Quantum gravity in terms of topological observables

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Abstract

We recast the action principle of four dimensional General Relativity so that it becomes amenable for perturbation theory which doesn't break general covariance. The coupling constant becomes dimensionless ($G_{Newton}\Lambda$) and extremely small 10^{-120}. We give an expression for the generating functional of perturbation theory. We show that the partition function of quantum General Relativity can be expressed as an expectation value of a certain topologically invariant observable. This sets up a framework in which quantum gravity can be studied perturbatively using the techniques of topological quantum field theory.

1 Introduction

It has long been believed that quantum general relativity is non-renormalizable. There is a solid argument for why it should be so. The coupling constant of General Relativity, the Newton constant, is dimensional and has to be multiplied by energy to form the actual coupling. Therefore, at high energies the theory becomes strongly coupled producing infinitely many types of divergent Feynman diagrams. This is understood as the breakdown of the theory below some scale sets by the Newton constant.

The above argument is not completely physical however. General Relativity is the theory with no preexisting metric. And we can ask: with respect to what metric we define the scale below which the theory breaks down? The answer is that this is the metric g_{0ab} that we use as the background for the expansion and defining the corrections to be quantized

$$g_{ab} = g_{0ab} + h_{ab}. \tag{1}$$

By fixing g_{0ab} we fix not only a particular classical solution, but also a coordinate system, and therefore break general covariance. Thus, the argument for non-renormalisability appears to rely on auxiliary structure which physics must be independent of.

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It is therefore worthwhile to explore the issue of renormalizability from a point of view which doesn't use any background metric. There are two possible outcomes of this effort. Either we get an argument for non-renormalizability which would use only relations between physical quantities, or we show that non-renormalizability is an artifact of non-covariant quantization based on (1) and goes away in a background independent quantization.

In a famous paper [1] Witten has shown that in 2+1 dimensional gravity, if we don't do the expansion (1), if we treat the whole geometry quantum mechanically, thus keeping the theory general covariant, we can avoid the ultraviolet problem. One can even show that the theory exactly soluble.

And the natural question that arises, which is also addressed in [1], is whether we can do the same in 3+1 dimensions. The immediate problem then is the following. If we look at the action of 2+1 dimensional General Relativity in the triad-Palatini representation

$$S = \int \mathrm{tr}(e \wedge dA + e \wedge A \wedge A) \tag{2}$$

we see that the lowest order term in it is quadratic. Thus the theory is nearly linear and we can apply standard quantum field theory techniques to it. On the other hand the action of 3+1 dimensional General Relativity in the tetrad-Palatini representation looks like

$$S = \int \mathrm{tr}(e \wedge e \wedge dA + e \wedge e \wedge A \wedge A). \tag{3}$$

The lowest order term in it is cubic. The standard quantum field theory techniques are not applicable anymore. The only way out seems to be to get a quadratic term in a action via the expansion (1), which leads to a non-renormalizable theory. The conclusion of [1] is that 3+1 dimensional General Relativity is non-renormalizable because it is too non-linear.

One of the questions that we address in this paper is: how non-linear is 3+1 dimensional General Relativity?

Below, to avoid complications with using a non-compact group we will consider Euclidian gravity with positive cosmological constant.

2 McDowell-Mansouri type BF-theory: How non-linear is 4 dimensional General Relativity?

There are several formulations of 4 dimensional general relativity known which do contain a quadratic term in the action [2]. They are based on BF-theory plus a term which breaks topological symmetry. A well known example is the so called Plebanski action[1]:

$$S = \int (B^{\mu\nu} \wedge F_{\mu\nu}(\omega) + \phi^{\mu\nu\alpha\beta} B_{\mu\nu} \wedge B_{\alpha\beta}) \tag{4}$$

[1] A similar formulation of gravity works in any dimension [3]

The first term in (4) is a BF-theory which is an exactly soluble theory and one could think that we can use it as a free field theory and treat the remaining term as a perturbation. There is a problem with such a perturbation theory. This is the fact that the second term actually imposes some constraints on $B^{\mu\nu}$, as $\phi^{\mu\nu\alpha\beta}$ is a Lagrangian multiplier. In a path integral this term becomes a delta function of the constraints, and to treat it as a perturbation we would have to expand a delta function in a power series around zero. But such an expansion doesn't exist. What we would need for a perturbation theory is an action principle in which General Relativity would be represented as an exactly soluble theory plus a regular interaction term. Such a formulation does exist and this is the McDowell-Mansouri formulation of General Relativity [4] rewritten as a BF-theory. Such kind of action principle was also considered in [5].

Our starting point will be the BF-theory for $SO(5)$ group. Let $T^{IJ} = -T^{JI}$ be ten generators of $so(5)$ Lie algebra, where $I, J = 1, ..., 5$ (see appendix A for our conventions and physical interpretation of other gauge groups). The basic dynamical variables are $so(5)$-connection A^{IJ} and $so(5)$-valued 2-form field B^{IJ}. The action principle is then

$$S = \int B^{IJ} \wedge F_{IJ}. \tag{5}$$

Here $F_{IJ} = dA_{IJ} + A_I^K \wedge A_{KJ}$ is the $so(5)$ curvature.

The equations of motion following from the action (5)

$$\begin{aligned} F_{IJ} &= 0 \\ d_A B_{IJ} &= 0 \end{aligned} \tag{6}$$

mean that the connection A^{IJ} is flat.

Now the statement is that if we break the $SO(5)$ symmetry in the theory (5) down to $SO(4)$ we get the action of General Relativity.

We add an extra term to the action (5) which depends only on B-field and contains a fixed $SO(5)$ vector v^A pointing in some preferred direction.

$$S_1 = \int (B^{IJ} \wedge F_{IJ} - \frac{1}{2} B^{IJ} \wedge B^{KL} \epsilon_{IJKLM} v^M). \tag{7}$$

The $SO(5)$ symmetry is not a symmetry of the action (7). It is broken down to $SO(4)$, the subgroup of $SO(5)$ rotations which leaves v^I immovable. For simplicity we choose $v^I = (0, 0, 0, 0, \alpha/2)$, where α is a fixed dimensionless constant. The action (7) then becomes

$$S_1 = \int (B^{IJ} \wedge F_{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL5}), \tag{8}$$

To show that (8) is the action of General Relativity we introduce the following notations for $4+1$-decomposition. Let $i, j = 1, 2, ..4$ be four dimensional indices such that $\epsilon^{ijkl} = \epsilon^{ijkl5}$. Then we can introduce $so(4)$-connection $\omega^{ij} = A^{ij}$ and its curvature $R^{ij}(\omega) = d\omega^{ij} + \omega^i_k \wedge \omega^{kj}$. Also, we can introduce a frame field $e^i = lA^{i5}$, where l is a constant of the dimension of length, giving rise to a four-dimensional metric $g_{\mu\nu} = e^i_\mu e_{\nu i}$.

In the above notations we have the following decomposition of $so(5)$-curvature:

$$F^{ij}(A) = R^{ij}(\omega) - \frac{1}{l^2} e^i \wedge e^j$$
$$F^{i5}(A) = \frac{1}{l} d_\omega e^i. \tag{9}$$

The equations of motion of (8) for B^{5i} impose the torsion to vanish $d_\omega e^i = 0$, this determines uniquely the connection ω to be the spin connection. Since the action is quadratic in the fields B^{ij} we can solve the equations of motion for B^{ij} and substituting them back into action, we find

$$S_1 = \frac{1}{4\alpha} \int F^{ij} \wedge F^{kl} \epsilon_{ijkl}, \tag{10}$$

where we used the notations introduced above. Finally, using (9) one can rewrite (10) as

$$\begin{aligned} S_1 &= \frac{1}{4\alpha} \int (R^{ij} - \frac{1}{l^2} e^i \wedge e^j) \wedge (R^{kl} - \frac{1}{l^2} e^k \wedge e^l) \epsilon_{ijkl} \\ &= S_P + \frac{1}{4\alpha} \int R^{ij}(\omega) \wedge R^{kl}(\omega) \epsilon_{ijkl} \end{aligned} \tag{11}$$

Here

$$S_P = -\frac{1}{2G} \int (R^{ij}(\omega) \wedge e^k \wedge e^l - \frac{\Lambda}{6} e^i \wedge e^j \wedge e^k \wedge e^l) \epsilon_{ijkl} \tag{12}$$

is the Palatini action[2] of General Relativity with nonzero cosmological constant. The role of the Newton constant[3] is played by $G = \alpha l^2$ and the cosmological constant is $\Lambda = 3/l^2$. The constant α in (5) is the square of the ratio of the Planck length over the cosmological radius, $\alpha = G\Lambda/3 \sim 10^{-120}$ is dimensionless and extremely small which makes it a good parameter for perturbative expansion.

The second term in the r.h.s. of (11) is the integral of the Euler class. It is topological and its variation vanishes identically due to Bianchi identity. Thus the action (7) indeed describes General Relativity.

The main result of this section is that General Relativity in four dimensions admits an action principle (8) which is just barely non-linear, exactly as non-linear as that of 3 dimensional gravity. Also, (8) has a form of exactly soluble theory plus a small correction. The correction is so small that even if we neglect it we should give a good approximation to the observed reality. And indeed we do, because the equations of motion in this case are (6) which in the case of $SO(4,1)$ gauge group have the only solution which is the deSitter spacetime, which is very close to what we see. This formulation of gravity is strikingly similar to a formulation of QCD in terms of the Lagrangian $tr(B \wedge F + g^2 B \wedge \star B)$. The

[2]The normalizations are such that when written in the metric variables the Palatini action is of the usual form

$$S_P = -\frac{1}{G} \int \sqrt{g}(R - 2\Lambda), \tag{13}$$

R being the scalar curvature.

[3]We work in units where c and $16\pi\hbar = 1$, so G means $16\pi G\hbar = l_p$ which is the Planck length.

difference coming from having a quadratic form contracting the B fields which is strictly positive and background dependent.

Despite the smallness of α, however, there are many situations in which the second term in the action (8) leads to noticeable effects. This happens when some of the components of B-field are large. Then the second term in (8) which is quadratic in B cannot be neglected as compared to the first term which is linear in B, even though multiplied by a tiny constant. In classical theory B field becomes large, for example, when we couple gravity to massive matter sources [6].

In quantum theory we have to take into account large fluctuations of B-field, thus including the regime in which the theory becomes strongly coupled. This may lead to a breakdown of perturbation theory. Most 'visible' is the contribution from the components of B which form the orbit of the 'translational' gauge group of the free field theory, $B = d_A\phi$, which is broken by the interaction term. To avoid this problem we need to find a way to suppress large fluctuations of B-field in a path integral. This can be done by a very natural modification of the action principle considered in the next section.

2.1 Introducing the Immirzi Parameter

In the previous section we have described gravity in terms of a symmetry breaking perturbation of topological BF theory. In this section we generalize this construction to the case where the topological field theory is BF with a 'cosmological term'. As we will see this is necessary in order to regulate in a physical way our perturbative expansion. We will also see that at the classical level this allows us to introduce naturally an other dimensionless parameter which appears in $4D$ gravity, the so called Barbero-Immirzi parameter.

The action principle for $SO(5)$ BF theory with a cosmological term is [7]

$$S = \int B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ}. \tag{14}$$

The equations of motion following from this action are

$$\begin{aligned} F_{IJ} &= \beta B_{IJ}, \\ d_A B_{IJ} &= 0. \end{aligned} \tag{15}$$

Note that the first equation implies the second one. This theory is invariant under local $SO(5)$ transformation, it is topological due to the additional 'translational' symmetry labelled by a one form Φ^{IJ} valued in the Lie algebra[4]

$$\delta A^{IJ} = \beta \Phi^{IJ}, \qquad \delta B^{IJ} = d_A \Phi^{IJ}. \tag{18}$$

[4]The non linear transformations corresponding to this infinitesimal symmetry are given by

$$A \to A + \beta\phi, \tag{16}$$

$$B \to B + d_A\phi + \frac{\beta}{2}[\phi, \phi] = B + \frac{F(A+\beta\phi) - F(A)}{\beta}. \tag{17}$$

The gauge invariant observables of this theory are therefore gauge invariant functions of $B^{IJ} - F^{IJ}/\beta$. As before, we add an extra term to the action which breaks the gauge symmetry down to $SO(4)$ gauge symmetry and also breaks translational symmetry. Our proposal for a gravity action is

$$S_2 = \int (B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL5}). \tag{19}$$

We can solve the equations of motion[5] for B^{IJ}

$$B^{ij} = \frac{1}{\alpha^2 - \beta^2} (\frac{\alpha}{2} \epsilon^{ijkl} F_{kl} - \beta F^{ij}), \tag{20}$$

$$B^{5i} = \frac{1}{\beta} F^{5i}, \tag{21}$$

and substitute them back into the action (19), we get

$$S_2 = \int \left(\frac{\alpha}{4(\alpha^2 - \beta^2)} F^{ij} \wedge F^{kl} \epsilon_{ijkl} - \frac{\beta}{2(\alpha^2 - \beta^2)} F^{ij} \wedge F_{ij} + \frac{1}{\beta} F^{5i} \wedge F_{5i} \right). \tag{22}$$

Using (9) and introducing the Nieh-Yan class $C = d_\omega e^i \wedge d_\omega e_i - R^{ij} \wedge e_i \wedge e_j$ [8], we can rewrite this action in terms of gravity variables

$$S_2 = \widetilde{S}_P + \int \left(\frac{\alpha}{4(\alpha^2 - \beta^2)} R^{ij}(\omega) \wedge R^{kl}(\omega) \epsilon_{ijkl} - \frac{\beta}{2(\alpha^2 - \beta^2)} R^{ij}(\omega) \wedge R_{ij}(\omega) + \frac{1}{\beta} C \right). \tag{23}$$

The last term is an integral of a linear combination of the Euler class, the Pontryagin class and the Nieh-Yan class. These are integer valued topological invariants with trivial local variation. The first term of action (23)

$$\widetilde{S}_P = -\frac{1}{2G} \int \left(R^{ij}(\omega) \wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{\Lambda}{6} e^i \wedge e^j \wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{2}{\gamma} R^{ij}(\omega) \wedge e_i \wedge e_j \right) \tag{24}$$

is the Cartan-Weyl action of General Relativity with nonzero cosmological constant and a nonzero Immirzi parameter γ, which is dimensionless [9]. The initial parameters α, β, l are related to the physical parameters as follows

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \qquad \alpha = \frac{G\Lambda}{3(1-\gamma^2)}, \qquad \beta = \frac{\gamma G \Lambda}{3(1-\gamma^2)}. \tag{25}$$

Even if the term proportional to γ is not topological (its variation is non zero), it doesn't affect the classical equation of motion when $\gamma^2 \neq 1$. It plays no role in the classical theory of gravity and it is therefore not constrained experimentally. It is important to note however that this term, similarly to the theta term in non abelian gauge theory, breaks CP symmetry. Since this fact seems to have been unnoticed let us explain it in more details. Suppose that we perform an orientation reversing diffeomorphism of our

[5] We restrict to the case $\alpha^2 \neq \beta^2$. Considering this case will lead to a self dual formulation of gravity.

spacetime, lets call it a C-transformation. All the terms in the action change signs since they are 4-forms, so C is not a symmetry of our gravity action. Lets now consider the discrete Lorentz transformations $g_{ij} = \mathrm{diag}(-+++)$ or $g_{ij} = \mathrm{diag}(+---)$, that we respectively called T or P transformation. The T transformation changes only the sign of e^0 and ω^{0i} leaving all the other fields invariant. The first two terms in the action change sign under P or T since they contain one epsilon tensor contracting the Lorentz indices but the last term does not. The action is not invariant under P or T but if we now consider CP (or CT) we see that the first two terms in the action are left invariant whereas the last one changes sign. In other word CP does not affect G or Λ but changes the sign of the Immirzi parameter. The CP symmetry is therefore realized only if $\gamma = 0$ or ∞. When $\gamma = 0$, which is the case studied previously, we recover the case of usual metric gravity, where the torsion is identically 0. When $\gamma = \infty$ we recover the usual Cartan-Weyl gravity where the torsion is free to fluctuate. Any other value of γ leads to a CP violation mechanism in quantum gravity which is worth exploring.

Even if it doesn't affect the classical theory the Immirzi parameter deeply affects the quantum theory and labels inequivalent quantizations in the context of kinematical loop quantum gravity. Indeed it is known for a long time (see [10] for a review) that this parameter modifies the symplectic structure and this modification is not unitarily implementable at the quantum level. It affects the prediction of the spectra of geometrical operators and plays a key role in the black hole entropy calculation [11]. This calculation suggests that γ and $1-\gamma^2$ are of order unity. One should keep in mind however, that the above conclusions are based on kinematical considerations, i.e. before Hamiltonian constraint is applied. And one open problem in this context is to understand wether the Immirzi parameter really leads to inequivalent quantization once the dynamics is fully taken into account or wether it can just be reabsorb into a redefinition of the Newton constant. This point have already been raised at the kinematical level in [12] where a seemingly more covariant approach to loop gravity leads to a geometrical spectra independent of the Immirzi parameter.

A more direct way to understand why the Immirzi parameter should affects quantization is to remark that $2/\gamma$ is proportional to the torsion square since $\int R^{ij} \wedge e_i \wedge e_j = \int d_\omega e^i \wedge d_\omega e_i$ up to a boundary term. γ therefore controls the width of fluctuation of the torsion at the quantum level. We have already remarked that if $\gamma = 0$, which is the case of metric gravity studied in the previous section, the torsion is not allowed to fluctuate. The mean value of the torsion is always equal to 0 irrespective of γ, this is why it doesn't affect classical gravity. However a naive semiclassical calculation shows that one should expect the two point function of the torsion to be proportional to γ. Therefore γ controls how strongly we suppress or not the torsion fluctuations in the path integral.

In our context the Immirzi parameter appears to act as a physical regulator. The role of the Immirzi parameter in this theory and its relevance to the physical predictions will be explored in more detail in our next paper [13].

There is not yet any preferred experimental value for γ, wether it is 0, ∞ or the value suggested by loop quantum gravity. Anyway, in all these cases[6] α and β are at most of the order $G\Lambda$ hence tiny numbers.

[6]if $\gamma \to \infty$ both α and β are sent to zero while the ratio β^2/α tends to a finite value $G\Lambda/3$

3 Formal setup for perturbation theory

We first concentrate on the case $\beta = 0$. Let us rewrite the action (8) in an index free form

$$S_{GR} = \int \mathrm{tr}(B \wedge F(A) - \frac{\alpha}{4} B \wedge B\gamma_5). \tag{26}$$

Here $B = B^{IJ} T_{IJ}/2$ and $A = A^{IJ} T_{IJ}/2$, where

$$T_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J] \tag{27}$$

are $so(5)$-generators in the fundamental representation and γ_I are γ-matrices satisfying $\{\gamma_I, \gamma_J\} = 2\delta_{IJ}$. The insertion of γ_5 in the second term of (26) breaks $SO(5)$ symmetry down to $SO(4)$.

We will be calculating the path integral for the action (26)

$$Z_{GR} = \int \mathcal{D}A\mathcal{D}B \exp(S_{GR}). \tag{28}$$

Following [14] we will treat the BF term in (26) as free field theory and the second term as a perturbation. Define the generating functional which is the path integral for the BF theory with an extrinsic source as

$$Z(J) = \int \mathcal{D}A\mathcal{D}B \exp\Big(\int \mathrm{tr}(B \wedge F(A) - B \wedge J)\Big), \tag{29}$$

where J is an $so(5)$-valued 2-form field. Then the path integral for General Relativity can be obtained by including the interaction by differentiating with respect to the sources.

$$Z_{GR} = \exp\Big(\int \mathrm{tr}(\frac{\alpha}{4}\frac{\delta}{\delta J} \wedge \frac{\delta}{\delta J}\gamma_5)\Big) Z(J)\Big|_{J=0}. \tag{30}$$

The perturbation theory can be obtained by expanding the exponent in (30) in a power series

$$Z_{GR} = \sum_n \frac{1}{n!}\Big(\int \mathrm{tr}(\frac{\alpha}{4}\frac{\delta}{\delta J} \wedge \frac{\delta}{\delta J}\gamma_5)\Big)^n Z(J)\Big|_{J=0}. \tag{31}$$

As $\alpha \ll 1$ we expect the sum to be dominated by the lowest order terms.

3.1 Computing the generating functional

We know show that the generating BF functional can be exactly evaluated. We start with

$$Z(J) = \int \mathcal{D}A\mathcal{D}B \exp\Big(i \int \mathrm{tr}(B \wedge F(A) - \frac{\beta}{2} B \wedge B - B \wedge J)\Big). \tag{32}$$

The action is quadratic in the B field we can integrate it out by replacing it by its classical value

$$\beta B^{IJ} = F^{IJ}(A) - J^{IJ}, \tag{33}$$

the action becomes

$$S_J = \frac{1}{2\beta}\int \mathrm{tr}\Big((F(A)-J)\wedge(F(A)-J)\Big). \tag{34}$$

Its equations of motion are

$$d_A J = 0, \tag{35}$$

and we denote by $\mathcal{M}_J$ the solution space. In order to solve these equations lets introduce a linear operator mapping Lie algebra valued 1-forms to Lie algebra valued three forms

$$\begin{aligned} L_J : \Omega_1(\mathcal{G}) &\rightarrow \Omega_3(\mathcal{G}) && (36)\\ A &\rightarrow [J, A]. && (37) \end{aligned}$$

The space of three forms $L_{\mu\nu\rho} \in \Omega_3(\mathcal{G})$ is isomorphic to the space of densitized vector $\tilde{L}^\alpha = 1/2\epsilon^{\alpha\mu\nu\rho}L_{\mu\nu\rho}$. L_J is a square matrix whose matrix elements can be explicitly written as

$$\tilde{L}_J^{(\nu AB)}\binom{\mu}{CD} = \epsilon^{\alpha\beta\mu\nu}J^{[A}_{\alpha\beta[C}\delta^{B]}_{D]}. \tag{38}$$

For a generic J, we expect L_J to be invertible, in this case there is a unique connection solution of 35

$$a_J = L_J^{-1}(dJ) \tag{39}$$

We expand $A = a_J + a$ and the action around this solution

$$\begin{aligned} 2\beta S_J = \int_M \mathrm{tr}\Big((F(a_J)-J)\wedge(F(a_J)-J) + 2(d_{a_J}a + \frac{1}{2}[a,a])\wedge(F(a_J)-J) && (40)\\ +(d_{a_J}a + \frac{1}{2}[a,a])\wedge(d_{a_J}a + \frac{1}{2}[a,a]).\Big) && (41) \end{aligned}$$

This expansion can be drastically simplified. First we can integrate by part the second term in the action using the equation of motion and the Bianchi identity $d_{a_J}J = d_{a_J}F(a_J) = 0$. We can also integrate by part the third term in the action by introducing the Chern-Simons functional

$$CS_J(a) = \mathrm{tr}\big(a\wedge d_{a_J}a + \frac{1}{3}a\wedge[a,a]\big), \tag{42}$$

its derivative is given by

$$dCS_J(a) = \mathrm{tr}\big((d_{a_J}a + \frac{1}{2}[a,a])\wedge(d_{a_J}a + \frac{1}{2}[a,a]) + [a,a]\wedge F(a_J)\big). \tag{43}$$

The action (40) can then be written as a sum of a boundary term

$$\int_{\partial M} CS_J(a) + 2\mathrm{tr}\big(a\wedge(F(a_J)-J)\big) \tag{44}$$

and a bulk action which remarkably is quadratic

$$2\beta S_J = \int_M \mathrm{tr}\big((F(a_J)-J)\wedge(F(a_J)-J) + a\wedge[J,a]\big). \tag{45}$$

We can then get an exact expression for the generating functional

$$Z(J) = \frac{\exp\left(\frac{i}{2\beta}\int_M \mathrm{tr}(F(a_J) - J) \wedge (F(a_J) - J)\right)}{\sqrt{\det L_J}}. \tag{46}$$

In the denominator we have the determinant of the operator $L_J(x, y) \equiv L_J(x)\delta(x, y)$.

If J is such that L_J is not invertible we can still carry out the computation. In this case $\mathcal{M}_J$, the space of solutions of (35), is an affine space of non zero dimension, its tangent space is the kernel of L_J. The action (34) now possess an extra gauge invariance

$$\delta A = a, \quad \text{with } a \in \ker(L_J). \tag{47}$$

We denote by a_J any solution of (35), and expand as before $A = a_J + a$, we still get the quadratic action (45). The integration over a now gives

$$Z(J) = \int_{\mathcal{M}_J/G_J} da_J \frac{\exp\left(\frac{i}{2\beta}\int_M \mathrm{tr}(F(a_J) - J) \wedge (F(a_J) - J)\right)}{\sqrt{\det' L_J}}. \tag{48}$$

where $\det'$ denotes the determinant of L_J acting on a orthogonal subspace of $Ker(L_J)$, $G_J = \{g/gJg^{-1} = J\}$ is the subgroup preserving J, and $\mathcal{M}_J/G_J$ is the space of solution modulo gauge transformation.

4 Topological effective action

We now want to study further the path integral and the effect of the gauge symmetry breaking term. We suppose in the following that $\beta = 0$. The gauge parameters are pairs g, ϕ were $g \in SO(5)$ and ϕ is a one form Lie algebra valued. The BF action $S_{BF}(B, A)$ is invariant under the transformation

$$A \rightarrow {}^gA = gAg^{-1} + gdg^{-1}, \tag{49}$$

$$B \rightarrow g(B - d_A\phi)g^{-1}. \tag{50}$$

We can split the integration over A, B into an integration over the gauge equivalence class $[A], [B]$ of the BF symmetry and an integration over the gauge parameters g, ϕ.

The integration measure decomposes, by the standard Faddev-Popov argument as $\mathcal{D}A\mathcal{D}B = \mathcal{D}[A]\mathcal{D}[B]\mathcal{D}g\mathcal{D}\phi$ and the path integral becomes

$$Z = \int \mathcal{D}[A]\mathcal{D}[B]e^{S_{BF}(B,A)+s(A,B)}. \tag{51}$$

where $s(A, B)$ is an effective action obtained by integration over the gauge degree of freedom, explicitly

$$e^{is(A,B)} = \int \mathcal{D}g\mathcal{D}\phi e^{-\frac{\alpha}{4}\int d^4x \mathrm{tr}(|B - d_A\phi|\gamma(g))} \tag{52}$$

Where we define $\gamma(g) \equiv g^{-1}\gamma_5 g$ which is a unit vector[7] in $\mathbb{R}^5$ and for every $SO(5)$ valued 2 form B we define the vector density $|B|(x) = |B|_M(x)\gamma^M$ by[8]

$$\mathrm{tr}(B \wedge B\gamma_M) = |B|_M d^4x \tag{54}$$

In order to understand the form of this effective action we will look at the partial effective action obtained by integrating only g and only ϕ. As we will see each partial integration is one loop exact, which means that it localizes on its classical solutions and that its evaluation is given by its stationary phase evaluation (see [15] for a discussion of localization in QFT). This is clear for the integration over ϕ since the action is quadratic in ϕ but it also happens for the integration over g.

4.1 spontaneous symmetry breaking

The action $\int \mathrm{tr}(|B - d_A\phi|\gamma(g))$ is ultralocal for g, it doesn't contain any derivative acting on g and we can therefore understand the localization property of the path integral by looking at the final dimensional analog

$$\int_{SO(4)} dg e^{i\mathrm{tr}(|B|\gamma(g))} = \frac{e^{i||B||}}{2||B||} - \frac{e^{-i||B||}}{2||B||}. \tag{55}$$

The RHS of this expression is the semi classical evaluation of the integral. This is clear since the equation of motion of $\mathrm{tr}(|B|\gamma(g))$ gives $[\gamma(g), |B|] = 0$. The solutions when $||B||^2 \equiv |B|_M|B|^M \neq 0$ are given by

$$\gamma(g) = \pm\frac{|B|}{||B||}. \tag{56}$$

The action evaluated on this solution is $\pm||B||$ which reproduces the terms in the exponential, the denominator comes from the evaluation of the quadratic fluctuations around the solution.

Similarly, the equations of motion of the continuum action $\int \mathrm{tr}(|B|\gamma(g))$ (supposing $\phi = 0$ for simplicity) are $[\gamma(g), |B|] = 0$. The solutions when $||B||^2 \equiv |B|_M|B|^M \neq 0$ are given by

$$\gamma(g)(x) = \pm\frac{|B|(x)}{||B||(x)}. \tag{57}$$

The sign is a priori x dependent, but if we restrict to continuous solution for g we have only two solutions[9].

[7] γ_5 is left invariant by an $SO(4)$ subgroup so $\gamma(g)$ determines a point in $S^4 = SO(5)/SO(4)$

[8] If we spell out the indices this reads

$$|B| = \gamma^M \epsilon_{IJKLM}\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \tag{53}$$

[9] this problem disappear when we consider negative cosmological constant, in this case $\gamma(g)$ belongs to an hyperboloid and there is only one solution to the equation $[\gamma(g), |B|] = 0$.

The localization property of the integral therefore suggest that the gravity effective action obtained by integration over g is (if one keep one branch)

$$S_{GR} = \int \mathrm{tr}(B \wedge F(A)) - \frac{\alpha}{4}||B||(x) d^4x. \tag{58}$$

This action is now $SO(5)$ invariant whereas gravity is only $SO(4)$ invariant, it sounds therefore strange at first that we can recover gravity from this action. In order to understand this let us first check that the equation of motion for this action are equivalent to Einstein's equation.

The action is defined for all B, however it is differentiable only when $||B||(x) \neq 0$ which we now suppose holds true. The equations of motion are given by

$$\begin{aligned} F^{IJ} &= \frac{\alpha}{2}\epsilon^{IJKLM} B_{KL} n_M, & (59)\\ d_A B^{IJ} &= 0. & (60)\end{aligned}$$

where we denote $n_M \equiv \frac{|B|_M}{||B||}$. Given this unit vector we define

$$\begin{aligned} e^I &\equiv d_A n^I = dn^I + A^{IJ} n_J, \quad \omega^{IJ} \equiv A^{IJ} + n^I e^J - n^J e^I, & (61)\\ b^I &\equiv B^{IJ} n_J, \qquad\qquad b^{IJ} \equiv B^{IJ} + n^I b^J - n^J b^I. & (62)\end{aligned}$$

We denote F^{IJ} (respectively R^{IJ}) the curvature of the connection A (respectively ω). We have the following identities

$$F^{IJ} n_J = d_A e^I = d_\omega e^I, \tag{63}$$

$$R^{IJ} - 2e^{[I} \wedge e^{J]} = F^{IJ} - 2d_\omega e^{[I} n^{J]}, \tag{64}$$

the bracket denotes antisymetrisation. From these identities it is clear that $R^{IJ} n_J = 0$, also $d_\omega n_I = 0$, so ω is a $SO(4)$ connection preserving the direction n^I. In terms of the variables (61, 62) the equations of motion read

$$\begin{aligned} d_\omega e^I &= 0, & (65)\\ \frac{1}{2\alpha}\epsilon_{IJKLM}(R^{IJ} - 2e^I \wedge e^J) n^M &= b_{KL}, & (66)\\ d_A B^{IJ} n_J = d_\omega b^I - b^{IJ} \wedge e_J &= 0, & (67)\\ d_\omega b^{IJ} + 2e^{[I} \wedge b^{J]} &= 0. & (68)\end{aligned}$$

The first equation tells us that ω is the spin connection if the frame field e is invertible. If one take the derivative d_ω of (66) one obtain that $d_\omega b_{KL} = 0$ since $d_\omega R^{IJ} = 0$ by Bianchi identity, $d_\omega e^I = 0$ by the torsion free equation and $d_\omega n_I = 0$ by construction. The equation (68) then imply that $b^I = 0$ when e^I is invertible. This means that $b^{IJ} \wedge e_J = 0$ by equation (67) which is equivalent, due to (66), to the Einstein equation

$$\epsilon_{ijkl}(R^{ij} - 2e^i \wedge e^l) \wedge e^k = 0, \tag{69}$$

where the indices i, j, k label vectors orthogonal to n.

One sees that the equation of motion of the $SO(5)$ invariant theory are equivalent to Einstein equation when e is invertible. Even if the action is invariant under $SO(5)$ gauge symmetry the solutions of this action spontaneously breaks this symmetry by choosing a preferred direction in the internal space proportional to $|B|$. The same results apply for $\beta \neq 0$.

On shell we have that $||B||d^4x = \frac{1}{\alpha^2}|\epsilon_{ijkl}F^{ij} \wedge F^{kl}|$. Any $SO(4)$ bivector B^{ij} can be decomposed into a self dual and anti self dual part $B = B_+ + B_-$, using this decomposition for spatial and internal indices of $F^{ij}_{\mu\nu} =$ we can write decompose F as $F = W_+ + W_- + \phi + \phi_0$ where W^{ij}_- is a symmetric traceless tensor labelling the 5 self dual components of the Weyl tensor, ϕ^{ij} a traceless tensor labelling the trace free part of the Ricci tensor and ϕ_0 is the scalar curvature. In term of these components we have $||B|| = 4!det(e)/\alpha^2((W_+)^2 + (W_-)^2 + (\phi_0)^2 - (\phi)^2)$ The components ϕ, ϕ_0 are zero by the Einstein equation, thus $||B||$ is zero if and only if The Weyl tensor vanish that is only if $F = 0$ and our spacetime is spherical[10] . We therefore see that the presence of a spontaneous symmetry breaking is equivalent in the Euclidean case to the existence of a non trivial gravitational field.

4.2 Gravity as a non local topological theory

We now consider the construction of the effective action coming from the integration of the translational symmetry parameter for g fixed. We discuss the case $\beta = 0$.

$$e^{is(A,B)=}\int \mathcal{D}\phi e^{i\frac{\alpha}{4}\int d^4x \mathrm{tr}(|B - d_A\phi|\gamma_5)}. \tag{70}$$

This integral being quadratic the integral localizes on the classical solution if any. The equation of motion are given by

$$d_A\{B, \gamma_5\} = \Delta_A \phi, \tag{71}$$

where Δ_A is the differential operator $\Delta_A\phi = d_A\{d_A\phi, \gamma_5\}$. If one uses the $4+1$ decomposition $A^{IJ} = (\omega^{ij}, e^i)$ and $\phi^{IJ} = (\phi^{ij}, \phi^i)$ we can write these equation in components.

$$d_\omega B^{ij} = d_w(d_\omega \phi^{ij} - 2e^{[i} \wedge \phi^{j]}) \tag{72}$$

$$\epsilon_{ijkl}B^{ij} \wedge e^k = \epsilon_{ijkl}(d_\omega\phi^{ij} - 2e^i \wedge \phi^j) \wedge e^k. \tag{73}$$

If Δ_A is invertible we can uniquely solve this equation. Lets denote $\varphi \equiv \Delta_A^{-1}(d_A\{B, \gamma_5\})$ a solution of these equations and define

$$\bar{B} = B - d_A\varphi, \tag{74}$$

by construction $\bar{B}$ is a solution of $d_A\{\bar{B}, \gamma_5\} = 0$ and if we insert the previous decomposition in the integral (70) we can factorize $\bar{B}$ out of the integral

$$e^{is(A,B)} = e^{i\frac{\alpha}{4}\int \mathrm{tr}(\bar{B}\wedge\bar{B}\gamma_5)} \int \mathcal{D}\phi e^{-\frac{\alpha}{4}\int d^4x \mathrm{tr}(|d_A\phi|\gamma_5)} = \frac{e^{i\frac{\alpha}{4}\int \mathrm{tr}(\bar{B}\wedge\bar{B}\gamma_5)}}{\sqrt{\det(\Delta_A)}} \tag{75}$$

[10] in the Lorentzian case the condition is less restrictive since $W_\pm$ are complex conjugate

where Δ_A is the differential operator $\Delta_A \phi = d_A\{d_A\phi, \gamma_5\}$.

One sees that the integration over the gauge modes produces for us an effective action

$$\tilde{s}(A,B) = \frac{\alpha}{4}\int \mathrm{tr}(\bar{B}\wedge\bar{B}\gamma_5) = \frac{\alpha}{4}\int \mathrm{tr}\Big((B\wedge B\gamma_5) + \frac{1}{2}(d_A\{B,\gamma_5\}\Delta_A^{-1}d_A\{B,\gamma_5\})\Big). \quad (76)$$

This action is invariant under the translational gauge symmetry of BF theory which is the symmetry that makes BF theory topological but by construction its partition function is the one of gravity. BF theory does not carry local degrees of freedom whereas gravity does, the catch is that the effective action is a non local observable for BF theory since it involves the propagator of Δ_A. It is important to remark that this action is still quadratic in B since Δ_A is a linear operator. It is not clear however, wether we can explicitely do the g integration of the action (76).

In the derivation of the effective action we have assumed that Δ_A is an invertible operator that is that there is no non trivial solution to the equation

$$d_A\{d_A\phi, \gamma_5\} = 0. \quad (77)$$

We expect it to be true for a generic choice of A (as long as $e = d_A\gamma_5$ is invertible). We are now going to give an argument in favor of this claim, keeping in mind that it will be interesting to have a proper characterization of the connections for which it holds.

Before doing so, let us first study a particular case where on the contrary Δ_A is not invertible. We will now show that if A is a flat $SO(5)$, then the gravitational waves around this connection are in one to one correspondence with the kernel of Δ_A. If we start from a Cartan Weyl formulation of gravity (12), with $\Lambda = 3$, the equation of motions are

$$d_\omega(R^{ij} - e^{[i}\wedge e^{j]}) = 0, \quad \epsilon_{ijkl}(R^{ij} - e^i\wedge e^j)\wedge e^k = 0. \quad (78)$$

In the first equation which comes from variation with respect to ω^{ij} we have added for convenience a term trivial by Bianchi identity. These equations can be written in a compact form

$$d_A\{F(A), \gamma_5\} = 0, \quad (79)$$

by using the notation of eq.(9). Given a gravity solution $A = (\omega^{ij}, e^i)$ we can look for 'graviton solution', i-e infinitesimal perturbation δA such that $A + \delta A$ is a solution of Einstein equations to first order. The equation for the perturbation is

$$\Delta_A \delta A = [\{F(A), \gamma_5\}, \delta A]. \quad (80)$$

Therefore, if the original space time is a four sphere: $F(A) = 0$, and δA is a graviton solution, then $\phi = \delta A$ is in the kernel of Δ_A. Even in Euclidean space where there is no graviton Δ_A is not invertible around a flat $SO(5)$ connection since infinitesimal diffeomorphism $\delta A = \mathcal{L}_\xi A$ are in the kernel of Δ_A. Away from a spherical space this is no longer true: the graviton Laplacian is now $\tilde{\Delta}_A = \Delta_A + [\{F(A), \gamma_5\}, \cdot]$, infinitesimal diffeomorphisms $\delta A = \mathcal{L}_\xi A$ are in the kernel of $\tilde{\Delta}_A$ but not of Δ_A. This can be easily understood from the fact that the action $S(A,\phi) = \int \mathrm{tr}(d_A\phi^{ij}\wedge d_A\phi^{kl})\epsilon_{ijkl}$ is not invariant under diffeomorphism or $SO(4)$ gauge transformation acting on ϕ alone unless A is chosen to be fixed by a combination of diffeomorphism and gauge transformation. This is the case for a flat connection since we have in this case $\mathcal{L}_\xi A - d_A(i_\xi A) = i_\xi(F(A)) = 0$, with ξ a four vector and i_ξ denotes the interior product[11]. In general Δ_A being non invertible means that $S(A,\phi)$ possess some gauge invariance. We expect all possible gauge invariance of such an action to come from a restriction of the diffeomorphism group times the local rotation group. Only some special connection will have such a subgroup of invariance like the flat connection but also some special holonomy connections. For a generic A there is no such invariance and therefore we expect Δ_A to be invertible.

4.3 Towards background independent perturbation theory

In the previous section we have shown that the partition function of quantum General Relativity can be expressed as an expectation value of a topologically invariant observable given by (76). The insertion of such an observable into topological field theory can be understood as introducing an infinite-dimensional moduli space labelled by ϕ and $\gamma(g)$ and then integrating over it. This moduli space contains all the physical degrees of freedom of four dimensional General Relativity. Looking at the effective action (76) one can see that unlike in ordinary quantum field theory the gravitational degrees of freedom are generically non-local and do not allow to form point-like excitation. The non-locality of fundamental gravitational excitation was argued on a different basis in [16]. This is a realization of equivalence principle in quantum gravity saying that a local free falling observer can never see gravitational effects.

For practical calculations we need a perturbation theory which would decompose the infinite dimensional moduli space defined into a set of finite dimensional moduli spaces in each of which the integration can be explicitly performed. This requires the exchange of the expansion of the interaction term in a power series and integration over the broken gauge degrees of freedom. Such an exchange is valid if all the resulting integrals converge. The later is not always the case. This can be seen if $\beta = 0$, in this case we can perform directly the integral over ϕ. If we do the perturbative expansion before integrating these degrees of freedom we get meaningless expression since the integral over ϕ diverges for large values of ϕ.

The story is however quite different if $\beta \neq 0$. First, at some naive level we see that in this case the large fluctuation of B-field are suppressed due to strong oscillatory behavior of the integral by the quadratic term proportional to β which is a non positive but definite quadratic form. This strongly suggest that Immirzi parameter act as a physical regulating parameter of our perturbation theory.

One can also understand this by looking at the action for the BF gauge degrees of freedom which is, in the case $\beta \neq 0$, given by (see footnote 4)

$$\frac{\alpha}{4}\int d^4x \mathrm{tr}(|B - \frac{F(A)}{\beta} + \frac{F(A+\beta\phi)}{\beta}|\gamma(g)). \tag{81}$$

[11] In the case of Flat $SO(5)$ connection this action is clearly invariant under the transformation $\delta\phi = d_A\psi$

In this case the action is non quadratic in ϕ and we cannot hope to solve it directly. The only possibility is then to construct perturbatively the partition function and do the perturbative expansion before the integration. One clearly sees from this expression that integration over $\beta\phi$ should now be understood as a integration over a $SO(5)$ connection. In order to do the integration we need to specify the measure we use to integrate over the space of gauge invariant connection[12]. We need a measure which is diffeomorphism invariant and normalized, such a measure has been proven (under additional technical hypothesis) to exist and to be unique for a compact gauge group [17]: this is the Ashtekar-Lewandowski measure. This suggests that the techniques of loop quantum gravity and spin foam model are adapted [14] to describe our perturbative expansion and lead to a finite result.

In section 3.1 we have calculated the generating functional needed for such a perturbation theory. One can notice that the resulting perturbative expansion we are proposing is non standard, since the insertion of a non zero J breaks a gauge symmetry which is restored in the limit $J \to 0$. At each order in perturbation theory the derivatives over J insert an operator which is slightly breaking the original BF gauge symmetry.

At zero-order the result of the computation is simply given by $Z(0) = \int_{\mathcal{A}/G} dA$ which is the integral over all gauge connection. If one uses the Ashtekar-Lewandowski measure this gives $Z(0) = 1$, which is the correct result for trivial topology. An other way to get the same result is to consider that this integral should be gauged fixed by the topological BF symmetry and as such it is just equal to 1. At each order of the perturbation theory one would then compute the expectation value in BF theory of a operator which is of polynomial order in the B fields and gauge fixed the residual BF symmetry. The gauge fixing procedure of BF is now well understood in the spin foam context [18]. This shows once again,that the techniques of spin foam models are well adapted [14] to describe our perturbative expansion. At each order of perturbation theory a triangulation can be chosen to do the explicit computation, and the topological symmetry which is acting away from the source should insure that the result is independent of the triangulation. This is contrast with the usual spin foam model which usually depends on the chosen triangulation, and one has to average over triangulations using the perturbative expansion of an auxiliary Field theory [19].

We can also understand in this context why the case $\beta \neq 0$ is much more regular. It has been conjectured for a long time that the computation of the partition function of BF model with non zero 'cosmological term' is realized by a state sum model build on a quantum group with $q = \exp(i\beta)$ roots of unity. This has been proven recently by Barrett et al [20] for the case of the group $SU(2)$, presumably this results holds for any compact group. Therefore the sums entering the computation of the BF observables in the spin foam context are all expected to be finite.

Our next paper [13] is devoted to study in more details the perturbation theory in the context of spin foam.

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[12]the action is gauge invariant after integration over g

References

[1] E.E. Witten, "(2+1)-Dimensional Gravity As An Exactly Soluble System," Nucl. Phys. B **311**, 46 (1988).

[2] J. F. Plebanski, "On the separation between Einsteinien substuctures", J. Math. Phys. (12) 1977, 2511. R. Capovilla, T. Jacobson, J. Dell and L. Mason, "Selfdual two forms and gravity," Class. Quant. Grav. **8** (1991) 41. R. De Pietri and L. Freidel, "so(4) Plebanski Action and Relativistic Spin Foam Model," Class. Quant. Grav. **16**, 2187 (1999) [arXiv:gr-qc/9804071]. M. P. Reisenberger, "Classical Euclidean general relativity from *left-handed area = right-handed area*," arXiv:gr-qc/9804061.

[3] L. Freidel, K. Krasnov and R. Puzio, "BF description of higher-dimensional gravity theories," Adv. Theor. Math. Phys. **3**, 1289 (1999) [arXiv:hep-th/9901069].

[4] S.W. MacDowell, F. Mansouri, "Unified Geometric theory of gravity and supergravity". Phys.Rev.Lett.38:739,1977., Erratum-ibid.38:1376,1977.

[5] L. Smolin and A. Starodubtsev, "General relativity with a topological phase: An action principle," arXiv:hep-th/0311163.

[6] L. Freidel, J. Kowalski-Glikman, A. Starodubtsev, "Background independent perturbation theory for gravity coupled to particles: classical analysis". To appear.

[7] J. C. Baez, "Four-Dimensional BF theory with cosmological term as a topological quantum field theory," Lett. Math. Phys. **38**, 129 (1996) [arXiv:q-alg/9507006].

[8] H. T. Nieh and M. L. Yan, "An Identity In Riemann-Cartan Geometry," J. Math. Phys. **23**, 373 (1982). O. Chandia and J. Zanelli, "Torsional topological invariants (and their relevance for real life)," arXiv:hep-th/9708138.

[9] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action," Phys. Rev. D **53**, 5966 (1996) [arXiv:gr-qc/9511026].

[10] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21**, R53 (2004) [arXiv:gr-qc/0404018].

[11] A. Ashtekar, J. C. Baez and K. Krasnov, "Quantum geometry of isolated horizons and black hole entropy," Adv. Theor. Math. Phys. **4**, 1 (2000) [arXiv:gr-qc/0005126]. M. Domagala and J. Lewandowski, "Black hole entropy from quantum geometry," Class. Quant. Grav. **21**, 5233 (2004) [arXiv:gr-qc/0407051].

[12] S. Alexandrov and E. R. Livine, "SU(2) loop quantum gravity seen from covariant theory," Phys. Rev. D **67**, 044009 (2003) [arXiv:gr-qc/0209105].

[13] L. Freidel and A. Starodubtsev, "Perturbative gravity via spin foam" To appear.

[14] L. Freidel and K. Krasnov, "Spin foam models and the classical action principle," Adv. Theor. Math. Phys. **2**, 1183 (1999) [arXiv:hep-th/9807092].

[15] E. Witten, "Two-dimensional gauge theories revisited," J. Geom. Phys. **9** (1992) 303 [arXiv:hep-th/9204083].

[16] A. Starodubtsev "Topological excitations around the vacuum of quantum gravity I: the symmetries of the vacuum", hep-th/0306135.

[17] H. Sahlmann, "Some comments on the representation theory of the algebra underlying loop quantum gravity," arXiv:gr-qc/0207111. H. Sahlmann and T. Thiemann, "Irreducibility of the Ashtekar-Isham-Lewandowski representation," arXiv:gr-qc/0303074. A. Okolow and J. Lewandowski, "Diffeomorphism covariant representations of the holonomy-flux *-algebra," Class. Quant. Grav. **20**, 3543 (2003) [arXiv:gr-qc/0302059].

[18] N. Kawamoto, N. Sato and Y. Uchida, "4-dimensional BF gravity on the lattice," Nucl. Phys. B **574**, 809 (2000) [arXiv:hep-th/9911228]. L. Freidel and D. Louapre, "Diffeomorphisms and spin foam models," Nucl. Phys. B **662**, 279 (2003) [arXiv:gr-qc/0212001].

[19] R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, "Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space," Nucl. Phys. B **574**, 785 (2000) [arXiv:hep-th/9907154].

[20] J. W. Barrett, J. M. Garcia-Islas and J. F. Martins, "Observables in the Turaev-Viro and Crane-Yetter models," arXiv:math.qa/0411281.

A SO(5) conventions

$T_{IJ} = -T_{JI}$ with $I = 1, \cdots, 5$ are the ten generators of $so(5)$. They satisfy the algebra

$$[T_{IJ}, T_{KL}] = \eta_{JK}T_{IL} - \eta_{IK}T_{JL} + \eta_{IL}T_{JK} - \eta_{JL}T_{IK}, \tag{82}$$

where $\eta_{IJ} = \delta_{IJ}$ in the case of $so(5)$. The corresponding theory of gravity is Euclidean with a positive cosmological constant, i-e 'spherical gravity'. This is the one we focus on in the main text. If we want to describe Lorentzian gravity and/or other sign of cosmological constant one should consider metric of different signatures, the cases of interest for gravity are: $SO(4,1)$, where $\eta = diag(++++-)$ which describes Euclidean gravity with a negative cosmological constant (i-e 'hyperbolic gravity'). $SO(1,4)$ where $\eta = diag(-++++)$ which describes Lorentzian gravity with a positive cosmological constant, i-e 'de Sitter gravity'. $SO(3,2)$ where $\eta = diag(-+++-)$ which describes Lorentzian gravity with a negative cosmological constant, i-e 'AdS gravity'. One can split the generators of in terms of $so(4)$ generators T_{ij}, $i = 1, \cdots, 4$ and 'translation' generators $P_i = T_{i5}/l$, where l is a length scale (cosmological length scale in our context). The algebra reads

$$\begin{aligned} [T_{ij}, T_{kl}] &= \eta_{jk}T_{il} + \cdots, & (83)\\ [T_{ij}, P_k] &= \eta_{ik}P_j - \eta_{jk}P_i, & (84)\\ [P_i, P_j] &= -\frac{\eta_{55}}{l^2}T_{ij}. & (85) \end{aligned}$$

The $so(5)$ can be represented in terms of γ matrix

$$T_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J] \tag{86}$$

where γ_I are γ-matrices satisfying $\{\gamma_I, \gamma_J\} = 2\eta_{IJ}$.