The Existence of Cartan Connections and Geometrizable Principle Bundles

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Abstract

The aim of this article is to proof a necessary and sufficient condition for the existence of a Cartan connection on a principal bundle, cf. Theorem 3.2. After collecting the essentially well known facts[1] to fix the terminology, abstract soldering forms and geometrizable principle bundles are defined to finally prove the existence criterion.

1 Principal bundles and associated bundles.

Throughout the paper let $H \to P \xrightarrow{\pi} M$ denote a *principal H-bundle* over the *base manifold* M, where P is a manifold, and H a Lie group with a smooth *free proper right* action on P

$$R : \left\{ \begin{array}{ccc} H & \to & \mathrm{Diff}(P); \\ h & \mapsto & R_h, \end{array} \right.$$

and $M := P/H$ the quotient manifold.

Further let F be a *H-manifold*, i.e. a manifold with a smooth *left* H action. For $p \in P$ and $\xi \in F$ denote by

$$p\xi := \{(ph, h^{-1}\xi) |\ h \in H\} \tag{1}$$

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[1]Cf. [KoNo] and [Sh].

the orbit of (p, ξ) under the *natural right* action of H on $P \times F$. Then the right orbit space

$$E = E(P, F) := P \times_H F := (P \times F)/H = \{p\xi |\ p \in P,\ \xi \in F\} \qquad (2)$$

is called the *associated fiber bundle to P with typical fiber F*, and denoted by $F \to E \overset{\pi_E}{\to} M$.

Every $p \in P$ with $\pi(p) = x \in M$ defines a diffeomorphism

$$p : \left\{ \begin{array}{ccl} F & \to & F_x := \pi_E^{-1}(x); \\ \xi & \mapsto & p\xi. \end{array} \right. \qquad (3)$$

Moreover we have: $(ph)\xi = p(h\xi)$ for every $h \in H$.

1.1 Lemma *The following are in bijective correspondence:*

(i) A global *section $f : M \to E$.*

*(ii) A H-*equivariant *map $\tilde{f} : P \to F$, i.e. $\tilde{f}(ph) = h^{-1}\tilde{f}(p)$ for all $h \in H$.*

1.2 Definition (Reduction)
Let K be a Lie subgroup of H. The principal H-bundle P is called *K-reducible*, if there exists a principal bundle $K \to Q \overset{\pi'}{\to} M$ and a K-*equivariant* embedding $\iota : Q \to P$, i.e. an embedding satisfying $\iota(qk) = \iota(q)k$ for all $q \in Q$ and $k \in K$. Q is called a *K-reduction* of P.

1.3 Definition (Symmetry breaking)
Let K a Lie subgroup of H. A *H-equivariant* map $f : P \to H/K$ is called a *K-symmetry breaking*.

1.4 Theorem *Let K a closed Lie subgroup of H. The following are in pairwise bijective correspondence:*

(i) A K-reduction $\iota : Q \to P$.

(ii) A global *section $f : M \to P/K$.*

(iii) A K-symmetry breaking $\tilde{f} : P \to H/K$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is given by factoring out the free action of K on Q and P. The equivalence (ii) $\Leftrightarrow$ (iii) is established by the fact that $P/K \cong P \times_H H/K = E(P, H/K)$ and Lemma 1.1. □

1.5 Definition (H-Structure)
Let M be a n-dimensional manifold and H be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$. The principal bundle $H \to P \overset{\pi}{\to} M$ is called a *H-structure*, if P is a H-reduction of the frame bundle $\mathrm{GL}(\mathbb{R}^n) \to L(M) \to M$.

1.6 Definition (Pseudotensorial, Tensorial)
Let V be a H-module via the representation $\rho : H \to \mathrm{GL}(V)$. A V-valued r-Form $\phi : \wedge^r TP \to V$ is called *pseudotensorial* of type (V, ρ) if it is H-equivariant, i.e. if

$$R_h^* \phi = \rho(h^{-1})\phi, \quad \forall h \in H.$$

A pseudotensorial r-form is thus an element of fixed space $\Gamma(\wedge^r T^*P \otimes V)^H$. ϕ is called *tensorial* if it is pseudotensorial and *horizontal*[2] meaning that

$$\phi(X_1, \dots, X_r) = 0$$

whenever at least one of the X_i's is *vertical*, i.e. tangent to the fiber. The set of tensorial r-forms is denoted by $\Gamma_{\mathrm{hor}}(\wedge^r T^*P \otimes V)^H$.

A tensorial r-form factors over the induced projection $TP \stackrel{\pi_*}{\to} TM$ and we obtain the natural identification

$$\Gamma_{\mathrm{hor}}(\wedge^r T^*P \otimes V)^H \cong \Gamma(\wedge^r \pi^* T^*M \otimes V)^H. \tag{4}$$

Further if the action of H on V is *trivial*, then every tensorial form is the pullback to the principal bundle P of a V-valued form on the base[3] manifold M, i.e. $\Gamma_{\mathrm{hor}}(\wedge^r T^*P \otimes V)^H \cong \pi^*\Gamma(\wedge^r T^*M \otimes V)$.

1.7 Definition (Bundle valued)
Let P, V be as above and $V \to E \stackrel{\pi_E}{\to} M$ the associated vector bundle. Elements of $\Gamma(\wedge^r T^*M \otimes E)$ are called *bundle valued* r-forms.

Now we can state the key lemma used in proof of the main theorem:

1.8 Lemma *Let P, V and E be as above. The following are in bijective correspondence:*

(i) A bundle valued r-form on M.

(ii) A tensorial V-valued r-form on P.

Thus we have the natural identification:

$$\Gamma(\wedge^r T^*M \otimes E) \cong \Gamma_{\mathrm{hor}}(\wedge^r T^*P \otimes V)^H. \tag{5}$$

Proof. See [KoNo], Example II.5.2. □

[2] This notion is found in [KoNo], Chapter II.5. Sharpe [Sh] uses '*semibasic*' instead.
[3] This motivates the notion '*basic*' for such tensorial forms, cf. [Sh], Definition 1.5.23 and Lemma 1.5.25. See also [KoNo], Example II.5.1.

2 Connections

For the rest of the paper let $\mathfrak{h}$ denote the Lie algebra of H.

2.1 Definition (Ehresmann connection)
A $\mathfrak{h}$-valued 1-form $\gamma : TP \to \mathfrak{h}$ is called an *Ehresmann connection* or simply a *connection*, if the following two conditions hold:

(i) $R_h^* \gamma = \mathrm{Ad}(h^{-1})\gamma$ for all $h \in H$.

(ii) $\gamma(X^\dagger) = X$ for all $X \in \mathfrak{h}$.

By $X^\dagger$ we denote the *fundamental vector field* of X induced by the infinitesimal action of $\mathfrak{h}$ on P induced from the action of H.

Because of (i) a connection is pseudotensorial of type $(\mathfrak{h}, \mathrm{Ad})$ and because of (ii) it is not tensorial.

2.2 Definition (Soldering form)
Let M be a n-dimensional manifold, $H \to P \xrightarrow{\pi} M$ and $\rho : H \to \mathrm{GL}(\mathbb{R}^n)$ a representation. A surjective $\mathbb{R}^n$-valued tensorial 1-form: $\theta : TP \twoheadrightarrow \mathbb{R}^n$ is called a *soldering form*[4].

2.3 Example (Fundamental form)
Every H-structure P has a natural soldering form, the so called *fundamental form*, defined by

$$\theta_u(X) := u^{-1}\pi_*(X) \quad \forall u \in P \text{ and } X \in TP,$$

where u^{-1} denotes the inverse of the isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$ defined in (3). ρ is given by the natural action of $H \leq \mathrm{GL}(\mathbb{R}^n)$ on $\mathbb{R}^n$.

The definition of the fundamental form relies on the fact, that the tangent bundle TM is associated to P. This directly motivates the next definition, which seems to be new.

2.4 Definition (Geometrizable principal bundle)
Let M be a n-dimensional manifold. We call the principal bundle $H \to P \xrightarrow{\pi} M$ *geometrizable* if the tangent bundle TM is associated to P, i.e. if there exists a representation $\rho : H \to \mathrm{GL}(\mathbb{R}^n)$ turning $\mathbb{R}^n$ into a H-module, such that the associated bundle satisfies the *soldering condition*:

$$P \times_H \mathbb{R}^n \cong TM. \tag{6}$$

We call P *first order* geometrizable if ρ is faithful, otherwise *higher order*. Once we fix a representation fulfilling the soldering condition, we call P *geometrical*.

[4]In [Ko] θ is called a form of *soudure* (cf. Theorem 2 and the above lines). See also [AM], 2.4, 5.1.

2.5 Example

- P is a H-structure, iff P is a *first order* geometrizable principal H-bundle. (Cf. Theorem 3.2 and [Sh], Exercise 5.3.21.)
- The trivial bundle $S^2 \times \mathrm{SO}(\mathbb{R}^2)$ is not geometrizable.
- A n-dimensional manifold M ($n \geq 3$) is spin, iff there exists a geometrizable principal $\mathrm{Spin}_n(\mathbb{R})$-bundle over M.

It is worth mentioning, that, in contrast to relativistic gravitation theories, most of the principle bundles appearing in classical gauge theories are *not* geometrizable, unless TM is trivial.

3 Cartan Connections

Now we come to the main object of this paper.

3.1 Definition (Cartan connection)
Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{h} \leq \mathfrak{g}$ and $\dim \mathfrak{g} = \dim P$. A $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$ satisfying[5]

(i) $\omega : T_pP \to \mathfrak{g}$ is an isomorphism for all $p \in P$. (*trivialization* of $TP \to P$)

(ii) $R_h^* \omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$. (*pseudotensorial* of type $(\mathfrak{g}, \mathrm{Ad})$)

(iii) $\omega(X^\dagger) = X$ for all $X \in \mathfrak{h}$.

We call a Cartan connection *reductive*, if $\mathfrak{h}$ has a H-invariant complement[6] $\mathfrak{p}$ in $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} \oplus_H \mathfrak{p}$.

In the definition of a Cartan connection we do not make use of the Lie algebra structure of $\mathfrak{g}$. We only use the H-module structure[7] of $\mathfrak{g}$ and the fact that $\mathfrak{h}$ is a H-submodule of $\mathfrak{g}$. The first place where we do need the Lie algebra structure on $\mathfrak{g}$ is in the definition of *curvature* of a Cartan connection. This is exactly where the power and flexibility of this notion lies, cf. [Ruh] and [Sh].

3.2 Theorem *The following statements are equivalent:*

(i) P has a Cartan connection for some $\mathfrak{g} \geq \mathfrak{h}$.

(ii) P has a reductive Cartan connection for some $\mathfrak{g} \geq \mathfrak{h}$.

(iii) P has a reductive Cartan connection for some $\mathfrak{g} \geq \mathfrak{h}$ with $[\mathfrak{p}, \mathfrak{p}] = 0$.

(iv) P has a soldering form.

(v) P is a geometrizable bundle.

[5] For an alternative definition see [AM], 2.1, 2.2.
[6] We do not require that $\mathfrak{p}$ is a Lie subalgebra.
[7] and the induced $\mathfrak{h}$-module structure

Proof. (v) $\Rightarrow$ (iv): TM is associated to P via a H-module structure of $\mathbb{R}^n$. The image of the *identity section* $\mathrm{id} \in T^*M \otimes TM \cong \mathrm{End}TM$ under the bijective correspondence (5) for $r = 1$, $E = TM$ and $V = \mathbb{R}^n$ (carrying the above H-module structure) is the desired soldering form. (iv) $\Rightarrow$ (iii): Let θ be the soldering form. Due to [KoNo], Theorem II.2.1 there exists a connection form γ. $\omega = \gamma + \theta$ is a reductive Cartan connection with $\mathfrak{g} := \mathfrak{h} \ltimes \mathbb{R}^n$, and $\mathbb{R}^n =: \mathfrak{p}$ viewed as a H-module (and $\mathfrak{h}$-module) via the representation ρ appearing in the definition of the soldering form θ. (iii) $\Rightarrow$ (ii): Trivial. (ii) $\Rightarrow$ (i): Trivial. (i) $\Rightarrow$ (v): [Sh], Theorem 5.3.15. □

Note, for a Cartan connection ω the composition $\omega_{\mathfrak{g}/\mathfrak{h}} := TP \xrightarrow{\omega} \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ is a soldering form, providing a direct proof for (i) $\Rightarrow$ (iv). As mentioned in the proof, $\omega_{\mathfrak{g}/\mathfrak{h}}$ is identified via (5) with the identity section $\mathrm{id} \in \mathrm{End}TM$. Cf. [Ko], Theorem 2, and [AM], 2.4, 5.1.

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