unitary matrix S such that

$$\lambda^a_{\mathbf{r}} = S\lambda^a_{\overline{\mathbf{r}}}S^{-1} = S(-(\lambda^a_{\mathbf{r}})^T)S^{-1}. \tag{60}$$

In this case we have

$$d^{abc} = \mathrm{Tr}\lambda^{\mathrm{a}}_{\mathbf{r}}\{\lambda^{\mathrm{b}}_{\mathbf{r}}, \lambda^{\mathrm{c}}_{\mathbf{r}}\} = -\mathrm{Tr}(\lambda^{\mathrm{a}}_{\mathbf{r}})^{\mathrm{T}}\{(\lambda^{\mathrm{b}}_{\mathbf{r}})^{\mathrm{T}}, (\lambda^{\mathrm{c}}_{\mathbf{r}})^{\mathrm{T}}\} = -\mathrm{Tr}\{\lambda^{\mathrm{b}}_{\mathbf{r}}, \lambda^{\mathrm{c}}_{\mathbf{r}}\}\lambda^{\mathrm{a}}_{\mathbf{r}} = -\mathrm{d}^{\mathrm{abc}} \tag{61}$$

from which we conclude that $d^{abc} = 0$.

Groups which have only real or pseudo-real representations and hence no gauge anomalies are $SU(2)$, $SO(2n+1)$ for $n \geq 2$, $Sp(2n)$ for $n \geq 3$ and G_2, F_4, E_7 and E_8. The remaining Lie groups in Cartan's classification, $U(1)$,$SU(n)$ for $n \geq 3$, E_6 and $SO(4n+2)$ all have potential anomalies. Except for $U(1)$, the groups with potential anomalies are also those groups for which $\pi_5(G) \neq 0$, a fact which is related to a topological characterization of anomalies which is reviewed in [2] and described briefly in sec 3.6.

2.3 't Hooft matching conditions

One of the more useful applications of anomalies in $D = 4$ arose in a study of the bound state spectrum of confining gauge theories by 't Hooft[26]. Given that nuclei are bound states of neutrons and protons and neutrons and protons in turn bound states of quarks, it is natural to ask whether quarks, or quarks and leptons, could themselves be bound states of some other objects (often called preons). On obvious objection to this idea is that there is no experimental evidence for substructure up to around the TeV scale, while the masses of light quarks and leptons are in the MeV range, much lighter than any possible scale of compositeness. How could there possibly be such light bound states if the scale of confinement or binding is so large? Such a theory would have to behave much differently from QCD where the mass scale of hadrons is comparable to the scale $\Lambda_{\rm QCD}$ where QCD becomes strongly interacting.

The only plausible mechanism that could give rise to such light states would be a theory which confines but does not break chiral symmetries. The bound state fermions could then be light because of the unbroken chiral symmetries. The argument of 't Hooft, based on anomalies, strongly constrains such a possibility. His argument goes as follows:

Consider a gauge theory with chiral fermions and an unbroken, anomaly-free, global symmetry group $\mathcal{G}$. Anomaly-free means that there are no $\mathcal{G}$—gauge—gauge anomalies so that $\mathcal{G}$ is a valid symmetry even in the presence of background gauge fields. Suppose further that the triangle diagram with three $\mathcal{G}$ currents is anomalous, that is the the coefficients d^{ijk} are non-zero, where i, j, k run over $1 \ldots \dim \mathcal{G}$.

Now imagine that we add a set of massless, gauge-singlet (spectator) fermions which contribute $-d^{ijk}$ to the $\mathcal{G}^3$ anomaly. Once we have done this we could gauge $\mathcal{G}$ because it is now completely free of anomalies. Imagine we have done this. Now, if the orginal gauge theory confines, we could study the low-energy effective action that describes the massless excitations of the resulting theory. In general this theory could contain massless Nambu-Goldstone bosons, but since we assume that $\mathcal{G}$ is not spontaneously broken, there are no such bosons that are relevant to our analysis of $\mathcal{G}$. The spectator fermions will remain massless since they were massless originally and being gauge singlets, are not affected by the dynamics of confinement. And then finally, there could be massless bound states.

Now the effective low-energy theory must be consistent since we started with a consistent theory. But the only way it can be consistent, that is anomaly free, is if the anomaly of the spectator fermions is canceled by an anomaly coming from massless bound states. Thus we are led to 't Hooft's conclusion, that there must be a set of massless bound states which have the same anomaly d^{ijk} as the original fundamental fields.

This argument did not involve the value of the gauge coupling when we gauged $\mathcal{G}$, so we could just as well take it to be zero. This decouples the $\mathcal{G}$ gauge fields, so we conclude that 't Hooft's condition must also be true in the original theory with $\mathcal{G}$ a global symmetry and no spectator fermions. For a version of this argument which relies more heavily on the analytical aspects of the anomaly see [27].

I will give one application of this condition. Consider QCD with three flavors of quarks in the massless limit $m_u = m_d = m_s = 0$. This theory has a $\mathcal{G} = SU(3)_L \times SU(3)_R \times U(1)_V$ global symmetry which has no anomalies involving gauge currents. There are however $SU(3)_R$—$SU(3)_R$—$U(1)_V$ and $SU(3)_L$—$SU(3)_L$—$U(1)_V$ anomalies as well as $SU(3)_L^3$ and $SU(3)_R^3$ anomalies. Assume that QCD confines into color singlets without spontaneous breaking of $\mathcal{G}$. Then you can check that there is no massless bound state spectrum which satisfies the 't Hooft anomaly matching conditions. Therefore the assumption that QCD confines without breaking $\mathcal{G}$ must be false. Of course we know in the real world that chiral symmetry is spontaneously broken. What is interesting about this argument is that it shows that in part this is unavoidable. It isn't simply a consequence of complicated dynamics which could have happened one way or the other. Rather, it follows from consistency of the theory. To be clear, anomalies in this model do not completely dictate the pattern of chiral symmetry breaking observed in the real world, they simply say that not all of the chiral symmetry can remain unbroken.

2.4 Exercises for Lecture 2

- Verify that all anomalies cancel in the Standard Model and also for $G = SU(5)$ with fermions in the representation $\overline{5} \oplus 10$ (the latter fact of course implies the former).

- 't Hooft's matching conditions play an important role in checking the consistency of certain dual descriptions of gauge theories. Work through the arguments in [29] to determine the anomaly free (that is no gauge anomaly) global chiral symmetries and check that the cubic global anomalies match for the proposed Seiberg duals of supersymmetric generalizations of QCD.

3 Lecture 3: Mathematical aspects of anomalies

One of the themes running through these lectures will be a relation between gauge anomalies in $2n$ dimensions and chiral anomalies in $2n+2$ dimensions. By gauge anomalies we mean the anomaly in the divergence of a current coupled to a gauge field alluded to in sec. 2.2. but not worked out in detail. By the chiral anomaly we mean the anomaly in the divergence of the axial current. In this section we will briefly explore the mathematical origin of this connection. In the fourth lecture we will discuss a physical model which provides a more direct interpretation of the result.

The objects such as $\mathrm{Tr}(\mathrm{F}^n)$ which we have encountered in the study of anomalies are examples of a general class of mathematical objects called characteristic classes. Manipulations with characteristic classes play an important role in the study of anomalies and in the relation between gauge and chiral anomalies, so we will take a detour here to review some of the basic material used in the study of these objects. Further details may be found in [30].

3.1 Characteristic classes

To start, suppose that α is a $k \times k$ complex matrix and $P(\alpha)$ is a polynomial in the components of α. We can act on α by elements g of $GL(k, \mathbb{C})$,

$$g : \alpha \to \alpha^g = g^{-1} \alpha g. \tag{62}$$

We will say that $P(\alpha)$ is a characteristic polynomial if P is invariant under $GL(k, \mathbb{C})$ transformations on α, that is if $P(\alpha^g) = P(\alpha)$. We can also consider characteristic polynomials for subgroups of $GL(k, \mathbb{C})$ such as $U(k)$, $GL(k, \mathbb{R})$, $O(k)$ and $SO(k)$.

Examples of characteristic polynomials are easy to come by. The canonical example arises by expanding the determinant

$$\mathrm{Det}(1 + \alpha) = 1 + \mathrm{S_1}(\lambda) + \mathrm{S_2}(\lambda) + \cdots + \mathrm{S_k}(\lambda). \tag{63}$$

where $\lambda_1, \cdots \lambda_k$ are the eigenvalues of α and $S_j(\lambda)$ is the j^{th} symmetric polynomial,

$$S_j(\lambda) = \sum_{i_1 < i_2 < \cdots i_j} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}. \tag{64}$$

In general characteristic polynomials for $GL(k, \mathbb{C})$ are expressible as polynomials in the $S_j(\lambda)$.

To obtain characteristic classes we substitute $k \times k$ matrix valued 2-form curvatures for α. We will denote a generic curvature and connection by (Ω, ω). When we have specific examples to discuss we will use the gauge curvature and connection (F, A) or the gravitational curvature and connection (R, ω).

We first consider complex bundles with $g \in GL(k, \mathbb{C})$ or $g \in U(k)$. The characteristic polynomials are the same, and from the physics point of view $U(k)$ is more natural since we naturally use compact gauge groups in physics. The curvature Ω can be taken to be anti-Hermitian. Substituting $i\Omega/2\pi$ for α in (63) gives the total Chern form

$$c(\Omega) = \mathrm{Det}(1 + \frac{\mathrm{i}\Omega}{2\pi}) = 1 + \mathrm{c_1}(\Omega) + \mathrm{c_2}(\Omega) + \cdots + \mathrm{c_k}(\Omega) \tag{65}$$

where the j^{th} Chern form is a polynomial in Ω of degree j:

$$c_1(\Omega) = \frac{i}{2\pi} \mathrm{Tr}\Omega \tag{66a}$$

$$c_2(\Omega) = \frac{1}{8\pi^2} (\mathrm{Tr}\Omega^2 - (\mathrm{Tr}\Omega)^2) \tag{66b}$$

and so on. If we consider instead real bundles with transition functions in $GL(k, \mathbb{R})$ or $O(k)$ then the curvature Ω is a real antisymmetric matrix and it is natural to consider the total Pontrjagin class defined by

$$p(\Omega) = \mathrm{Det}(1 - \frac{\Omega}{2\pi}) = 1 + \mathrm{p}_1(\Omega) + \mathrm{p}_2(\Omega) + \cdots + \mathrm{p}_\mathrm{k}(\Omega). \qquad (67)$$

In physics applications we will typically encounter Chern classes in computing topological invariants of $U(k)$ gauge theory and Pontrjagin classes when we compute topological invariants of gravitational theories where $O(k)$ or $O(k-1,1)$, depending on signature, act as local Lorentz transformations.

For real bundles Ω is an antisymmetric $k \times k$ matrix. If k is an even integer, $k = 2r$, then we can put Ω in the form

$$\frac{i\Omega}{2\pi} = \begin{pmatrix} 0 & x_1 & & & & \\ -x_1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & x_r \\ & & & & -x_r & 0 \end{pmatrix} \qquad (68)$$

in which case we have

$$p_1 = \sum_a x_a^2 \qquad (69a)$$

$$p_2 = \sum_{a<b} x_a^2 x_b^2 \qquad (69b)$$

and so on. If k is an odd integer then we can put Ω in the form (68) but with an extra row and column of zeroes.

With $k = 2r$ if we demand only $SO(2r)$ invariance rather than $O(2r)$ invariance (that is we have an oriented real vector bundle) then there is one additional characteristic class we can define. If α is a $2r \times 2r$ antisymmetric matrix, then putting α in the form (68) we can define the Pfaffian of α by

$$Pf(\alpha) = x_1 x_2 \cdots x_n. \qquad (70)$$

Substituting the curvature Ω for α gives a characteristic class called the Euler class $e(\Omega)$. Note that e is the square root of the top Pontrjagin class, $e^2(\Omega) = p_k(\Omega)$.

Combinations of characteristic classes often appear in the computation of topological invariants. These include the A-roof genus given by

$$\hat{A}(R) = \prod_a \frac{x_a/2}{\sinh x_a/2} = 1 - \frac{p_1}{24} + \frac{1}{16}\left(\frac{7}{360}p_1^2 - \frac{1}{90}p_2\right) + \cdots \qquad (71)$$

and the Hirzebruch L-polynomial

$$L(\Omega)=\prod_j \frac{x_j}{\tanh x_j}=1+\frac{1}{3}p_1+\frac{1}{45}\left(7p_2-p_1^2\right)+\cdots. \tag{72}$$

For later use we also mention the behavior of characteristic classes for sums of vector bundles. Given two vector bundles E, F with connections there is a natural vector bundle $E\oplus F$ called the Whitney sum with the curvature of $E\oplus F$ the direct sum of the curvatures of E and F. The total Chern and Pontrajin classes obey the relations

$$P(E\oplus F)=p(E)p(F), \qquad c(E\oplus F)=c(E)c(F) \tag{73}$$

thus

$$p_1(E\oplus F)=p_1(E)+p_1(F), \qquad p_2(E\oplus F)=p_2(E)+p_2(F)+p_1(E)p_1(F), \tag{74}$$

and so on.

3.2 Properties of characteristic classes

Having introduced characteristic classes, we now need to study some of their properties. In particular we would like to show

- That $P(\Omega)$ is closed.
- That integrals of $P(\Omega)$ are topological invariants.

To show that P is closed it suffices to show this for $P_m\equiv \mathrm{Tr}\Omega^{\mathrm{m}}$ since a general invariant polynomial can be expressed in terms of sums and products of the P_m. In this section we denote a generic connection by ω and its curvature by Ω. Using the chain rule we have

$$dP_m=m\mathrm{Trd}\Omega\Omega^{\mathrm{m}-1} \tag{75}$$

which we can write using the Bianchi identity, $D\Omega=d\Omega+\omega\Omega-\Omega\omega=0$, as

$$dP_m=m\mathrm{Tr}(\Omega\omega-\omega\Omega)\Omega^{\mathrm{m}-1}=0 \tag{76}$$

where in the last step we have used cyclicity of the trace.

To show that the integrals of P are topological invariants it will again be sufficient to establish this for P_m. To do this we consider two connections ω_0,ω_1 with the same transition functions and will show that the difference

$P_m(\Omega_1) - P_m(\Omega_0)$ is exact. The integral of P_m over a closed manifold is then independent of variations of the connection (while keeping the transition functions fixed) and is thus a topological invariant.

It is useful to construct an interpolation between the two connections. Let $t \in [0,1]$ be a real parameter and consider the connection and curvature

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0), \tag{77a}$$
$$\Omega_t = d\omega_t + \omega_t^2 \tag{77b}$$

A small amount of algebra shows that

$$\frac{\partial \Omega_t}{\partial t} = d(\omega_1 - \omega_0) + [\omega_t, \omega_1 - \omega_0] = D_t(\omega_1 - \omega_0), \tag{78}$$

with D_t the covariant derivative with respect to ω_t. We thus have

$$\frac{\partial}{\partial t} P_m(t) = m\mathrm{Tr}\frac{\partial \Omega_t}{\partial t}\Omega_t^{m-1} = m\mathrm{Tr}D_t(\omega_1 - \omega_0)\Omega_t^{m-1} = m d\mathrm{Tr}(\omega_1 - \omega_0)\Omega_t^{m-1} \tag{79}$$

where we have used the Bianchi identity to pull D_t out of the trace and then gauge invariance to reduce D_t to d acting on the trace. Integrating (79) with respect to t then yields the desired result

$$P_m(\Omega_1) - P_m(\Omega_0) = md\int_0^1 dt \mathrm{Tr}(\omega_1 - \omega_0)\Omega_t^{m-1}. \tag{80}$$

In applications to anomalies the most important fact about characteristic classes is that the obey a set of equations called the descent equations. I will prove these equations for characteristic classes constructed from the connection and curvature in gauge theory. It turns out that a natural and elegant way to do this involves the use of ghosts and the BRST formalism. To explain why the BRST formalism is particulary useful in the study of anomalies I first take a short detour by introducing the Wess-Zumino consistency conditions and then translating these into the language of BRST cohomology.

3.3 WZ condition and BRST cohomology

We have found that the path integral over fermion fields can lead to a violation of gauge invariance in theories with chiral gauge currents. Following Wess and Zumino[31] we will first show that this violation obeys a certain condition. The effective action that results from integrating out the fermions is defined by

$$e^{-W[A]} = \int D\psi D\overline{\psi} e^{-\int \overline{\psi} i\gamma^\mu D_\mu \psi}. \tag{81}$$

Under an infinitesimal gauge transformation we have

$$\delta_v A = dv + [A, v] = Dv, \tag{82}$$

so the gauge variation of $W[A]$ can be written as

$$\begin{aligned} \delta_v W[A] &= W[A + Dv] - W[A] = \int (D_\mu v)^a \frac{\delta W[A]}{\delta A^a_\mu} && \text{(83a)} \\ &= -\int v^a \left(D_\mu \frac{\delta W[A]}{\delta A_\mu} \right)^a = -\int v^a D^\mu J^a_\mu && \text{(83b)} \\ &= -\int v^a \mathcal{A}^a[x, A] && \text{(83c)} \end{aligned}$$

where the current is $J^a_\mu = \delta W / \delta A^{\mu a}$ and since we choose gauge variations which vanish at infinity, we have also freely integrated by parts.

The generator of gauge transformations acting on functionals of the gauge field is thus

$$v^a D_\mu \frac{\delta}{\delta A^a_\mu(x)} \equiv -v^a \mathcal{J}^a(x). \tag{84}$$

A short calculation shows that the generators obey the algebra

$$\left[\mathcal{J}^a(x), \mathcal{J}^b(y)\right] = i f^{abc} \delta^4(x - y) \mathcal{J}^c(x) \tag{85}$$

and this, along with the definition of the anomaly $\mathcal{A}^a$ as the covariant divergence of the current implies the Wess-Zumino condition

$$\mathcal{J}^a(x) \mathcal{A}^b[y, A] - \mathcal{J}^b(y) \mathcal{A}^a[x, A] = i f^{abc} \delta^4(x - y) \mathcal{A}^c[y, A]. \tag{86}$$

This condition obviously follows just from the algebra of gauge variations and the fact that the anomaly arises from variation of an effective action, $W[A]$, but nonetheless it is quite useful. Historically it was useful in sorting out the proper calculation of gauge anomalies from Feynman diagrams. More recently it has played an important role in understanding the cohomological interpretation of anomalies as we now discuss.

We will introduce ghost fields and the BRST operator into our discussion. There are at least three reasons for doing this. First, to actually make sense of the path integral in gauge theories we need to gauge fix and replace gauge invariance by BRST invariance. Thus we should really formulate gauge anomalies in this language as well. Second, it is technically useful in demonstrating the descent equations, and third there is an elegant interpretation of the Wess-Zumino consistency condition in this language[32,33].

Recall that in the BRST formalism we introduce ghosts $\omega^a(x)$ which are Grassmann valued spin zero fields in the adjoint representation of the gauge group (there are also anti-ghosts but they will not play in the following discussion). The BRST operator $\mathcal{S}$ acts on the gauge fields and ghosts as

$$\mathcal{S}A^a_\mu = \partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c, \tag{87a}$$

$$\mathcal{S}\omega^a = -\frac{1}{2} f^{abc} \omega^b \omega^c. \tag{87b}$$

Note that the action on A^a_μ is that of a gauge transformation, but with the ghost field as anti-commuting gauge parameter. Geometrically ghosts should be thought of as 1-forms in the space $\mathcal{G}$ of gauge transformations [34]. Consistent with this we will write the gauge field as a Lie-algebra valued 1-form $A = A^a_\mu \lambda^a dx^\mu$ and use a formalism where the 1-forms on spacetime, dx^μ, anticommute with the 1-forms ω^a on $\mathcal{G}$:

$$dx^\mu \omega^a(x) + \omega^a(x) dx^\mu = 0, \tag{88}$$

and similarly we will take the BRST operator $\mathcal{S}$ and the exterior derivative d to be anticommuting:

$$\mathcal{S}d + d\mathcal{S} = 0. \tag{89}$$

In fact, $\mathcal{S}$ can be thought of as the exterior derivative on $\mathcal{G}$. The gauge transformation law of the ghost field can then be interpreted as the Mauer-Cartan equation on $\mathcal{G}$.

As a first application of this formalism we consider the anomaly with the anomaly parameter v^a replaced by the ghost field ω^a,

$$\mathcal{A}[\omega, A] = \int d^4x \; \omega^a(x) \mathcal{A}^a[x, A]. \tag{90}$$

A short calculation shows that the Wess-Zumino consistency condition (86) is equivalent to the statement that $\mathcal{S}\mathcal{A}[\omega, A] = 0$, that is that the anomaly (90) is BRST closed. Since $\mathcal{S}^2 = 0$, this can obviously be satisfied by setting $\mathcal{A} = \mathcal{S}F[A]$, but then we could simply add $F[A]$ to the action and cancel the anomaly. Since anomalies are defined precisely up to the choice of regulator, or equivalently up to the ability to add such a local counterterm to the action (see e.g. the discussion in sec 1.2), possible anomaly terms are classified by the cohomology of the BRST operator (in the space of local functionals) at ghost number one.

3.4 Descent formalism for anomalies

We now have the necessary machinery to demonstrate the descent equations. These state that if α_{2n+2} is a characteristic $2n+2$-form then we have locally

$$\alpha_{2n+2} = d\alpha^{(0)}_{2n+1} \tag{91a}$$
$$\delta\alpha^{(0)}_{2n+1} = d\alpha^{(1)}_{2n} \tag{91b}$$
$$\delta\alpha^{(1)}_{2n} = d\alpha^{(2)}_{2n-1} \tag{91c}$$

In these equations δ indicates the gauge variation, and the superscript indicates whether the quantity is independent of the parameter of the gauge variation (0), first order in the gauge parameter (1), and so on.

The descent formula follows quite easily from the BRST formalism and we will see as an extra bonus that the BRST formalism also gives us a natural candidate for the gauge anomaly. We will prove the descent formalism for the particular case that $P(F) = \mathrm{TrF^{n+1}} \equiv \alpha_{\mathrm{2n+2}}$. Other characteristic classes for the gauge field can be written as sums and products of these so it suffices to show the descent formalism for this case.

First, we have already seen that α_{2n+2} is closed. Locally, or on a simply connected manifold, this implies the first step in the descent formula,

$$\alpha_{2n+2} = d\alpha^{(0)}_{2n+1}. \tag{92}$$

Now since α_{2n+2} is gauge invariant and independent of ghost fields, we also have $\mathcal{S}\alpha_{2n+2} = 0$. Turning to the next step in the descent formalism, we consider the gauge variation of $\alpha^{(0)}_{2n+1}$ with an anti-commuting gauge parameter, that is $\mathcal{S}\alpha^{(0)}_{2n+1}$. This is also closed because

$$d\mathcal{S}\alpha^{(0)}_{2n+1} = -\mathcal{S}d\alpha^{(0)}_{2n+1} = -\mathcal{S}\alpha_{2n+2} = 0. \tag{93}$$

Therefore we have shown that, locally,

$$\mathcal{S}\alpha^{(0)}_{2n+1} = d\alpha^{(1)}_{2n} \tag{94}$$

where $\alpha^{(1)}_{2n}$ has ghost number one. We can continue this process, for example we also have

$$d\mathcal{S}\alpha^{(1)}_{2n} = -\mathcal{S}d\alpha^{(1)}_{2n} = -\mathcal{S}^2\alpha^{(0)}_{2n+1} = 0, \tag{95}$$

which shows that

$$\mathcal{S}\alpha^{(1)}_{2n} = d\alpha^{(2)}_{2n-1}. \tag{96}$$

Equations (92) , (94) , and (96) constitute a demonstration that α_{2n+2} obeys the descent equations. We have expressed these equations in the BRST formalism, but having derived them, we can replace the BRST variation by the gauge variation and the ghost fields by parameters of the gauge variation and they still hold true. In addition, (96) shows that the spacetime integral of $\alpha_{2n}^{(1)}$ provides a candidate for the anomaly in $2n$ spacetime dimensions since it is BRST closed and has ghost number one. In fact, the spacetime integral of $\alpha_{2n}^{(1)}$ is precisely the gauge anomaly in $2n$ dimensions up to numerical factors which will be discussed in the following lecture where a physical model of this connection will be presented. Note that if we work strictly in $2n$ dimensions we would have to add two extra dimensions to carry out this procedure in order to make sense of $2n+2$-forms.

3.5 Determinant line bundle

Anomalies have a deep connection to the topology of the configuration space of gauge theories. This connection was understood in [35,36,37,38]. A very useful review can be found in [2] and as a result I will be rather brief here.

We are interested in anomalies in the effective action which arises from integrating out the fermions in a theory coupled to gauge theory or gravity. We consider left-handed Weyl fermions in some representation $\mathbf{r}$ of a gauge group G. In $D = 0$ mod 4 dimensions we write all fermions fields as left-handed since complex conjugation relates left- and right-handed fields. In $D = 2$ mod 4 dimensions this is not the case and right-handed fermions must be treated separately from left-handed fermions. The results will however only differ from the analysis below by the overall sign of the anomaly. The fermion effective action is

$$e^{-W[A]} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-\int d^{2n}x\overline{\psi}i\gamma^\mu D_{\mu_+}\psi} \tag{97}$$

One often writes $e^{-W[A]} = \det i\gamma^\mu D_{\mu_+}$ but this is not completely correct. To define a determinant one needs an operator which maps a vector space to itself while $\gamma^\mu D_{\mu_+}$ maps left-handed fermions to right-handed fermions. This can be dealt with in various ways. One possibility discussed in [2] is to add to $\gamma^\mu D_{\mu_+}$ a free operator ∂_- which acts on right-handed fermions and to define the effective action to be the determinant of $\gamma^\mu D_{\mu_+} + \partial_-$. This changes the effective action only by an overall constant which is independent of the gauge fields and so does not affect any possible anomalies in the gauge variation of the efective action.

Another subtlety can appear in $D = 2, 10$ where one can have real Majorana-Weyl fermions. In this case integrating out the fermions give the square root of the determinant or more precisely the Pfaffian of $\gamma^\mu D_{\mu_+} + \partial_-$. This will result in factors of $1/2$ is the formula for the anomaly.

If we consider fermions in the representation $\mathbf{r}$ and in the complex conjugate representation $\bar{\mathbf{r}}$ then it follows from the form of the action that $W_{\bar{\mathbf{r}}}[A] = (W_{\mathbf{r}}[A])^*$. Thus

$$2\mathrm{Re}W_{\mathbf{r}}[A] = W_{\mathbf{r}}[A] + W_{\bar{\mathbf{r}}}[A] = W_{\mathbf{r}\oplus\bar{\mathbf{r}}}[A]. \tag{98}$$

Since $W_{\mathbf{r}\oplus\bar{\mathbf{r}}}[A]$ is the effective action for a fermion in a real representation, it can be regulated in a gauge invariant way. Therefore there can be no anomaly in the real part of the effective action so the only anomaly can be in the phase of $\det i\gamma^\mu D_\mu$.

In gauge theory we have the space of all gauge connections $\mathcal{A}$, and the group of gauge transformations $\mathcal{G}$ (these are maps from spacetime into the gauge group G). To define the fermion effective action we must define $e^{-i\mathrm{Im}W[A]}$ for each point in $\mathcal{C} = \mathcal{A}/\mathcal{G}$, since as discussed above, the real part of $W[A]$ is clearly well defined. Now $e^{-i\mathrm{Im}W[A]}$ is a phase, that is an element of the group $U(1)$. We should view this in the language of principal fibre bundles. The base space is $\mathcal{C}$ and we try to patch together $U(1)$ fibres to define the effective action. Now if we can assign a unique value to the effective action for each point in $\mathcal{C}$ then this $U(1)$ bundle has a global section. Conversely, if there is no global section then we cannot uniquely define the effective action for each point in the configuration space and the theory is anomalous.

Now the existence of a global section is equivalent to the statement that the $U(1)$ bundle is trivial. Thus we have turned the question of anomalies into a topological question since obstructions to the triviality of the bundle are topological in nature.

A famous example of a topologically non-trivial $U(1)$ bundle is the Dirac-Wu-Yang monopole bundle over S^2. This suggests that we look for a non-trivial two-sphere in $\mathcal{C}$. It should be noted that $\mathcal{A}$ is contractible since any gauge field can be shrunk down to zero continuously. Thus $\mathcal{A}$ has trivial topology and any non-trivial topology of $\mathcal{C}$ must arise from taking the quotient by $\mathcal{G}$. An analysis along these lines relates the existence of a global section to the question of whether or not $\pi_5(G)$ is non-trivial or not (for theories in four dimensions) and eventually leads to an identification between the chiral anomaly and the gauge anomaly as given by the descent procedure, a connection which will be explained in the following lecture. For further details on this approach see [2].

In the Fujikawa analysis of the chiral anomaly in $D = 4$ we encountered the quantity

$$A(x) = \sum_n \phi_n^\dagger(x)\overline{\gamma}\phi_n(x) \tag{99}$$

where $\phi_n(x)$ are the eigenfunctions of $i\gamma^\mu D_\mu$. Up to omission of the factor of $\alpha(x)$, this is the trace of the matrix denoted by C_{mn} in sec 1.3.

Now, if ϕ_n is an eigenfunction of $i\gamma^\mu D_\mu$ with non-zero eigenvalue λ_n, then since $\overline{\gamma}$ and $i\gamma^\mu D_\mu$ anti-commute, it follows that $\overline{\gamma}\phi_n$ is an eigenfunction with eigenvalue $-\lambda_n$. Thus, since eigenfunctions with different eigenvalues are orthogonal, the integral of $A(x)$ only receives contributions from the eigenfunctions with zero eigenvalues,

$$\int d^4x A(x) = \int d^4x \sum_i (\phi_0^i(x))^\dagger \overline{\gamma}(\phi_0^i(x)) \tag{100}$$

where i labels the zero-modes. Now we can choose the zero modes to be eigenfunctions of $\overline{\gamma}$. Let n_+ be the number of zero-modes with eigenvalue $+1$ and n_- the number of zero-modes with eigenvalue -1. Then since the non-zero modes cancel, the right hand side of (100) is formally $n_+ - n_-$, giving

$$\int d^4x A(x) = n_+ - n_- \tag{101}$$

The problem with this argument is that it is purely formal, we are canceling off an infinite number of zero terms. To make careful sense of this we need to regularize so that the sum converges, and then take the limit as the regulator goes to infinity. This allows us to make the above argument precise, but we also saw above that introducing the regulator allows us to compute $A(x)$ in terms of the background gauge fields and yields a finite result when the regulator goes to infinity. Carrying this out exactly as before we conclude that

$$n_+ - n_- = \frac{1}{32\pi^2}\int d^4x \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}F_{\mu\nu} \tag{102}$$

Equation (102) is precisely the Atiyah-Singer index theorem for the Dirac operator coupled to gauge fields in four dimensions. Note that the quantity under the integral in (102) is one half of the chiral anomaly (43) .

Redoing this calculation on a $2n$-dimensional manifold $\mathcal{M}$ equipped with a spin connection ω with curvature R and for fermions in the representation $\mathbf{r}$ of a

$SU(N)$ gauge connection, A, with curvature F gives the general Atiyah-Singer index theorem:

$$n_+ - n_- = \int_{\mathcal{M}} \left[\hat{A}(R)ch(F)\right]_{2n} \tag{103}$$

where the $2n$ subscript indicates that we keep only the $2n$-form part of the expression. Here the combination of Pontragin classes that contributes to the index, $\hat{A}(R)$, is the A-roof genus defined previously in (71) , and $ch(F)$ is the Chern character given by

$$ch(F) = \mathrm{Tr}_{\mathbf{r}} e^{iF/2\pi} = \mathrm{dim}\mathbf{r} + c_1(F) + \cdots . \tag{104}$$

The index density $\hat{A}(R)ch(F)$ is in general equal to one half of the chiral anomaly in $2n$ dimensions as can be seen by keeping careful track of the factors of two in the Fujikawa analysis of the chiral anomaly. As we will see in the following lecture, the index density in $2n$ dimensions also provides the starting point for a derivation of the gauge anomaly in $2n-2$ dimensions.

3.7 Exercises for Lecture 3

- Consider an $SO(k)$ bundle N with Pontragin classes $p_k(N)$. In many physics applications one also encounters the associated complex spin bundle $S(N)$ (for example, in a gravitational theory fermion fields take values in $S(N)$). Compute the Chern classes c_2 and c_4 for $S(N)$ in terms of the Pontragin classes $p_1(N)$ and $p_2(N)$.
- Work out the gauge anomaly in $d = 2, 4$ by starting from the chiral anomaly in $d = 4, 6$ and applying the descent formalism.

4 Lecture 4: Anomaly inflow

In the previous lecture we found that the descent procedure provides a candidate for the gauge anomaly in $2n$ dimensions starting from the $2n+2$-form $\mathrm{Tr}F^{n+1}$ in $2n+2$ dimensions. From the results of the first lecture we also know that this $2n+2$-form is proportional to the chiral anomaly in $2n+2$ dimensions.

In this lecture we will explore a physical model which explains the connection between gauge anomalies in $2n$ dimensions and chiral anomalies in $2n+2$ dimensions in more physical terms[39] and gives us the precise numerical factors which relate the two anomalies. We will discover that there can be interactions in non-anomalous theories in $2n+2$ dimensions which are anomalous in the presence of $2n$-dimensional topological defects. The anomaly of the bulk interactions is localized on the defect and expressed in terms of the chiral anomaly in $2n+2$ dimensions via the descent procedure which encodes the topology of the defect. The anomaly from these bulk interactions is cancelled by an equal but opposite anomaly arising from fermion zero modes localized on the defect.

The cancellation of anomalies between bulk terms and local terms coming from zero modes on defects has many applications in string theory and M theory, some of which we will discuss. There are also interesting applications in condensed matter physics [40,41] and lattice gauge theory [42].

4.1 Axion electrodynamics

We will start by considering Dirac fermions in $D=4$ coupled vectorially to a $U(1)$ gauge connection A (which we will refer to as electromagnetism) and a complex scalar field $\Phi = \Phi_1 + i\Phi_2$. We will later generalize to include coupling to gravity through the spin connection ω and also to higher dimensions. Since we are dealing with Dirac fermions the theory can be regulated while maintaining gauge invariance (e.g. by Pauli-Villars regularization) and so there are no gauge anomalies in this theory.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \overline{\psi} i\gamma^\mu D_\mu \psi - \overline{\psi}(\Phi_1 + i\overline{\gamma}\Phi_2)\psi + \partial_\mu \Phi^* \partial^\mu \Phi - V(|\Phi|^2) \quad (105)$$

We assume that the potential $V(|\Phi|^2)$ is chosen so that $|\Phi| = v$ at the minimum for some non-zero v.

This theory has a global $U(1)$ Peccei-Quinn symmetry under which the fields transform as

$$\Phi \quad \rightarrow \quad e^{i\alpha}\Phi, \quad (106a)$$
$$\psi \quad \rightarrow \quad e^{-i\alpha\overline{\gamma}/2}\psi. \quad (106b)$$

Classically the chiral current j^A_μ associated to this symmetry is conserved, but as we saw in the first lecture it is anomalous at the quantum level with

$$\partial_\mu j^\mu_A = \frac{1}{8\pi^2} F^{\mu\nu}\tilde{F}_{\mu\nu}. \quad (107)$$

At the classical level there is a Nambu-Goldstone boson due to spontaneous breaking of the $U(1)$ Peccei-Quinn symmetry. We can identify this field by writing fluctuations of Φ about the vacuum as $\Phi = ve^{ia}$ with a the axion field. The axion couples to the fermion fields through the interaction $\overline{\psi}e^{ia\overline{\gamma}}\psi$. This coupling can be removed by a chiral redefinition of the fermion fields, but because of the anomaly (107) it reappears as a coupling to $F\tilde{F}$. We can thus reduce the theory to a low-energy effective theory of the photon and axion (axion QED) which will take the form

$$S^{\mathrm{axion-QED}} = \int d^4x \left(-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{a}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{v^2}{2} \partial_\mu a \partial^\mu a - V(a) \right) \tag{108}$$

In the theory at hand the potential for the axion vanishes, $V(a) \equiv 0$. In "real world" variants of this theory we would include the strong interactions. It is then thought that QCD effects generate a potential for the axion $V(a) \sim \Lambda^4_{QCD}(1 - \cos a)$. For our purposes we will ignore this complication.

Because of the axion coupling to $F\tilde{F}$, there are new sources of the electromagnetic current when the axion field varies in spacetime. Varying the action with respect to A_μ yields the equation of motion

$$\partial^\mu F_{\mu\nu} = j_\nu, \tag{109}$$

with

$$j_\nu = \frac{e^2}{16\pi^2} \partial^\mu (a \tilde{F}_{\nu\mu}). \tag{110}$$

Written in non-covariant form $j_\mu = (\rho, \vec{j})$ we have

$$\rho \sim a\vec{\nabla} \cdot \vec{B} + \vec{B} \cdot \vec{\nabla} a, \tag{111a}$$

$$\vec{j} \sim \dot{a}\vec{B} + \vec{\nabla} a \times \vec{E}. \tag{111b}$$

The first term in ρ is responsible for the Witten effect [43]—monopoles in the presence of an axion field or non-zero θ angle carry electric charge—while the last term in $\vec{j}$ gives a Hall-like contribution to the current which is perpendicular to the applied electric field.

Anomaly inflow arises by considering these contributions to the current in the presence of topological defects. For $V(a) = 0$ the vacuum manifold has non-trivial π_1: $\pi_1(\mathcal{M}_{\mathrm{vac}}) = \mathbb{Z}$ so the theory has axion strings. For an axion string of charge 1 the scalar field Φ has the form

$$\Phi = f(\rho) e^{i\theta} \tag{112}$$

where we are using polar coordinates with z along the string so that θ is the azimuthal angle and ρ the radial distance from the string. The function $f(\rho)$ approaches zero as $\rho \to 0$ and v as $\rho \to \infty$. The precise form of $f(\rho)$ will not be needed in what follows.

Now consider the axion contributions to the electromagnetic current in the presence of an axion string. If we apply an electric field along the string,

$\vec{E} = E\hat{z}$, then we find a current $\vec{j} \sim -E\hat{\rho}/\rho$. The current is directed radially inward and has a non-zero divergence on the string. Clearly electric charge can be conserved only if the axion string is capable of carrying electric charge.

To understand how this can come about we should go back to the Lagrangian describing the interaction of fermions with the axion string configuration,

$$\mathcal{L}_\psi = \overline{\psi} i\gamma^\mu D_\mu \psi + \overline{\psi}(\Phi_1 + i\overline{\gamma}\Phi_2)\psi. \tag{113}$$

We can show that the string can carry charge by exhibiting normalizable zero modes of the fermion equations of motion. To search for zero modes of the Dirac equation in the background (112) we first write $\overline{\gamma} = \gamma^{\rm int}\gamma^{\rm ext}$ with $\gamma^{\rm int} = -\gamma^0\gamma^1$ and $\gamma^{\rm ext} = -i\gamma^2\gamma^3$. We also decompose the coordinates into $x^a_{\rm int} = (x^0, x^1)$ and $x_{\rm ext} = (x^2, x^3)$. Writing ψ in terms of eigenfunctions $\psi_\pm$ of $\overline{\gamma}$ and looking for solutions independent of ϕ, the Dirac equation becomes

$$\begin{aligned} i\gamma^a\partial_a\psi_- + i\gamma^2(\cos\theta + i\gamma^{\rm ext}\sin\theta)\partial_\rho\psi_- &= f(\rho)e^{i\theta}\psi_+, & \text{(114a)} \\ i\gamma^a\partial_a\psi_+ + i\gamma^2(\cos\theta + i\gamma^{\rm ext}\sin\theta)\partial_\rho\psi_+ &= f(\rho)e^{-i\theta}\psi_-, & \text{(114b)} \end{aligned}$$

with solution

$$\psi_- = \eta(x_{\rm int})\exp\left[-\int_0^\rho f(\sigma)d\sigma\right], \qquad \psi_+ = -i\gamma^2\psi_-, \tag{115}$$

with $i\gamma^a\partial_a\eta = 0$ and $\gamma^{\rm int}\eta = -\eta$.

Thus we see that the zero modes on the string are chiral and since they couple to the pullback of the spacetime gauge field, a 1+1 dimensional observer will conclude that the electromagnetic current on the string is anomalous. Since the outside observer also sees an apparent violation of current conservation we suspect that these two facts are related and that in fact charge is conserved with an inflow of charge from the outside of the correct magnitude to account for the anomaly seen from the $1+1$ dimensional point of view.

It is actually more straightforward to demonstrate that the bulk plus string action is gauge invariant than to show that the current is properly conserved, although of course the two facts are related. A proper understanding of current conversation requires a careful study of the difference between the consistent and covariant forms of the anomaly and may be found in [44].

4.2 Anomaly inflow for axion strings

We now want to show explicitly that the bulk and world-sheet contributions to the anomaly cancel so that the overall theory is gauge invariant in the presence of an axion string. That is, we want to show that

$$\delta_{\text{gauge}}\left(S_{\text{bulk}}^{\text{axion}-\text{QED}}+S_{\text{string}}\right)=0. \tag{116}$$

where S_{string} is the effective action for the axion string zero-modes.

To do this in a way which makes contact with our discussion in the previous lecture, we first rewrite the relevant couplings in terms of characteristic classes. The chiral anomaly in $D=4$ is given by

$$2\omega_4 \equiv \frac{1}{4\pi^2}F\wedge F, \tag{117}$$

where

$$\omega_4 = ch(F)|_4 \tag{118}$$

is the Dirac index density in four dimensions. Clearly ω_4 is a characteristic class, so we can apply the descent procedure. An explicit calculation gives

$$\omega_3^{(0)} \quad = \quad \frac{1}{8\pi^2}A\wedge F \tag{119a}$$

$$\omega_2^{(1)} \quad = \quad \frac{1}{8\pi^2}\Lambda F \tag{119b}$$

where Λ is the parameter of the gauge transformation, and we have used the same notation for the descent procedure as in sec. 3.5. Now let us see how the descent procedure appears in the analysis of the gauge variation of the bulk coupling of the axion to gauge fields.

The bulk coupling of the axion to the gauge field given in (108) can be written in terms of differential forms as

$$\int_{M^4}\frac{a}{2}\,2\omega_4 = \int_{M^4} a\omega_4. \tag{120}$$

where the factor of $a/2$ arises because we need to do a chiral transformation with parameter $\alpha = a/2$ in order to remove the coupling of the axion to the fermion fields. We have also denoted the spacetime manifold by M^4.

Now in the presence of an axion string a is not single valued since it changes by 2π in going around the string. Therefore (120) doesn't really make sense in the presence of an axion string. However, da *is* single valued so we will integrate by parts and write the coupling (120) as

$$-\int_{M^4} da\wedge\omega_3^{(0)}. \tag{121}$$

If we now vary (121) with respect to a gauge transformation we have

$$\delta\left[-\int_{M^4} da\wedge\omega_3^{(0)}\right] = -\int_{M^4} da\wedge\delta\omega_3^{(0)} = -\int_{M^4} da\wedge d\omega_2^{(1)} = \int_{M^4} d^2a\wedge\omega_2^{(1)} \tag{122}$$

Now naively one might think that d^2 acting on any smooth function is zero and that as a result (122) vanishes. However, a is not a smooth function. It has winding number one around the origin, just like the polar angle, and hence is not well defined at the origin. A more precise way of saying this which is relevant to finishing the calculation of (122) is to consider the integral of da over any circle S^1 which encloses the axion string. Also, let D^2 be a disc whose boundary is this S^1. Then since a has winding number one and using Stoke's theorem we have

$$\int_{S^1} da = 2\pi = \int_{D^2} d^2a. \tag{123}$$

Since this is true for any S^1 enclosing the origin (that is the axion string, at least in the limit of an infinitely thin string), we must have

$$d^2a = 2\pi\delta_2(\Sigma^2\hookrightarrow M^4), \tag{124}$$

where $\delta_2(\Sigma^2\hookrightarrow M^4)$ is a 2-form delta function with integral one over the directions transverse to the string world-sheet Σ^2. In rectangular coordinates we would have

$$\delta_2(\Sigma^2\hookrightarrow M^4) = \delta(x)\delta(y)dx\wedge dy, \tag{125}$$

and

$$\int_{D^2}\delta_2 = 1. \tag{126}$$

The reader might be wary of these manipulations of singular functions, and indeed we will see later that this treatment of the string source is too naive for some purposes, but for now this representation will suffice.

We can now finish the calculation of (122) using (124) to find

$$\delta\left[-\int_{M^4} da\wedge\omega_3^{0)}\right] = \int_{M^4} 2\pi\delta_2(\Sigma^2\hookrightarrow M^4)\wedge\omega_2^{(1)} = 2\pi\int_{\Sigma^2}\omega_2^{(1)}. \tag{127}$$

Thus the total action will be gauge invariant if the anomaly due to the string zero modes is given by

$$-2\pi\int_{\Sigma^2}\omega_2^{(1)}. \tag{128}$$

That is, if it is given by the descent procedure described in the previous lecture, starting with the Dirac index density in four dimensions, and with a factor of 2π multiplying the final result. Since we know the theory must be consistent overall, we could view this as a *derivation* that the two-dimensional gauge anomaly is given by 2π times the descent of the two higher-dimensional Dirac index density.

We can clearly generalize this construction to a theory with general non-Abelian anomalies and to theories in higher dimensions. Thus, consider the Lagrangian (105) in $2n+2$ dimensions with electromagnetism replaced by an arbitrary gauge group, and with the fermions in a representation $\mathbf{r}$ of the gauge group. As before, we assume the global Peccei-Quinn symmetry is spontaneously broken, and isolate the axion field $a(x)$. In generalizing this construction to higher dimensions and particularly to superstring theory, it is useful to introduce the dual of the derivative of the axion field, $H_{2n+1} = {}^*da$. The theory has axion "strings," except that since we are now in $2n+2$ dimensions, these are actually $2n-1$-branes. They couple to B_{2n} with $H_{2n+1} = dB_{2n}$ such that

$$d^*H_{2n+1} = 2\pi\delta_2(\Sigma^{2n} \hookrightarrow M^{2n+2}). \tag{129}$$

This generalization has non-chiral fermions in $2n+2$ dimensions, but one can verify that on the axion $(2n-1)$-brane there are chiral fermions in the same representation $\mathbf{r}$ of the gauge group as the bulk fermions. These fermions have an anomaly which must be canceled by inflow from the bulk.

The bulk inflow is provided by a coupling which arises as before, by using the chiral anomaly of the bulk theory to integrate out the bulk fermions which leaves one with the bulk coupling

$$-\int_{M^{2n+2}} {}^*H_{2n+1} \wedge (ch(F))^{(0)}_{2n+1}\,. \tag{130}$$

Here we have introduced a generalization of our previous notation where the superscript (0) means the form one obtains by applying the first step of the descent procedure to the quantity in brackets (in this case the $2n+2$-form part of $ch(F)$). Working out the gauge variation of this term in the presence of the $2n-1$ brane as before shows that the inflow precisely cancels the zero mode anomaly provided that the zero mode anomaly is given by the descent procedure just discussed.

4.3 Gravitational anomaly cancellation

One interesting generalization of the previous results arises when we couple the theory to gravity. In doing this we will encounter subtleties which presage some of the problems which arise in the theory of fivebranes in string theory and M theory. This analysis also provides a nice application of some of the formalism we developed in lecture 3 involving the manipulation of characteristic classes.

The gravitational contribution to the Dirac index density in $D = 4$ is the 4-form part of $\hat{A}(R)$ which from (71) is $-p_1(R)/24$. There will thus be a coupling to the axion of the form (120) but with

$$\omega_4 = \frac{1}{8\pi^2} F \wedge F - \frac{p_1(R)}{24}. \tag{131}$$

To figure out the implications of this new term for anomaly cancellation, we first need to ask about the symmetries of the axion string configuration. Of the $SO(3,1)$ local Lorentz symmetry, only $SO(1,1) \times SO(2)$ leaves the axion string configuration invariant. Thus we should check that these symmetries are not anomalous. Geometrically, $SO(1,1)$ transformations act on the tangent bundle to the string world-sheet, $T\Sigma^2$, while the $SO(2)$ transformations act as gauge transformations on N, the normal bundle to the string world-sheet. Corresponding to this decomposition, we can decompose the tangent bundle restricted to the string world-sheet as

$$TM|_{\Sigma^2} = T\Sigma^2 \oplus N. \tag{132}$$

We can then use (74) to deduce that $p_1(TM) = p_1(T\Sigma^2) + p_1(N)$. And, following through the rest of the calculation for anomaly inflow for the axion string, we see that there is a gravitational inflow contribution to the anomaly given by the descent of

$$I^{\text{inflow}} = -\left(p_1(T\Sigma^2) + p_1(N)\right)/24. \tag{133}$$

Now let us compare this to the anomaly of the chiral zero modes, again focusing only on the gravitational contribution. The normal bundle N has an $SO(2)$ structure group. The curvature can be represented by a 2×2 antisymmetric matrix. Let the skew eigenvalues be $\pm x$. Thus $p_1(N) = x^2$. However, the fermion zero modes transform in the spinor representation of $SO(2)$, $S(N)$, so the curvature in this complex representation has eigenvalues $\pm x/2$. The total gravitational anomaly of the fermion zero modes is thus given by descent of

$$-\frac{1}{2} chS(N)\hat{A}(T\Sigma^2)|_4. \tag{134}$$

From the discussion above we have

$$chS(N) = e^{x/2} + e^{-x/2} = 2 + x^2/4 + \cdots = 2 + \frac{p_1(N)}{4} + \cdots \tag{135}$$

which gives for the total anomaly

$$I^{\text{zeromode}} = \frac{p_1(T\Sigma^2)}{24} - \frac{p_1(N)}{8}. \tag{136}$$

Adding (133) and (136) gives

$$I^{\text{total}} = I^{\text{inflow}} + I^{\text{zeromode}} = -\frac{p_1(N)}{6}. \tag{137}$$

So, while the tangent bundle anomaly cancels, it appears that the normal bundle anomaly does not! The resolution of this puzzle was first pointed out in [45] in an analysis of fivebrane anomalies in IIA string theory. One important ingredient is to note that since N is an even-dimensional bundle, the top Pontrajin class can be factorized in terms of the Euler class, $p_1(N) = e^2(N)$ with $e(N)$ the Euler class. We can thus write the uncanceled anomaly as $-e^2(N)/6$.

A second key ingredient to canceling the normal bundle anomaly is to realize that the definition of $\delta_2(\Sigma^2 \hookrightarrow M^4)$ requires modification in a theory that includes gravity. The modifications are somewhat complicated and will be discussed in detail in the following lecture. For now, the only fact we will need is that the modifications ensure that the connection A on the normal bundle is such that

$$dH_1|_{\Sigma^2} = \delta_2|_{\Sigma^2} = e(F), \tag{138}$$

where $e(F)$ is the representative of the Euler class for the connection A with curvature F which is induced from the spin-connection in spacetime.

As a result of these two facts, we see that there is a local counterterm we can add to the effective action on the axion string which will cancel the normal bundle anomaly. Namely, we add the term

$$\int_{\Sigma^2} \frac{1}{6} H_1|_{\Sigma^2} \wedge (e(F))_1^{(0)}. \tag{139}$$

Computing the variation of this term under gauge transformations of the normal bundle and using (138) we see that its variation precisely cancels the anomaly (137) .

4.4 Anomalous couplings on D-branes

Another nice example of the inflow mechanism occurs in the study of Chern-Simons or anomalous couplings on D-branes [46,47,48]. Consider type II string theory and define a formal sum of RR potentials as

$$C = C_1 + C_3 + \cdots \qquad IIA \tag{140a}$$

$$C = C_0 + C_2 + \cdots \qquad IIB \tag{140b}$$

In the absence of D-branes, the corresponding field strength is $H = dC$. On a Dp-brane with worldvolume B^{p+1} there is a coupling of C to anomalous or Chern-Simons terms given by

$$\int_{B^{p+1}} C \wedge ch(F) \frac{\sqrt{\hat{A}(TB^{p+1})}}{\sqrt{\hat{A}(NB^{p+1})}} \tag{141}$$

This coupling was deduced using anomaly inflow arguments [46,47,48] and played an important role in suggesting the importance of K-theory in the classification of D-brane charges [48,49].

We will discuss part of this coupling by considering the special case of two D5-branes in IIB string theory which intersect along a 1-brane, $D5_1 \cap D5_2 = I1$. For example we can take the $D5_1$ worldvolume Σ_1^6 to lie along the $0,1,2,3,4,5$ directions, and the $D5_2$ worldvolume Σ_2^6 to lie along the $0,1,6,7,8,9$ directions and then the $I1$ brane at the intersection has its worldvolume Σ^2 along $0,1$. We will also focus only on the terms involving the gauge field, leaving the generalization to the gravitational couplings as an exercise. Assume that $D5_{1,2}$ has Chan-Paton labels $N_{1,2}$ so that there is a $U(N_{1,2})$ gauge group on $D5_{1,2}$. The zero modes that are localized on the intersection come from open string which start on one $D5$ and end on the other. They thus transform in the $(\overline{N}_1, N_2) + (N_1, \overline{N}_2)$ representation of the $U(N_1) \times U(N_2)$ gauge group. By working through the standard quantization of open strings, or by thinking about the zero modes resulting from the supersymmetries broken by the intersection, one sees that these zero modes are chiral on the $I1$ worldvolume. They thus have a gauge anomaly which is determined by descent from the 4-form

$$\begin{aligned} I^{\text{zeromodes}} &= \frac{1}{2}\Big(ch_{(\overline{N}_1,N_2)}(F_1) + ch_{(N_1,\overline{N}_2)}(F_2)\Big)\Big|_4 && \text{(142a)} \\ &= ch_{N_1}(F_1) ch_{N_2}(F_2)\Big|_4 && \text{(142b)} \\ &= N_1 c_2(F_2) + N_2 c_2(F_1) + c_1(F_1) c_1(F_2). && \text{(142c)} \end{aligned}$$

The factor of $1/2$ in (142) accounts for the reality of the fermion representation.

Since there are no gauge fields in the ten-dimensional bulk, the anomaly (142) must be cancelled by inflow from the $D5$-branes onto the $I1$-brane at the intersection. To see how this happens we study the possible anomalous couplings of the RR potentials to gauge fields. We assume that the D5-branes have couplings of the form

$$S_{\text{anom}} = \sum_i \left[-\frac{1}{2} \int_{\Sigma_i^6} \Big(N_i C - H \wedge Y_i^{(0)}(F_i) \Big) \right] \tag{143}$$

where $i = 1, 2$ labels the two $D5$-branes.

Some words of explanation are in order regarding this ansatz. The first term in (143) simply expresses the fact that the $D5$-brane acts as a source for C (C_6 to be precise) with strength N_i. The second term involves a characteristic form $Y(F) = dY(F)^{(0)}$ and an integration by parts as compared to this term written in terms of C for the same reason as in (121) : C is not single-valued in the presence of D-branes. Finally, the factor of $1/2$ accounts for the fact that this action is written in terms of both "electric" and "magnetic" potentials. See [47] for details.

The action (143) leads to the equations of motion/Bianchi identities

$$dH = -\delta_4(\Sigma_1^6 \hookrightarrow M^{10}) \wedge Y_1^{(0)}(F_1) - \delta_4(\Sigma_2^6 \hookrightarrow M^{10}) \wedge Y_2^{(0)}(F_2) \qquad (144)$$

Gauge invariance of H then requires that C vary under gauge transformations as

$$\delta C = \delta_4(\Sigma_1^6 \hookrightarrow M^{10}) \wedge Y_1^{(1)} + \delta_4(\Sigma_2^6 \hookrightarrow M^{10}) \wedge Y_2^{(1)}. \qquad (145)$$

Using this, we can compute the gauge variation of (143) :

$$\delta S_{\rm anom} = -\int_{M^{10}} \delta_4(\Sigma_1^6 \hookrightarrow M^{10}) \wedge \delta_4(\Sigma_2^6 \hookrightarrow M^{10}) \wedge (Y_1 \wedge Y_2)^{(1)} \qquad (146a)$$

$$= -\int_{\Sigma^2} (Y_1 \wedge Y_2)^{(1)} \qquad (146b)$$

We thus see that the anomaly will cancel between (146) and (142) provided that $Y(F) = ch(F)$.

By doing such an analysis systematically and including tangent and normal bundle gravitational anomalies one can derive the full set of couplings given in (141) . For details see [46,47,48]. The dynamics of this system and the role played by anomaly inflow have recently been analyzed in great detail in [51].

4.5 Exercises for Lecture 4.

- Verify that (115) is a normalizable solution to (114) .
- Compute the gravitational contribution to the anomaly of the zero modes on the intersection of two $D5$-branes, and verify that the anomaly is cancelled by inflow from the interaction (141) .

5 Lecture 5: M5-brane anomalies

Since M theory is thought to be the mother theory from which all string theories arise, it is natural to study the cancellation of anomalies for the extended objects of M theory. M theory is known to have two types of BPS extended objects, membranes and fivebranes which we will denote as M2 and M5. Since the M2-brane has an odd-dimensional world-volume it does not have anomalies in continuous symmetries. There is a parity anomaly which is connected to the quantization condition of the 4-form field strength of M theory [50].

The M5 brane is a more interesting and subtle object. For charge $Q_5 = 1$ it has zero modes which comprise a tensor multiplet of six-dimensional $(2,0)$ supersymmetry [52,53]. The free field theory of these zero modes has a $Spin(5)_R$ symmetry and the tensor multiplet contains a $Spin(5)_R$ singlet, a 2-form B_2^+ with self-dual field strength, $H_3 = dB_2^+ = {}^*H_3$, chiral fermions ψ transforming in the spinor representation of $Spin(5)_R$, and five scalar fields transforming as the vector of $Spin(5)_R$.

For fivebrane charge $Q_5 > 1$ the M5-brane is not well understood. A first principles definition of the theory does not exist, although there are partial results on the primary conformal fields and correlations functions which follow from a Matrix theory formulation of the theory and from application of the AdS/CFT correspondence to the $AdS_7 \times S^4$ near horizon geometry of the M5-brane. We will see that anomaly inflow yields some additional information about this theory.

So far we have only discussed anomalies for chiral spin 1/2 fermions. It is clear however that there are potential anomalies for other chiral fields. For example, in $1+1$ dimensions we can have chiral scalars ϕ obeying $d\phi = \pm {}^* d\phi$ which are equivalent upon fermionization to chiral fermions and therefore must contribute to the gravitational anomaly. Similarly, in $D = 6, 10$ one can have bosonic 2-form and 4-form potentials with self-dual or anti-self-dual field strengths. The duality constraint means that the fields transform chirally under the Lorentz group. There is no way to regulate a theory with such fields without violating Lorentz invariance, so there is a potential gravitational anomaly. These gravitational anomalies were analyzed in [54] and we will simply apply their results here without further discussion.

5.1 Tangent bundle anomalies and bulk couplings

For a charge $Q_5 = 1$ M5-brane the gravitational anomaly on the M5 has contributions from the fermions and the self-dual 2-form. Since the world-volume W^6 is six-dimensional, the descent formalism implies that both contributions are summarized as in our previous discussions by an 8-form characteristic class. In this section we will focus only on the tangent bundle anomaly. The normal bundle anomaly will require a considerably more complicated treatment. The fermion contribution to the tangent bundle anomaly is given by

$$I_8^{\rm ferm} = 2\hat{A}(TW^6)|_{8-{\rm form}} = \frac{1}{5760}\left(14p_1^2(TW^6) - 8p_2(TW^6)\right) \tag{147}$$

The factor of two in front of $\hat{A}$ arises because there are four fermions (transforming as a **4** of $Spin(5)_R$), but they obey a Majorana constraint reflecting their origin as Majorana spinors in eleven dimensions which reduces the anomaly by a factor of 1/2.

It follows from the results of [54] that the 2-form contribution is given by

$$I_8^{B^+} = \frac{1}{5760}\left(16p_1^2(TW^6) - 112p_2(TW^6).\right) \tag{148}$$

The total tangent bundle anomaly is thus

$$I_8^{\rm total} = I_8^{\rm ferm} + I_8^{B^+} = \frac{1}{192}\left(p_1^2(TW^6) - 4p_2(TW^6).\right) \tag{149}$$

This anomaly must be cancelled by inflow from the bulk in a way which is quite analogous to what happens for the axion string. The M5-brane acts as a magnetic source of C_3 via

$$dG_4 = 2\pi\delta_5(W^6 \hookrightarrow M^{11}) \tag{150}$$

with G_4 the field strength for C_3. We can therefore cancel the anomaly via inflow [55] if there is a bulk coupling given by

$$\int_{M^{11}} C_3 \wedge X_8 \tag{151}$$

with

$$X_8 = -\frac{1}{192}\left(p_1^2(TM^{11}) - 4p_2(TM^{11})\right). \tag{152}$$

The verification that this cancels the tangent bundle anomaly is completely analogous to the demonstration for the axion string and is left as a small exercise for the reader.

5.2 The normal bundle anomaly

Although the previous section shows how the tangent bundle anomaly cancels between the world-brane and bulk contributions, as in our discussion of the axion string, we still need to analyze possible anomalies in diffeomorphisms which act as $SO(5)$ gauge transformations on the normal bundle as well as mixed tangent bundle-normal bundle anomalies. The M5-brane background breaks the $D = 11$ Lorentz symmetry $Spin(10,1)$ to $Spin(5,1) \otimes Spin(5)$ and corresponding to this we can decompose the restriction of the spacetime tangent bundle to the fivebrane world-volume as

$$TM^{11}|_{W^6} = TW^6 \oplus N \tag{153}$$

with N the normal bundle. The fact that there are diffeomorphisms acting as $SO(5)$ gauge transformations may be more familiar to some readers in the context of the AdS/CFT correspondence [56]. In the near horizon limit the M5 geometry becomes $AdS_7 \times S^4$ which can be viewed as an S^4 Kaluza-Klein compactification of $D = 11$ supergravity. The resulting supergravity on AdS_7 has an $SO(5)$ gauge group coming from the isometry group of S^4.

There are two obvious contributions to the normal bundle anomaly coming from the M5-brane zero modes and via inflow from the bulk term (151) determined above by cancellation of the tangent bundle anomaly.

The antisymmetric tensor field does not contribute to the normal bundle anomaly since it is a singlet under $SO(5)$. The fermion fields transform as a **4** under $Spin(5)$, that is they take values in the rank four spin bundle $S(N)$. The total anomaly due to the fermion fields is thus the 8-form part of

$$I_8^{\rm ferm} = \frac{1}{2} chS(N)\hat{A}(TW^6)|_8 \tag{154}$$

It is useful to represent the Chern classes of $S(N)$ in terms of Pontrajin classes. This can be done by writing the curvature of the normal bundle as in (68) and noting that the eigenvalues of the curvature of $S(N)$ will then be $\pm(x_1+x_2)/2$, $\pm(x_1 - x_2)/2$. Using this and the definition of $ch(F)$ from (104) gives

$$\begin{aligned} chS(N) &= e^{(x_1+x_2)/2} + e^{(x_1-x_2)/2} + e^{(-x_1+x_2)/2} + e^{(-x_1-x_2)/2} && \text{(155a)} \\ &= 4 + (x_1^2 + x_2^2)/2 + (x_1^4 + x_2^4 + 6x_1^2x_2^2)/96 + \cdots && \text{(155b)} \\ &= 4 + \frac{p_1(N)}{2} + \frac{p_1(N)^2}{96} + \frac{p_2(N)}{24} + \cdots && \text{(155c)} \end{aligned}$$

The contribution from the bulk term is easily computed using

$$p_1(TM^{11}|_{W^6}) = p_1(TW^6) + p_1(N) \tag{156}$$

and

$$p_2(TM^{11}|_{W^6}) = p_2(TW^6) + p_2(N) + p_1(TW^6)p_1(N) \tag{157}$$

to give

$$I_8^{\rm inflow} = -\frac{1}{48}\left(\frac{p_1(TW^6)^2 + p_1(N)^2 - 2p_1(TW^6)p_1(N)}{4} - p_2(TW^6) - p_2(N)\right) \tag{158}$$

Adding (158) and (154) using (155) gives for the total anomaly

$$I_8^{\rm ferm} + I_8^{\rm inflow} = \frac{p_2(N)}{24}. \tag{159}$$

So as in the axion string case, there is a normal bundle anomaly which does not cancel between the zero modes and bulk terms. However unlike the axion string case, here the solution is not so clear. Note that if we were considering the IIA fivebrane in $D = 10$ then the normal bundle would be a $SO(4)$ bundle, $p_2(N)$ would be the square of the Euler class of this bundle, and the anomaly could be cancelled by adding a term to the fivebrane worldvolume in analogy to what we did for the axion string as was first shown in [45]. The relationship between the anomaly cancellation discussed below for the M5-brane and that for the IIA five-brane can be found in [57].

This will not work for the M5-brane. The Euler class vanishes for odd rank $SO(N)$ bundles and there is simply no way to factorize the uncancelled anomaly. Thus something new is required. What this something new is, is not a priori obvious. One might think that since the extremal M5-brane is a smooth solution to $D = 11$ supergravity, we should try to study the diffeomorphism invariance directly in supergravity. This would involve things like the study of the Rarita-Schwinger operator in the M5-brane background geometry. However we have seen that the study of anomalies really requires a study of families of backgrounds, and perturbations of the M5-brane geometry are generically singular. Thus studying the Rarita-Schwinger operator in a sufficiently general background is a daunting task. To my knowledge no serious attempts have been made to study the problem this way. One might also hope to address the problem in Matrix Theory since it claims to be a fundamental formulation of M theory [58]. It is clear though that this is not practical with current technology. We don't even have a proof of $D = 11$ Lorentz invariance in Matrix Theory so we are ceratinly not in a position to be looking for quantum violations of local Lorentz invariance.

There is a formalism which allows one to cancel the remaining normal bundle anomaly which we will now discuss. It involves a more careful treatment of the M5-brane source term and the structure of $D = 11$ supergravity in the presence of fivebranes. It probably should not be viewed as a final understanding of the problem. One would eventually hope for a microscopic formulation of M theory which makes some of the manipulations in the following section appear more natural.

5.3 Bump forms and Thom classes

So far in our discussion of M5-brane anomalies, and in fact for anomalies on all branes, we have treated the branes as singular sources. Thus we have treated the M5-brane as a singular magnetic source for the 3-form field with

$$dG_4 = 2\pi\delta_5(W^6 \hookrightarrow M^{11}). \tag{160}$$

In our discussion of normal bundle anomalies for the axion string we already saw that this was not completely correct, or at least not completely well specified without a more precise definition of δ_5, because the source term must depend in a non-trivial way on the connection on the normal bundle. In addition, we might expect that singular sources might be problematic in situations where the equations of motion are non-linear. In M theory this occurs even for the 3-form field alone due to the Chern-Simons like term in the action

$$S_{CS} = -\frac{2\pi}{6}\int_{M^{11}} \frac{C_3}{2\pi}\wedge\frac{G_4}{2\pi}\wedge\frac{G_4}{2\pi}. \tag{161}$$

In this section we will describe a method of smoothing out the source term and including the dependence on the connection on the normal bundle. This will involve a very lowbrow presentation of mathematics which is covered more rigorously but still in an accessible way in [59]. We will see that it also becomes necessary to modify the term (161) in the presence of an M5-brane, or it might be more precise to say, give a careful definition of (161) in the presence of a M5-brane. In the following section we show that this modification or definition cancels the remaining anomaly (159) .

To begin with we will make sure that bulk integrals in the presence of an M5-brane are well defined. We do this by cutting out a region of radius ϵ around the M5-brane. At each point on W^6 we thus have a disk of radius ϵ. We will denote the total space of this disk bundle over W^6 by $D_\epsilon(W^6)$. To define bulk integrals we cut out this "tubular neighborhood" of the fivebrane and then take the limit as $\epsilon \to 0$:

$$\int_{M^{11}} \mathcal{L}_{bulk} \equiv \lim_{\epsilon\to 0}\int_{M^{11}-D_\epsilon(W^6)} \mathcal{L}_{bulk} \tag{162}$$

Later on we will see that certain bulk terms are boundary terms and we will then use the fact that the boundary of $D_\epsilon(W^6)$ is $S_\epsilon(W^6)$, the sphere bundle of radius ϵ over W^6.

Next we smooth out the fivebrane source. The anomaly should be insensitive to the precise profile of the fivebrane, so we introduce a function of the radial direction away from the fivebrane, $\rho(r)$ with $\rho \to -1$ as $r \to 0$ and with

ρ vanishing for r larger than some finite value. The derivative $d\rho$ is called a bump form and integrates to 1 in the radial direction, $\int d\rho = 1$.

If we were considering a fivebrane in "flat space" with a vanishing connection on the normal bundle then we could write the smoothed out source term as

$$\delta_5(W^6 \to M^{11}) \sim d\rho \epsilon_{a_1 \cdots a_5} d\hat{y}^{a_1} \cdots d\hat{y}^{a_4} \hat{y}^{a_5} \tag{163}$$

where $\hat{y}^a = y^a/r$ are isotropic coordinates normal to the M5-brane. We choose the proportionality constant in (163) so that the r.h.s. has integral one over the space transverse to the fivebrane. This can then be viewed as an approximation to a delta function by choosing ρ to be supported in an arbitrarily small neighborhood of the origin.

To study normal bundle anomalies we need to vary the normal bundle connection and we need a generalization of (163) which is properly covariant under $SO(5)$ gauge transformations of the normal bundle. We will write the fivebrane source equation as

$$d(G_4/2\pi) = d\rho \wedge e_4/2 \tag{164}$$

and determine e_4 by demanding that it obey the following conditions.

- Taking the exterior derivative of (164) implies that $de_4 = 0$.
- e_4 should be covariant under $SO(5)$ gauge transformations.
- The integral of $e_4/2$ over the fibres of $S_\epsilon(W^6)$ should equal one in order that we have the correctly normalized fivebrane charge.
- The expression for $e_4/2$ should be proportional to $\epsilon_{a_1 \cdots a_5} d\hat{y}^{a_1} \cdots d\hat{y}^{a_4} \hat{y}^{a_5}$ when the $SO(5)$ connection on N is trivial.

To make e_4 covariant it is natural to replace ordinary derivatives by covariant derivatives. Recall that for a principal fibre bundle with connection Θ^{ab} we can split the tangent space at a point x into vertical and horizontal components $T_x = V_x \otimes H_x$ with basis $(\partial/\partial\hat{y}^a, D_\mu = \partial/\partial x^\mu - \Theta^{ab}_\mu \hat{y}^a \partial/\partial\hat{y}^b)$.

To construct e_4 we want to generalize $d\hat{y}^a$ to the 1-form $(D\hat{y})^a$ in T^*_x which is dual to $\partial/\partial\hat{y}^b$, that is which satisfies

$$\langle (D\hat{y})^a, D_\mu \rangle = 0, \qquad \langle (D\hat{y})^a, \partial/\partial\hat{y}^b \rangle = \delta^{ab}. \tag{165}$$

This gives $(D\hat{y})^a = d\hat{y}^a - \Theta^{ab}\hat{y}^b$.

It thus seems natural to guess that

$$e_4 \sim \epsilon_{a_1 \cdots a_5} (D\hat{y})^{a_1} \cdots (D\hat{y})^{a_4} \hat{y}^{a_5}. \tag{166}$$

However this is not correct since it is easy to see that (166) is not closed using

$$d(D\hat{y})^a = \Theta^{ab}(D\hat{y})^b - F^{ab}\hat{y}^b \tag{167}$$

where $F^{ab} = d\Theta^{ab} - \Theta^{ac} \wedge \Theta^{cb}$ is the field strength of Θ. Nonetheless it is possible to add additional terms to cancel these unwanted terms and to construct a form for e_4 which is both covariant and closed. This leads to[a]

$$e_4 = \frac{1}{32\pi^2}\epsilon_{a_1\cdots a_5}[(D\hat{y})^{a_1}\cdots(D\hat{y})^{a_4}\hat{y}^{a_5} \tag{168a}$$

$$- 2F^{a_1a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4}\hat{y}^{a_5} + F^{a_1a_2}F^{a_3a_4}\hat{y}^{a_5}] \tag{168b}$$

This expression for e_4 satisfies all the conditions above so we take as our smoothed out, covariant fivebrane source $d\rho\wedge e_4/2$. The quantity $e_4/2$ is known in the mathematics literature as the global angular form and the source term $d\rho\wedge e_4/2$ is a particular representative of the Thom class of the normal bundle.

5.4 Anomaly cancellation

We have found that the properly defined fivebrane source has metric dependence. It particular it will vary under diffeomorphisms which act as $SO(5)$ gauge transformations on the normal bundle. We can derive this variation by applying the descent procedure to e_4 since it is a characteristic class. We thus have

$$e_4 = de_3^{(0)} \tag{169a}$$

$$\delta e_3^{(0)} = de_2^{(1)} \tag{169b}$$

We can then reexpress the uncancelled normal bundle anomaly by using a result of Bott and Cattaneo [60]:

$$\int_{S_\epsilon(W^6)} e_4 \wedge e_4 \wedge e_2^{(1)} = 2\int_{W^6} p_2^{(1)}(N). \tag{170}$$

The left hand side of (170) gives a cubic form for the uncanceled anomaly which is reminiscent of the Chern-Simons coupling of $D = 11$ supergravity

$$S_{CS} = -\frac{2\pi}{6}\int_{M^{11}}(C_3/2\pi)\wedge d(C_3/2\pi)\wedge d(C_3/2\pi). \tag{171}$$

and so suggests that we think carefully about the proper definition of this term in the presence of M5-branes.

[a] This corrects [57] the expression given in [63] by a factor of two.

In the absence of M5-branes we have $G_4 = dC_3$ but in the presence of M5-branes we have found a smoothed out source equation

$$d(G_4/2\pi) = d\rho \wedge e_4/2 \tag{172}$$

which is not compatible with $G_4 = dC_3$. We can integrate (172) to find

$$(G_4/2\pi) = d(C_3/2\pi) + A\rho e_4/2 - Bd\rho e_3^{(0)}/2 \tag{173}$$

with $A + B = 1$. Here C_3 describes flucutations about the M5-brane background. Since we have smoothed out the source, physical quantities like G_4 should be smooth at the origin. This requires that $A = 0$ since e_4 is singular at $r = 0$ much like $d\theta$ is singular at the origin in polar coordinates. So in the presence of a M5-brane the correct relation between G_4 and C_3 is

$$(G_4/2\pi) = d(C_3/2\pi) - d\rho \wedge e_3^{(0)}/2. \tag{174}$$

This is roughly analogous to the relation $H_3 = dB_2 - \omega_3$ which occurs in the low-energy theory of the heterotic string and which plays an important role in the Green-Schwarz anomaly cancellation mechanism[61]. In particular, for G_4 to be invariant under $SO(5)$ gauge transformations, C_3 must vary according to

$$\delta(C_3/2\pi) = -d\rho \wedge e_2^{(1)}/2 \tag{175}$$

and this implies the descent relations

$$\begin{aligned} (G_4/2\pi) - \rho e_4/2 &= d((C_3/2\pi) - \sigma_3) && \text{(176a)} \\ \delta((C_3/2\pi) - \sigma_3) &= -d(\rho e_2^{(1)}/2) && \text{(176b)} \end{aligned}$$

where we have defined $\sigma_3 \equiv \rho e_3^{(0)}/2$.

We now have to decide what the proper form of (171) should be in the presence of a fivebrane. There is clearly some ambiguity since dC_3 and G_4 differ in the presence of a fivebrane. A natural choice which also cancels the anomaly is to preserve the Chern-Simons form of (171) , that is we try to generalize (171) in the presence of a fivebrane so that it still has the form $\int x \wedge dx \wedge dx$ for some 3-form x. Using the descent relation (176) suggests that the correct form for the Chern-Simons coupling is

$$S'_{CS} = \lim_{\epsilon \to 0} -\frac{2\pi}{6} \int_{M^{11} - D_\epsilon(W^6)} ((C_3/2\pi) - \sigma_3) \wedge d((C_3/2\pi) - \sigma_3) \wedge d((C_3/2\pi) - \sigma_3) \tag{177}$$

where we have also recalled the definition of bulk integrals (162) .

We can now compute the variation of the modified Chern-Simons term under diffeomorphsims acting as gauge transformations on the $SO(5)$ normal bundle. Using (176) we have

$$\delta S'_{CS} = \lim_{\epsilon\to 0}\int_{M^{11}-D_\epsilon(W^6)} \frac{2\pi}{6} d(\rho e_2^{(1)}/2)\wedge((G_4/2\pi)-\rho e_4/2)^2 \quad (178a)$$

$$= -\lim_{\epsilon\to 0}\int_{S_\epsilon(W^6)} \frac{2\pi}{6}\rho\frac{e_2^{(1)}}{2}\wedge\rho\frac{e_4}{2}\wedge\rho\frac{e_4}{2}+O(\epsilon) \quad (178b)$$

$$= 2\pi\int_{W^6}\frac{p_2^{(1)}(N)}{24} \quad (178c)$$

where we have integrated by parts, used the fact that G_4 is smooth near the fivebrane, and applied the Bott-Catteneo formula (170) .

We thus find that the normal bundle anomaly does indeed cancel, but only after properly including the normal bundle dependence in the fivebrane source and modifying the supergravity action in the presence of a fivebrane.

Although this mechanism must be correct in some sense given the intricate way that the different pieces fit together, it raises as many questions as it answers. First, the description we have given is in the spirit of a low-energy effective action. It would be nice to have a microscopic derivation of the modified source terms and Chern-Simons couplings. For a treatment of these questions in simpler models see [66,67]. These modifications are of course not unique, and some other forms have been proposed which also cancel the anomaly [68]. Some hints at a more elegant formulation of anomaly cancellation can be found in [69]. Second, we have modified some of the bosonic terms in the supergravity Lagrangian but supersymmetry will clearly require modifications to fermion terms as well. These have not been worked out to my knowledge.

5.5 Applications of M5-brane anomaly cancellation

The cancellation of anomalies for the M theory fivebrane has some interesting applications. These will be described only briefly here. For further details see [62,70,71,72].

Earlier we used the known zero modes of the charge $Q_5 = 1$ M5-brane to deduce the bulk coupling $\int C_3 X_8$ given in (152) . In the previous section we have understood all the bulk contributions necessary to cancel both the tangent and normal bundle anomalies, again for $Q_5 = 1$. For $Q_5 > 1$ we do not know very much about the $(2,0)$ supersymmetric theory of the zero modes of the M5-brane. In particular we do not know how to compute the zero mode contribution to the tangent and normal bundle anomalies. However we do understand the anomaly inflow of the bulk couplings for $Q_5 > 1$ and so we can use these to predict what the zero mode contributions must be, assuming that the anomalies do indeed cancel.

For a charge Q_5 fivebrane G_4 obeys the equation

$$dG_4 = 2\pi Q_5 d\rho \wedge e_4/2 \tag{179}$$

that is, G_4 scales linearly with Q_5. The two bulk terms which contribute to the anomaly are the $\int C_3 X_8$ term which scales like Q_5 and the Chern-Simons term which scales like Q_5^3. Anomaly cancellation therefore predicts that the M5-brane zero mode contribution to the anomaly should be

$$I_8^{zm}(Q_5) = Q_5 I_8^{zm}(1) + (Q_5^3 - Q_5)\frac{p_2(N)}{24} \tag{180}$$

with

$$I_8^{zm}(1) = \frac{1}{48}\left[p_2(N) - p_2(TW^6) + \frac{1}{4}(p_1(TW^6) - p_1(N))^2\right] \tag{181}$$

In the $(2,0)$ theory the $SO(5)$ R symmetry current, whose anomaly we have just deduced, is in the same supermultiplet as the energy-momentum tensor. So in principle the anomaly (180) has interesting implications for various correlation functions of the $(2,0)$ theory. In particular, the Q_5^3 dependence in (180) is consistent with the Q_5^3 dependence found in the entropy of black holes related to the $(2,0)$ theory[73,74] and calculations of the conformal anomaly using the AdS/CFT correspondence[75].

We have assumed above that the theory is at the origin of moduli space, with all Q_5 M5-branes coincident and the $SO(5)_R$ symmetry unbroken. As we move away from the origin by separating the M5-branes we should integrate out the fields which become massive to derive the low-energy effective theory of separated fivebranes. This low-energy theory naively consists of Q_5 free $(2,0)$ theories. However, as is well known from analogous considerations in the analysis of chiral symmetries in QCD, the story is more complicated. Wess-Zumino terms are generated when we integrate out fermions which contribute to the anomaly, and these WZ terms have an anomalous variation which ensures that the anomaly matches throughout moduli space. These WZ terms for the $(2,0)$ theory were worked out in [70] and have some interesting implications for the structure of $(2,0)$ theories.

The $(2,0)$ theory is known to have self-dual string excitations[76,77,78]. From the spacetime point of view, these arise as the boundaries of M2-branes ending on M5-branes. These self-dual strings carry fermion zero modes with a non-zero anomaly, and techniques similar to those described here can be used to deduce information about the scaling of these anomalies [71] with Q_5 and Q_2, the number of M2-branes or equivalently, the charge of the self-dual string.

In [64] a model of black holes in theories with $N = 2$ supersymmetry was introduced which uses the properties of M theory fivebranes. One considers a compactification of M theory on a Calabi-Yau space X and considers a M5-brane which wraps a supersymmetric four-cycle $P_4 \subset X$. In the resulting five dimensional theory the wrapped fivebrane appears as a one-brane or string. If one takes the direction along the string to be a circle and gives the string non-zero momentum q along the string, then the description of this configuration in $D = 4$ is that of a extremal black hole with non-zero entropy.

At low-energies the effective theory on the string is given by a $(4, 0)$ SCFT, and the entropy of the black hole in this model is determined by the left-moving conformal anomaly c_L and the charge q. In [64] c_L was determined by an intricate index theory computation. It can however be determined purely by anomaly inflow. The $(4, 0)$ SCFT has a $SU(2)_R$ affine Lie algebra related to the normal bundle of the string. The level k of this $SU(2)_R$ is determined directly by the normal bundle anomaly and is related by supersymmetry to the right-moving central charge c_R. On the other hand cancellation of the tangent bundle anomaly of the string determines $c_R - c_L$ so c_L can be computed purely by anomaly inflow. Recently, similar techniques have been applied to explain spacetime corrections to black hole entropy coming from higher order terms in the spacetime effective action [72].

5.6 Exercises for Lecture 5.

- Show that $\alpha_{n-1} = \epsilon_{a_1 \cdots a_n} dy^{a_1} \cdots dy^{a_{n-1}} y^{a_n}$ is proportional to the volume form on the $(n-1)$ sphere with the y^{a_i} being Cartesian coordinates on R^n. Hint: Use Stokes theorem and the fact that $d\alpha_{n-1}$ is proportional to the volume form on R^n.

- Compute the contribution to the five-brane anomaly from the interaction (151) and verify that the tangent bundle anomaly from this term cancels the tangent bundle anomaly from the five-brane zero modes.

- Following the treatment in section 5.3, construct a smoothed out source for a one-brane in four dimensions, $\delta_2(\Sigma^2 \hookrightarrow M^4)$ and verify that $\delta_2|_{\Sigma^2} = e(F)$, a result which was used in the analysis of normal bundle anomaly cancellation for axion strings.

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