Ten Lectures on Jet Manifolds in Classical and Quantum Field Theory

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Abstract.
These Lectures summarize the relevant material on existent applications of jet manifold techniques to classical and quantum field theory. The following topics are included: *1. Fibre bundles, 2. Jet manifolds, 3. Connections, 4. Lagrangian field theory, 5. Gauge theory of principal connections, 6. Higher order jets, 7. Infinite order jets, 8. The variational bicomplex, 9. Geometry of simple graded manifolds, 10. Jets of ghosts and antifields.*

Introduction

Finite order jet manifolds [24, 42] provide the adequate mathematical formulation of classical field theory [17, 36, 38]. Infinite order jets and jets of odd variables find applications to quantum field theory, namely, to the field-antifield BRST model [5, 6, 10, 11]. Therefore, we aim to modify our survey hep-th/9411089 and to complete it with the relevant facts on infinite order jets, the variational bicomplex, and jets of graded manifolds [19, 20, 30, 41].

All morphisms throughout are smooth and manifolds are real, smooth, and finite-dimensional. Smooth manifolds are customarily assumed to be Hausdorff and second-countable topological space (i.e., have a countable base for topology). Consequently, they are paracompact, separable (i.e., have a countable dense subset), and locally compact topological spaces, which are countable at infinity. Unless otherwise stated, manifolds are assumed to be connected, i.e., are also arcwise connected.

1 Fibre bundles

A fibred manifold (or a fibration) over an n-dimensional base X is defined as a manifold surjection

$$\pi : Y \to X, \tag{1.1}$$

where Y admits an atlas of fibred coordinates (x^λ, y^i) such that (x^λ) are coordinates on the base X, i.e.,

$$\pi : Y \ni (x^\lambda, y^i) \mapsto (x^\lambda) \in X.$$

This condition is equivalent to π being a submersion, i.e., the tangent map $T\pi : TY \to TX$ is a surjection. It follows that π is also an open map.

A. Smooth fibre bundles

A fibred manifold $Y \to X$ is said to be a (smooth) fibre bundle if there exist a manifold V, called a typical fibre, and an open cover $\mathfrak{U} = \{U_\xi\}$ of X such that Y is locally diffeomorphic to the splittings

$$\psi_\xi : \pi^{-1}(U_\xi) \to U_\xi \times V, \tag{1.2}$$

glued together by means of transition functions

$$\rho_{\xi\zeta} = \psi_\xi \circ \psi_\zeta^{-1} : U_\xi \cap U_\zeta \times V \to U_\xi \cap U_\zeta \times V \tag{1.3}$$

on overlaps $U_\xi \cap U_\zeta$. It follows that fibres $Y_x = \pi^{-1}(x)$, $x \in X$, of a fibre bundle are its closed imbedded submanifolds. Transition functions $\rho_{\xi\zeta}$ fulfil the cocycle condition

$$\rho_{\xi\zeta} \circ \rho_{\zeta\iota} = \rho_{\xi\iota} \tag{1.4}$$

on all overlaps $U_\xi \cap U_\zeta \cap U_\iota$. We will also use the notation

$$\psi_\xi(x) : Y_x \to V, \qquad x \in U_\xi, \tag{1.5}$$

$$\rho_{\xi\zeta}(x) : V \to V, \qquad x \in U_\xi \cap U_\zeta. \tag{1.6}$$

Trivialization charts (U_ξ, ψ_ξ) together with transition functions $\rho_{\xi\zeta}$ (1.3) constitute a bundle atlas

$$\Psi = \{(U_\xi, \psi_\xi), \rho_{\xi\zeta}\} \tag{1.7}$$

of a fibre bundle $Y \to X$. Two bundle atlases are said to be equivalent if their union is also a bundle atlas, i.e., there exist transition functions between trivialization charts of different atlases. A fibre bundle $Y \to X$ is uniquely defined by a bundle atlas, and all its atlases are equivalent.

Throughout, only proper coverings of manifolds are considered, i.e., $U_\xi \neq U_\zeta$ if $\zeta \neq \xi$. A cover $\mathfrak{U}'$ is said to be a refinement of a cover $\mathfrak{U}$ if, for each $U' \in \mathfrak{U}'$, there exists $U \in \mathfrak{U}$ such that $U' \subset U$. Of course, if a fibre bundle $Y \to X$ has a bundle atlas over a cover $\mathfrak{U}$ of X, it admits a bundle atlas over any refinement of $\mathfrak{U}$. The following two theorems describe the particular covers which one can choose for a bundle atlas.

THEOREM 1.1. Every smooth fibre bundle $Y \to X$ admits a bundle atlas over a countable cover $\mathfrak{U}$ of X where each member U_ξ of $\mathfrak{U}$ is a domain (i.e., a contractible open subset) whose closure $\overline{U}_\xi$ is compact [21]. $\Box$

Proof. The statement at once follows from the fact that, for any cover $\mathfrak{U}$ of an n-dimensional smooth manifold X, there exists a countable atlas $\{(U'_i, \phi_i)\}$ of X such that: (i) the cover $\{U'_i\}$ refines $\mathfrak{U}$, (ii) $\phi_i(U'_i) = \mathbb{R}^n$, and (iii) $\overline{U}'_i$ is compact, $i \in \mathbb{N}$. **QED**

If the base X is compact, there is a bundle atlas of Y over a finite cover of X which obeys the condition of Theorem 1.1. In general, every smooth fibre bundle admits a bundle atlas over a finite cover of its base X, but its members need not be contractible and connected as follows.

THEOREM 1.2. Every smooth fibre bundle $Y \to X$ admits a bundle atlas over a finite cover $\mathfrak{U}$ of X. □

Proof. Let Ψ (1.7) be a bundle atlas of $Y \to X$ over a cover $\mathfrak{U}$ of X. For any cover $\mathfrak{U}$ of a manifold X, there exists its refinement $\{U_{ij}\}$, where $j \in \mathbb{N}$ and i runs through a finite set such that $U_{ij} \cap U_{ik} = \emptyset$, $j \neq k$. Let $\{(U_{ij}, \psi_{ij})\}$ be the corresponding bundle atlas of the fibre bundle $Y \to X$. Then Y has the finite bundle atlas

$$U_i \overset{\rm def}{=} \bigcup_j U_{ij}, \qquad \psi_i(x) \overset{\rm def}{=} \psi_{ij}(x), \qquad x \in U_{ij} \subset U_i.$$

QED

Without a loss of generality, we will further assume that a cover $\mathfrak{U}$ for a bundle atlas of $Y \to X$ is also a cover for a manifold atlas of the base X. Then, given a bundle atlas Ψ (1.7), a fibre bundle Y is provided with the associated bundle coordinates

$$x^\lambda(y) = (x^\lambda \circ \pi)(y), \qquad y^i(y) = (y^i \circ \psi_\xi)(y), \qquad y \in \pi^{-1}(U_\xi),$$

where x^λ are coordinates on $U_\xi \subset X$ and y^i are coordinates on the typical fibre V.

Morphisms of fibre bundles, by definition, preserve their fibrations, i.e., send a fibre to a fibre. Namely, a bundle morphism of a fibre bundle $\pi : Y \to X$ to a fibre bundle $\pi' : Y' \to X'$ is defined as a pair (Φ, f) of manifold morphisms which make up the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}, \qquad \pi' \circ \Phi = f \circ \pi,$$

i.e., Φ is a fibrewise morphism over f which sends a fibre Y_x, $x \in X$, to a fibre $Y'_{f(x)}$. A bundle diffeomorphism is called an isomorphism, or an automorphism if it is an isomorphism to itself. In field theory, any automorphism of a fibre bundle is treated as a gauge transformation. For the sake of brevity, a bundle morphism over $f = \mathrm{Id}\, X$ is often said to be a bundle morphism over X, and is denoted by $Y \underset{X}{\longrightarrow} Y'$. In particular, an automorphism over X is called a vertical automorphism or a vertical gauge transformation. Two different fibre bundles over the same base X are said to be equivalent if there exists their isomorphism over X. A bundle monomorphism $\Phi : Y \to Y'$ over X is called a subbundle of the fibre bundle $Y' \to X$ if $\Phi(Y)$ is a submanifold of Y'.

In particular, a fibre bundle $Y \to X$ is said to be trivial if it is equivalent to the Cartesian product of manifolds

$$X \times V \xrightarrow{\mathrm{pr}_1} X.$$

It should be emphasized that a trivial fibre bundle admits different trivializations $Y \cong X \times V$ which differ from each other in surjections $Y \to V$.

THEOREM 1.3. A fibre bundle over a contractible base is always trivial [45]. □

Classical fields are described by sections of fibre bundles. A section (or a global section) of a fibre bundle $Y \to X$ is defined as a manifold injection $s : X \to Y$ such that $\pi \circ s = \mathrm{Id}\, X$, i.e., a section sends any point $x \in X$ into the fibre Y_x over this point. A section s is an imbedding, i.e., $s(X) \subset Y$ is both a submanifold and a topological subspace of Y. It is also a closed map, which sends closed subsets of X onto closed subsets of Y. In particular, $\pi(X)$ is a closed submanifold of Y. Similarly, a section of a fibre bundle $Y \to X$ over a submanifold of X is defined. Let us note that by a local local section is customarily meant a section over an open subset of X. A fibre bundle admits a local section around each point of its base, but need not have a global section.

THEOREM 1.4. A fibre bundle $Y \to X$ whose typical fibre is diffeomorphic to an Euclidean space $\mathbb{R}^m$ has a global section. More generally, its section over a closed imbedded submanifold (e.g., a point) of X is extended to a global section [45]. □

Given a bundle atlas Ψ and associated bundle coordinates (x^λ, y^i), a section s of a fibre bundle $Y \to X$ is represented by collections of local functions $\{s^i = y^i \circ \psi_\xi \circ s\}$ on trivialization sets U_ξ.

In conclusion, let us describe two standard constructions of new fibre bundles from old ones.

- Given a fibre bundle $\pi : Y \to X$ and a manifold morphism $f : X' \to X$, the pull-back of Y by f is defined as a fibre bundle

$$f^*Y = \{(x', y) \in X' \times Y \ : \ \pi(y) = f(x')\} \tag{1.8}$$

  over X' provided with the natural surjection $(x', y) \mapsto x'$. Roughly speaking, its fibre over a point $x' \in X'$ is that of Y over the point $f(x') \in X$.

- Let Y and Y' be fibre bundles over the same base X. Their fibred product $Y \times Y'$ is a fibre bundle over X whose fibres are the Cartesian products $Y_x \times Y'_x$ of those of fibre bundles Y and Y'.

B. Vector and affine bundles

Vector and affine bundles provide a standard framework in classical and quantum field theory. Matter fields are sections of vector bundles, while gauge potentials are sections of an affine bundle.

A typical fibre and fibres of a smooth vector bundle $\pi : Y \to X$ are vector spaces of some finite dimension (called the fibre dimension $\mathrm{fdim}\, Y$ of Y), and Y admits a bundle atlas Ψ (1.7) where trivialization morphisms $\psi_\xi(x)$ (1.5) and transition functions $\rho_{\xi\zeta}(x)$ (1.6) are linear isomorphisms of vector spaces. The corresponding bundle coordinates (y^i) possess a linear coordinate transformation law

$$y'^i = \rho^i_j(x) y^j.$$

We have the decomposition $y = y^i e_i(\pi(y))$, where

$$\{e_i(x)\} = \psi_\xi^{-1}(x)\{v_i\}, \qquad x = \pi(y) \in U_\xi,$$

are fibre bases (or frames) for fibres Y_x of Y and $\{v_i\}$ is a fixed basis for the typical fibre V of Y.

By virtue of Theorem 1.4, a vector bundle has a global section, e.g., the canonical zero section $\widehat{0}(X)$ which sends every point $x \in X$ to the origin 0 of the fibre Y_x over x.

The following are the standard constructions of new vector bundles from old ones.

- Given two vector bundles Y and Y' over the same base X, their Whitney sum $Y \oplus Y'$ is a vector bundle over X whose fibres are the direct sums of those of the vector bundles Y and Y'.

- Given two vector bundles Y and Y' over the same base X, their tensor product $Y \otimes Y'$ is a vector bundle over X whose fibres are the tensor products of those of the vector bundles Y and Y'. Similarly, the exterior product $Y \wedge Y$ of vector bundles is defined. We call

$$\wedge Y = X \times \mathbb{R} \underset{X}{\oplus} Y \underset{X}{\oplus} \overset{2}{\wedge} Y \underset{X}{\oplus} \cdots \underset{X}{\oplus} \overset{m}{\wedge} Y, \qquad m = \mathrm{fdim}\, Y, \tag{1.9}$$

  the exterior bundle of Y.

- Let $Y \to X$ be a vector bundle. By $Y^* \to X$ is denoted the dual vector bundle whose fibres are the duals of those of Y. The interior product (or contraction) of Y and Y^* is defined as a bundle morphism

$$\rfloor : Y \otimes Y^* \underset{X}{\longrightarrow} X \times \mathbb{R}.$$

Vector bundles are subject to linear bundle morphisms, which are linear fibrewise maps. They possess the following property. Given vector bundles Y' and Y over the same base X, every linear bundle morphism

$$\Phi : Y'_x \ni \{e'_i(x)\} \mapsto \{\Phi_i^k(x) e_k(x)\} \in Y_x$$

over X defines a global section

$$\Phi : x \mapsto \Phi_i^k(x) e_k(x) \otimes e'^i(x)$$

of the tensor product $Y \otimes Y'^*$, and *vice versa.*

Given a linear bundle morphism $\Phi : Y' \to Y$ of vector bundles over X, its kernel $\mathrm{Ker}\,\Phi$ is defined as the inverse image $\Phi^{-1}(\widehat{0}(X))$ of the canonical zero section $\widehat{0}(X)$ of Y. If Φ is of constant rank, its kernel $\mathrm{Ker}\,\Phi$ and its image $\mathrm{Im}\,\Phi$ are subbundles of the vector bundles Y' and Y, respectively. For instance, monomorphisms and epimorphisms of vector bundles fulfil this condition. If Y' is a subbundle of the vector bundle $Y \to X$, the factor bundle Y/Y' over X is defined as a vector bundle whose fibres are the quotients Y_x/Y'_x, $x \in X$.

Let us consider a sequence

$$Y' \xrightarrow{i} Y \xrightarrow{j} Y''$$

of vector bundles over X. It is called exact at Y if $\mathrm{Ker}\, j = \mathrm{Im}\, i$. Let

$$0 \to Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \to 0 \tag{1.10}$$

be a sequence of vector bundles over X, where 0 denotes the zero-dimensional vector bundle over X. This sequence is called a short exact sequence if it is exact at all terms Y', Y, and Y''. This means that i is a bundle monomorphism, j is a bundle epimorphism, and $\mathrm{Ker}\, j = \mathrm{Im}\, i$. Then Y'' is the factor bundle Y/Y'. One says that the short exact sequence (1.10) admits a splitting if there exists a bundle monomorphism $s : Y'' \to Y$ such that $j \circ s = \mathrm{Id}\, Y''$ or, equivalently,

$$Y = i(Y') \oplus s(Y'') \cong Y' \oplus Y''.$$

THEOREM 1.5. Every exact sequence of vector bundles (1.10) is split [22]. □

Given an exact sequence of vector bundles (1.10), we have the (dual) exact sequence of the dual bundles

$$0 \to Y''^* \xrightarrow{j^*} Y^* \xrightarrow{i^*} Y'^* \to 0.$$

Let us turn to affine bundles. Given a vector bundle $\overline{Y} \to X$, an affine bundle modelled over $\overline{Y} \to X$ is a fibre bundle $Y \to X$ whose fibres Y_x, $x \in X$, are affine spaces modelled over the corresponding fibres $\overline{Y}_x$ of the vector bundle $\overline{Y}$, and Y admits a bundle atlas Ψ (1.7) whose trivialization morphisms $\psi_\xi(x)$ and transition functions functions $\rho_{\xi\zeta}(x)$

are affine maps. The corresponding bundle coordinates (y^i) possess an affine coordinate transformation law

$$y'^i = \rho^i_j(x)y^j + \rho^i(x).$$

There are the bundle morphisms

$$\begin{array}{ll} Y \underset{X}{\times} \overline{Y} \underset{X}{\longrightarrow} Y, & (y^i, \overline{y}^i) \mapsto y^i + \overline{y}^i, \\ Y \underset{X}{\times} Y \underset{X}{\longrightarrow} \overline{Y}, & (y^i, y'^i) \mapsto y^i - y'^i, \end{array}$$

where $(\overline{y}^i)$ are linear bundle coordinates on the vector bundle $\overline{Y}$. For instance, every vector bundle has a natural structure of an affine bundle.

By virtue of Theorem 1.4, every affine bundle has a global section.

One can define a direct sum $Y \oplus \overline{Y}'$ of a vector bundle $\overline{Y}' \to X$ and an affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$. This is an affine bundle modelled over the Whitney sum of vector bundles $\overline{Y}' \oplus \overline{Y}$.

Affine bundles are subject to affine bundle morphisms which are affine fibrewise maps. Any affine bundle morphism $\Phi : Y \to Y'$ from an affine bundle Y modelled over a vector bundle $\overline{Y}$ to an affine bundle Y' modelled over a vector bundle $\overline{Y}'$, yields the linear bundle morphism of these vector bundles

$$\overline{\Phi} : \overline{Y} \to \overline{Y}', \qquad \overline{y}'^i \circ \overline{\Phi} = \frac{\partial \Phi^i}{\partial y^j} \overline{y}^j. \tag{1.11}$$

The analogues of Theorems 1.1, 1.2 on a particular cover for atlases of vector and affine bundles hold.

C. Tangent and cotangent bundles

Tangent and cotangent bundles exemplify vector bundles. The fibres of the tangent bundle

$$\pi_Z : TZ \to Z$$

of a manifold Z are tangent spaces to Z. The peculiarity of the tangent bundle TZ in comparison with other vector bundles over Z lies in the fact that, given an atlas $\Psi_Z = \{(U_\xi, \phi_\xi)\}$ of a manifold Z, the tangent bundle of Z is provided with the holonomic atlas $\Psi = \{(U_\xi, \psi_\xi = T\phi_\xi)\}$, where by $T\phi_\xi$ is meant the tangent map to ϕ_ξ. Namely, given coordinates (z^λ) on a manifold Z, the associated bundle coordinates on TZ are holonomic coordinates $(\dot{z}^\lambda)$ with respect to the holonomic frames $\{\partial_\lambda\}$ for tangent spaces $T_z Z$, $z \in Z$. Their transition functions read

$$\dot{z}'^\lambda = \frac{\partial z'^\lambda}{\partial z^\mu} \dot{z}^\mu.$$

Every manifold morphism $f : Z \to Z'$ yields the linear bundle morphism over f of the tangent bundles

$$Tf : TZ \underset{f}{\longrightarrow} TZ', \qquad \dot z'^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial z^\mu}\dot z^\mu. \tag{1.12}$$

It is called the tangent map to f.

The cotangent bundle of a manifold Z is the dual

$$\pi_{*Z} : T^*Z \to Z$$

of the tangent bundle $TZ \to Z$. It is equipped with the holonomic coordinates $(z^\lambda, \dot z_\lambda)$ with respect to the coframes $\{dz^\lambda\}$ for T^*Z which are the duals of $\{\partial_\lambda\}$. Their transition functions read

$$\dot z'_\lambda = \frac{\partial z^\mu}{\partial z'^\lambda}\dot z_\mu.$$

A tensor product

$$T = (\overset{m}{\otimes} TZ) \otimes (\overset{k}{\otimes} T^*Z), \qquad m, k \in \mathbb{N}, \tag{1.13}$$

over Z of tangent and cotangent bundles is called a tensor bundle.

Tangent, cotangent and tensor bundles belong to the category of natural fibre bundles which admit the canonical lift of any diffeomorphism f of a base to a bundle automorphism, called the natural automorphism [24]. For instance, the natural automorphism of the tangent bundle TZ over a diffeomorphism f of its base Z is the tangent map Tf (1.12) to f. In view of the expression (1.12), natural automorphisms are also called holonomic transformations or general covariant transformations (in gravitation theory).

Let us turn now to peculiarities of tangent and cotangent bundles of fibre bundles.

Let $\pi_Y : TY \to Y$ be the tangent bundle of a fibre bundle $\pi : Y \to X$. Given bundle coordinates (x^λ, y^i) on Y, the tangent bundle TY is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot x^\lambda, \dot y^i)$. The tangent bundle $TY \to Y$ has the subbundle $VY = \mathrm{Ker}\, T\pi$ which consists of the vectors tangent to fibres of Y. It is called the vertical tangent bundle of Y, and is provided with the holonomic coordinates $(x^\lambda, y^i, \dot y^i)$ with respect to the vertical frames $\{\partial_i\}$.

Let $T\Phi$ be the tangent map to a bundle morphism $\Phi : Y \to Y'$. Its restriction $V\Phi$ to VY is a linear bundle morphism $VY \to VY'$ such that

$$\dot y'^i \circ V\Phi = \dot y^j \partial_j \Phi^i.$$

It is called the vertical tangent map to Φ.

In many important cases, the vertical tangent bundle $VY \to Y$ of a fibre bundle $Y \to X$ is trivial, and is equivalent to the fibred product

$$VY \cong Y \underset{X}{\times} \overline{Y} \tag{1.14}$$

of Y and some vector bundle $\overline{Y} \to X$. This means that VY can be provided with bundle coordinates $(x^\lambda, y^i, \overline{y}^i)$ such that a transformation law of coordinates $\overline{y}^i$ is independent of coordinates y^i. One calls (1.14) the vertical splitting.

For instance, every affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ admits the canonical vertical splitting (1.14) with respect to the holonomic coordinates $\dot{y}^i$ on VY, whose transformation law coincides with that of the linear coordinates $\overline{y}^i$ on the vector bundle $\overline{Y}$. If Y is a vector bundle, the vertical splitting (1.14) reads

$$VY \cong Y \underset{X}{\times} Y. \tag{1.15}$$

The vertical cotangent bundle $V^*Y \to Y$ of a fibre bundle $Y \to X$ is defined as the dual of the vertical tangent bundle $VY \to Y$. It is not a subbundle of the cotangent bundle T^*Y, but there is the canonical surjection

$$\zeta : T^*Y \ni \dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i \mapsto \dot{y}_i \overline{d}y^i \in V^*Y, \tag{1.16}$$

where $\{\overline{d}y^i\}$ are the bases for the fibres of V^*Y which are duals of the holonomic frames $\{\partial_i\}$ for the vertical tangent bundle VY. It should be emphasized that coframes $\{dy^i\}$ for T^*Y and $\{\overline{d}y^i\}$ for V^*Y are transformed in a different way.

With VY and V^*Y, we have the following two exact sequences of vector bundles over Y:

$$0 \to VY \hookrightarrow TY \xrightarrow{\pi_T} Y \underset{X}{\times} TX \to 0, \tag{1.17a}$$

$$0 \to Y \underset{X}{\times} T^*X \hookrightarrow T^*Y \xrightarrow{\zeta} V^*Y \to 0. \tag{1.17b}$$

In accordance with Theorem 1.5, they have a splitting which, by definition, is a connection on a fibre bundle $Y \to X$.

D. Composite fibre bundles

Let us consider the composition

$$\pi : Y \to \Sigma \to X, \tag{1.18}$$

of fibre bundles

$$\pi_{Y\Sigma} : Y \to \Sigma, \tag{1.19}$$

$$\pi_{\Sigma X} : \Sigma \to X. \tag{1.20}$$

It is called the composite fibre bundle. It is provided with bundle coordinates $(x^\lambda, \sigma^m, y^i)$, where (x^λ, σ^m) are bundle coordinates on the fibre bundle (1.20), i.e., transition functions of coordinates σ^m are independent of coordinates y^i.

The following two assertions make composite fibre bundles useful for numerous physical applications [17, 30].

PROPOSITION 1.6. Given a composite fibre bundle (1.18), let h be a global section of the fibre bundle $\Sigma \to X$. Then the restriction

$$Y_h = h^*Y \tag{1.21}$$

of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of the fibre bundle $Y \to X$. □

PROPOSITION 1.7. (i) Given a section h of the fibre bundle $\Sigma \to X$ and a section s_Σ of the fibre bundle $Y \to \Sigma$, their composition $s = s_\Sigma \circ h$ is a section of the composite fibre bundle $Y \to X$ (1.18).

(ii) Conversely, every section s of the fibre bundle $Y \to X$ is a composition of the section $h = \pi_{Y\Sigma} \circ s$ of the fibre bundle $\Sigma \to X$ and some section s_Σ of the fibre bundle $Y \to \Sigma$ over the closed imbedded submanifold $h(X) \subset \Sigma$. □

In field theory, sections h of the fibre bundle $\Sigma \to X$ play the role, e.g., of a Higgs field and a gravitational field.

E. Vector fields

A vector field on a manifold Z is defined as a global section of the tangent bundle $TZ \to Z$. The set $\mathcal{T}_1(Z)$ of vector fields on Z is a real Lie algebra with respect to the Lie bracket

$$[v, u] = (v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu)\partial_\mu, \quad v = v^\lambda \partial_\lambda, \quad u = u^\lambda \partial_\lambda.$$

Every vector field on a manifold Z can be seen as an infinitesimal generator of a local one-parameter group of diffeomorphisms of Z as follows [23]. Given an open subset $U \subset Z$ and an interval $(-\epsilon, \epsilon)$ of $\mathbb{R}$, by a local one-parameter group of diffeomorphisms of Z defined on $(-\epsilon, \epsilon) \times U$ is meant a map

$$G : (-\epsilon, \epsilon) \times U \ni (t, z) \mapsto G_t(z) \in Z$$

such that:

- for each $t \in (-\epsilon, \epsilon)$, the map G_t is a diffeomorphism of U onto the open subset $G_t(U) \subset Z$;
- $G_{t+t'}(z) = (G_t \circ G_{t'})(z)$ if $t, t', t + t' \in (-\epsilon, \epsilon)$ and $G_{t'}(z), z \in U$.

If G is defined on $(-\epsilon, \epsilon) \times Z$, it can be prolonged onto $\mathbb{R} \times Z$, and is called a one-parameter group of diffeomorphisms of Z. Any local one-parameter group of diffeomorphisms G on $U \subset Z$ defines a local vector field u on U by setting $u(z)$ to be the tangent vector to the curve $z(t) = G_t(z)$ at $t = 0$. Conversely, if u is a vector field on a manifold Z, there exists a unique local one-parameter group G_u of diffeomorphisms on a neighbourhood of every point $z \in Z$ which defines u. We will call G_u a flow of the vector field u. A vector field u on a manifold Z is called complete if its flow is a one-parameter group of diffeomorphisms of Z. For instance, every vector field on a compact manifold is complete [23].

A vector field u on a fibre bundle $Y \to X$ is an infinitesimal generator of a local one-parameter group G_u of isomorphisms of $Y \to X$ if and only if it is a projectable vector field on Y. A vector field u on a fibre bundle $Y \to X$ is called projectable if it projects onto a vector field on X, i.e., there exists a vector field τ on X which makes up the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{u} & TY \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ X & \xrightarrow{\tau} & TX \end{array}, \qquad \tau \circ \pi = T\pi \circ u.$$

A projectable vector field has the coordinate expression

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i, \qquad \tau = u^\lambda\partial_\lambda,$$

where u^λ are local functions on X. A projectable vector field is said to be vertical if it projects onto the zero vector field $\tau = 0$ on X, i.e., $u = u^i\partial_i$ takes its values in the vertical tangent bundle VY.

In field theory, projectable vector fields on fibre bundles play a role of infinitesimal generators of local one-parameter groups of gauge transformations.

In general, a vector field $\tau = \tau^\lambda\partial_\lambda$ on a base X of a fibre bundle $Y \to X$ gives rise to a vector field on Y by means of a connection on this fibre bundle (see the formula (3.6) below). Nevertheless, every natural fibre bundle $Y \to X$ admits the canonical lift $\widetilde{\tau}$ onto Y of any vector field τ on X. For instance, if Y is the tensor bundle (1.13), the above mentioned canonical lift reads

$$\widetilde{\tau} = \tau^\mu\partial_\mu + [\partial_\nu\tau^{\alpha_1}\dot{x}^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k} + \ldots - \partial_{\beta_1}\tau^\nu\dot{x}^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k} - \ldots]\frac{\partial}{\partial\dot{x}^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k}}. \tag{1.22}$$

In particular, we have the canonical lift

$$\widetilde{\tau} = \tau^\mu\partial_\mu + \partial_\nu\tau^\alpha\dot{x}^\nu\frac{\partial}{\partial\dot{x}^\alpha} \tag{1.23}$$

onto the tangent bundle TX, and that

$$\widetilde{\tau} = \tau^\mu\partial_\mu - \partial_\beta\tau^\nu\dot{x}_\nu\frac{\partial}{\partial\dot{x}_\beta} \tag{1.24}$$

onto the cotangent bundle T^*X.

F. Exterior forms

An exterior r-form on a manifold Z is a section

$$\phi = \frac{1}{r!}\phi_{\lambda_1\ldots\lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}$$

of the exterior product $\overset{r}{\wedge} T^*Z \to Z$. Let $\mathcal{O}^r(Z)$ denote the vector space of exterior r-forms on a manifold Z. By definition, $\mathcal{O}^0(Z) = C^\infty(Z)$ is the ring of smooth real functions on Z. All exterior forms on Z constitute the $\mathbb{N}$-graded exterior algebra $\mathcal{O}^*(Z)$ of global sections of the exterior bundle $\wedge T^*Z$ (1.9) with respect to the exterior product $\wedge$. This algebra is provided with the exterior differential

$$d : \mathcal{O}^r(Z) \to \mathcal{O}^{r+1}(Z),$$
$$d\phi = dz^\mu \wedge \partial_\mu \phi = \frac{1}{r!}\partial_\mu \phi_{\lambda_1\ldots\lambda_r} dz^\mu \wedge dz^{\lambda_1} \wedge \cdots dz^{\lambda_r},$$

which is nilpotent, i.e., $d \circ d = 0$, and obeys the relation

$$d(\phi \wedge \sigma) = d(\phi) \wedge \sigma + (-1)^{|\phi|}\phi \wedge d(\sigma).$$

The symbol $|\phi|$ stands for the form degree.

Given a manifold morphism $f : Z \to Z'$, any exterior k-form ϕ on Z' yields the pull-back exterior form $f^*\phi$ on Z by the condition

$$f^*\phi(v^1, \ldots, v^k)(z) = \phi(Tf(v^1), \ldots, Tf(v^k))(f(z))$$

for an arbitrary collection of tangent vectors $v^1, \cdots, v^k \in T_zZ$. The following relations hold:

$$f^*(\phi \wedge \sigma) = f^*\phi \wedge f^*\sigma, \qquad df^*\phi = f^*(d\phi).$$

In particular, given a fibre bundle $\pi : Y \to X$, the pull-back onto Y of exterior forms on X by π provides the monomorphism of exterior algebras

$$\pi^* : \mathcal{O}^*(X) \to \mathcal{O}^*(Y).$$

Elements of its image $\pi^*\mathcal{O}^*(X)$ are called basic forms. Exterior forms on Y such that $u\rfloor\phi = 0$ for an arbitrary vertical vector field u on Y are said to be horizontal forms. They are generated by horizontal one-forms $\{dx^\lambda\}$. For instance, basic forms are horizontal forms with coefficients in $C^\infty(X) \subset C^\infty(Y)$. A horizontal form of degree $n = \dim X$ is called a density. For instance, Lagrangians in field theory are densities. We will use the notation

$$\omega = dx^1 \wedge \cdots \wedge dx^n, \qquad \omega_\lambda = \partial_\lambda\rfloor\omega, \qquad \omega_{\mu\lambda} = \partial_\mu\rfloor\partial_\lambda\rfloor\omega. \tag{1.25}$$

The interior product (or contraction) of a vector field $u = u^\mu\partial_\mu$ and an exterior r-form ϕ on a manifold Z is given by the coordinate expression

$$u\rfloor\phi = \sum_{k=1}^{r} \frac{(-1)^{k-1}}{r!} u^{\lambda_k}\phi_{\lambda_1\ldots\lambda_k\ldots\lambda_r} dz^{\lambda_1} \wedge \cdots \wedge \widehat{dz}^{\lambda_k} \wedge \cdots \wedge dz^{\lambda_r} = \tag{1.26}$$
$$\frac{1}{(r-1)!} u^\mu \phi_{\mu\alpha_2\ldots\alpha_r} dz^{\alpha_2} \wedge \cdots \wedge dz^{\alpha_r},$$

where the caret $\widehat{}$ denotes omission. The following relations hold:

$$\phi(u_1,\ldots,u_r) = u_r\rfloor\cdots u_1\rfloor\phi, \tag{1.27}$$

$$u\rfloor(\phi\wedge\sigma) = u\rfloor\phi\wedge\sigma + (-1)^{|\phi|}\phi\wedge u\rfloor\sigma, \tag{1.28}$$

$$[u,u']\rfloor\phi = u\rfloor d(u'\rfloor\phi) - u'\rfloor d(u\rfloor\phi) - u'\rfloor u\rfloor d\phi, \qquad \phi\in\mathcal{O}^1(Z). \tag{1.29}$$

The Lie derivative of an exterior form ϕ along a vector field u is defined as

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi),$$

and fulfils the relation

$$\mathbf{L}_u(\phi\wedge\sigma) = \mathbf{L}_u\phi\wedge\sigma + \phi\wedge\mathbf{L}_u\sigma.$$

In particular, if f is a function, then

$$\mathbf{L}_u f = u(f) = u\rfloor df.$$

It is important for physical applications that an exterior form ϕ is invariant under a local one-parameter group of diffeomorphisms G_t of Z (i.e., $G_t^*\phi = \phi$) if and only if its Lie derivative $\mathbf{L}_u\phi$ along the vector field u, generating G_t, vanishes.

G. Tangent-valued forms

A tangent-valued r-form on a manifold Z is a section

$$\phi = \frac{1}{r!}\phi^\mu_{\lambda_1\ldots\lambda_r}dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_r}\otimes\partial_\mu \tag{1.30}$$

of the tensor bundle $\overset{r}{\wedge} T^*Z\otimes TZ\to Z$. Tangent-valued forms play a prominent role in jet formalism and theory of connections on fibre bundles.

In particular, there is one-to-one correspondence between the tangent-valued one-forms ϕ on a manifold Z and the linear bundle endomorphisms

$$\widehat\phi: TZ\to TZ, \qquad \widehat\phi: T_zZ\ni v\mapsto v\rfloor\phi(z)\in T_zZ, \tag{1.31}$$

$$\widehat\phi^*: T^*Z\to T^*Z, \qquad \widehat\phi^*: T^*_zZ\ni v^*\mapsto \phi(z)\rfloor v^*\in T^*_zZ, \tag{1.32}$$

over Z. For instance, the canonical tangent-valued one-form

$$\theta_Z = dz^\lambda\otimes\partial_\lambda \tag{1.33}$$

on Z corresponds to the identity morphisms (1.31) and (1.32).

The space $\mathcal{O}^*(Z)\otimes\mathcal{T}_1(Z)$ of tangent-valued forms is provided with the Frölicher–Nijenhuis bracket

$$[\,,\,]_{\rm FN}: \mathcal{O}^r(Z)\otimes\mathcal{T}_1(Z)\times\mathcal{O}^s(Z)\otimes\mathcal{T}_1(Z)\to\mathcal{O}^{r+s}(Z)\otimes\mathcal{T}_1(Z),$$

$$[\phi,\sigma]_{\rm FN} = \frac{1}{r!s!}(\phi^\nu_{\lambda_1\ldots\lambda_r}\partial_\nu\sigma^\mu_{\lambda_{r+1}\ldots\lambda_{r+s}} - \sigma^\nu_{\lambda_{r+1}\ldots\lambda_{r+s}}\partial_\nu\phi^\mu_{\lambda_1\ldots\lambda_r} - \tag{1.34}$$

$$r\phi^\mu_{\lambda_1\ldots\lambda_{r-1}\nu}\partial_{\lambda_r}\sigma^\nu_{\lambda_{r+1}\ldots\lambda_{r+s}} + s\sigma^\mu_{\nu\lambda_{r+2}\ldots\lambda_{r+s}}\partial_{\lambda_{r+1}}\phi^\nu_{\lambda_1\ldots\lambda_r})dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_{r+s}}\otimes\partial_\mu.$$

The following relations hold:

$$[\phi,\psi]_{\rm FN} = (-1)^{|\phi||\psi|+1}[\psi,\phi]_{\rm FN}, \tag{1.35}$$
$$[\phi,[\psi,\theta]_{\rm FN}]_{\rm FN} = [[\phi,\psi]_{\rm FN},\theta]_{\rm FN} + (-1)^{|\phi||\psi|}[\psi,[\phi,\theta]_{\rm FN}]_{\rm FN}. \tag{1.36}$$

Given a tangent-valued form θ, the Nijenhuis differential on $\mathcal{O}^*(Z)\otimes\mathcal{T}_1(Z)$ along θ is defined as

$$d_\theta\sigma = [\theta,\sigma]_{\rm FN}. \tag{1.37}$$

By virtue of the relation (1.36), it has the property

$$d_\phi[\psi,\theta]_{\rm FN} = [d_\phi\psi,\theta]_{\rm FN} + (-1)^{|\phi||\psi|}[\psi,d_\phi\theta]_{\rm FN}.$$

In particular, if $\theta = u$ is a vector field, the Nijenhuis differential is the Lie derivative of tangent-valued forms

$$\begin{aligned}\mathbf{L}_u\sigma = d_u\sigma = [u,\sigma]_{\rm FN} = (u^\nu\partial_\nu\sigma^\mu_{\lambda_1\ldots\lambda_s} - \sigma^\nu_{\lambda_1\ldots\lambda_s}\partial_\nu u^\mu + \\ s\sigma^\mu_{\nu\lambda_2\ldots\lambda_s}\partial_{\lambda_1}u^\nu)dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_s}\otimes\partial_\mu, \qquad \sigma\in\mathcal{O}^s(M)\otimes\mathcal{T}(M).\end{aligned} \tag{1.38}$$

Let $Y\to X$ be a fibre bundle. In the sequel, we will deal with the following classes of tangent-valued forms on Y:

- tangent-valued horizontal forms

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} TY,$$
$$\phi = \frac{1}{r!}dx^{\lambda_1}\wedge\ldots\wedge dx^{\lambda_r}\otimes[\phi^\mu_{\lambda_1\ldots\lambda_r}(y)\partial_\mu + \phi^i_{\lambda_1\ldots\lambda_r}(y)\partial_i];$$

- vertical-valued horizontal forms

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} VY,$$
$$\phi = \frac{1}{r!}\phi^i_{\lambda_1\ldots\lambda_r}(y)dx^{\lambda_1}\wedge\ldots\wedge dx^{\lambda_r}\otimes\partial_i;$$

- vertical-valued horizontal one-forms, called soldering forms,

$$\sigma = \sigma^i_\lambda(y)dx^\lambda\otimes\partial_i; \tag{1.39}$$

- basic vertical-valued horizontal forms

$$\phi = \frac{1}{r!}\phi^i_{\lambda_1\ldots\lambda_r}(x)dx^{\lambda_1}\wedge\ldots\wedge dx^{\lambda_r}\otimes\partial_i$$

  on an affine bundle which are constant along its fibres.

Any tangent valued form ϕ (1.30) on a manifold Z defines the vertical-valued form

$$\phi = \frac{1}{r!}\phi^\mu_{\lambda_1\ldots\lambda_r} dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_r}\otimes\dot\partial_\mu, \qquad \dot\partial_\mu = \frac{\partial}{\partial\dot z^\mu},$$

on the tangent bundle TZ. For instance, the canonical tangent-valued form θ_Z (1.33) on a manifold Z yields the canonical vertical-valued form

$$\dot\theta_Z = dz^\lambda\otimes\dot\partial_\lambda \tag{1.40}$$

on the tangent bundle TZ. By this reason, tangent-valued one-forms on a manifold Z are also called soldering forms.

2 Jet manifolds

Jet manifolds provide the standard language for theory of (non-linear) differential operators, the calculus of variations, Lagrangian and Hamiltonian formalisms [13, 17, 26, 34]. Here, we restrict our consideration to the notion of jets of sections of fibre bundles.

A. First order jet manifolds

Given a fibre bundle $Y\to X$ with bundle coordinates (x^λ, y^i), let us consider the equivalence classes j^1_xs of its sections s, which are identified by their values $s^i(x)$ and the values of their first order derivatives $\partial_\mu s^i(x)$ at a point $x\in X$. They are called the first order jets of sections at x. One can justify that the definition of jets is coordinate-independent. The key point is that the set J^1Y of first order jets j^1_xs, $x\in X$, is a smooth manifold with respect to the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ such that

$$\begin{aligned} & y^i_\lambda(j^1_xs) = \partial_\lambda s^i(x), \\ & {y'}^i_\lambda = \frac{\partial x^\mu}{\partial {x'}^\lambda}(\partial_\mu + y^j_\mu\partial_j){y'}^i. \end{aligned} \tag{2.1}$$

It is called the first order jet manifold of the fibre bundle $Y\to X$.

The jet manifold J^1Y admits the natural fibrations

$$\pi^1 : J^1Y\ni j^1_xs\mapsto x\in X, \tag{2.2}$$

$$\pi^1_0 : J^1Y\ni j^1_xs\mapsto s(x)\in Y. \tag{2.3}$$

A glance at the transformation law (2.1) shows that π^1_0 is an affine bundle modelled over the vector bundle

$$T^*X\underset{Y}{\otimes} VY\to Y. \tag{2.4}$$

It is convenient to call π^1 (2.2) the jet bundle, while π^1_0 (2.3) is said to be the affine jet bundle.

Let us note that, if $Y \to X$ is a vector or an affine bundle, the jet bundle π_1 (2.2) is so.

Jets can be expressed in terms of familiar tangent-valued forms as follows. There are the canonical imbeddings

$$\lambda_1 : J^1Y \underset{Y}{\hookrightarrow} T^*X \underset{Y}{\otimes} TY, \qquad \lambda_1 = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) = dx^\lambda \otimes d_\lambda, \tag{2.5}$$

$$\theta_1 : J^1Y \hookrightarrow T^*Y \underset{Y}{\otimes} VY, \qquad \theta_1 = (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \tag{2.6}$$

where d_λ are said to be total derivatives, and θ^i are called contact forms. Identifying the jet manifold J^1Y to its images under the canonical morphisms (2.5) and (2.6), one can represent jets $j^1_x s = (x^\lambda, y^i, y^i_\mu)$ by tangent-valued forms

$$dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) \quad \text{and} \quad (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i. \tag{2.7}$$

Sections and morphisms of fibre bundles admit prolongations to jet manifolds as follows.

Any section s of a fibre bundle $Y \to X$ has the jet prolongation to the section

$$(J^1s)(x) \overset{\mathrm{def}}{=} j^1_x s, \qquad y^i_\lambda \circ J^1s = \partial_\lambda s^i(x),$$

of the jet bundle $J^1Y \to X$. A section $\overline{s}$ of the jet bundle $J^1Y \to X$ is called holonomic or integrable if it is the jet prolongation of some section of the fibre bundle $Y \to X$.

Any bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f admits a jet prolongation to a bundle morphism over Φ of affine jet bundles

$$J^1\Phi : J^1Y \underset{\Phi}{\longrightarrow} J^1Y', \qquad {y'}^i_\lambda \circ J^1\Phi = \frac{\partial (f^{-1})^\mu}{\partial x'^\lambda} d_\mu \Phi^i.$$

Any projectable vector field $u = u^\lambda \partial_\lambda + u^i \partial_i$ on a fibre bundle $Y \to X$ has a jet prolongation to the projectable vector field

$$\begin{aligned} &J^1u = r_1 \circ J^1u : J^1Y \to J^1TY \to TJ^1Y, \\ &J^1u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu) \partial^\lambda_i, \end{aligned} \tag{2.8}$$

on the jet manifold J^1Y. In order to obtain (2.8), the canonical bundle morphism

$$r_1 : J^1TY \to TJ^1Y, \qquad \dot{y}^i_\lambda \circ r_1 = (\dot{y}^i)_\lambda - y^i_\mu \dot{x}^\mu_\lambda$$

is used. In particular, there is the canonical isomorphism

$$VJ^1Y = J^1VY, \qquad \dot{y}^i_\lambda = (\dot{y}^i)_\lambda. \tag{2.9}$$

B. Second order jet manifolds

Taking the first order jet manifold of the jet bundle $J^1Y \to X$, we obtain the repeated jet manifold J^1J^1Y provided with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda, \widehat{y}^i_\mu, y^i_{\mu\lambda})$, with transition functions

$$\widehat{y}'^i_\lambda = \frac{\partial x^\alpha}{\partial x'^\lambda} d_\alpha y'^i, \qquad {y'}^i_{\mu\lambda} = \frac{\partial x^\alpha}{\partial x'^\mu} d_\alpha {y'}^i_\lambda, \qquad d_\alpha = \partial_\alpha + \widehat{y}^j_\alpha \partial_j + y^j_{\nu\alpha} \partial^\nu_j.$$

There exist two different affine fibrations of J^1J^1Y over J^1Y:

- the familiar affine jet bundle (2.3):

$$\pi_{11} : J^1J^1Y \to J^1Y, \qquad y^i_\lambda \circ \pi_{11} = y^i_\lambda, \tag{2.10}$$

- and the affine bundle

$$J^1\pi^1_0 : J^1J^1Y \to J^1Y, \qquad y^i_\lambda \circ J^1\pi^1_0 = \widehat{y}^i_\lambda. \tag{2.11}$$

In general, there is no canonical identification of these fibrations. The points $q \in J^1J^1Y$, where $\pi_{11}(q) = J^1\pi^1_0(q)$, form the affine subbundle $\widehat{J}^2Y \to J^1Y$ of J^1J^1Y called the sesqui-holonomic jet manifold. It is given by the coordinate conditions $\widehat{y}^i_\lambda = y^i_\lambda$, and is coordinated by $(x^\lambda, y^i, y^i_\lambda, y^i_{\mu\lambda})$.

The second order jet manifold J^2Y of a fibre bundle $Y \to X$ can be defined as the affine subbundle of the fibre bundle $\widehat{J}^2Y \to J^1Y$ given by the coordinate conditions $y^i_{\lambda\mu} = y^i_{\mu\lambda}$. It is coordinated by $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu} = y^i_{\mu\lambda})$. The second order jet manifold J^2Y can also be introduced as the set of the equivalence classes j^2_xs of sections s of the fibre bundle $Y \to X$, which are identified by their values and the values of their first and second order partial derivatives at points $x \in X$, i.e.,

$$y^i_\lambda(j^2_xs) = \partial_\lambda s^i(x), \qquad y^i_{\lambda\mu}(j^2_xs) = \partial_\lambda\partial_\mu s^i(x).$$

Let s be a section of a fibre bundle $Y \to X$, and let J^1s be its jet prolongation to a section of the jet bundle $J^1Y \to X$. The latter gives rise to the section J^1J^1s of the repeated jet bundle $J^1J^1Y \to X$. This section takes its values into the second order jet manifold J^2Y. It is called the second order jet prolongation of the section s, and is denoted by J^2s.

PROPOSITION 2.1. Let $\overline{s}$ be a section of the jet bundle $J^1Y \to X$, and let $J^1\overline{s}$ be its jet prolongation to the section of the repeated jet bundle $J^1J^1Y \to X$. The following three facts are equivalent: (i) $\overline{s} = J^1s$ where s is a section of the fibre bundle $Y \to X$, (ii) $J^1\overline{s}$ takes its values into $\widehat{J}^2Y$, (iii) $J^1\overline{s}$ takes its values into J^2Y. $\square$

C. Higher order jet manifolds

The notion of first and second order jet manifolds is naturally extended to higher order jets (see Lecture 6 for a detailed exposition). The r-order jet manifold J^rY of a fibre bundle $Y \to X$ is defined as the disjoint union of the equivalence classes j^r_xs of sections s of $Y \to X$ identified by the $r+1$ terms of their Taylor series at points of X. It is a smooth manifold endowed with the adapted coordinates (x^λ, y^i_Λ), $0 \leq\mid \Lambda \mid\leq r$, where $\Lambda = (\lambda_k \ldots \lambda_1)$ denotes a multi-index modulo permutations and

$$y^i_{\lambda_k\cdots\lambda_1}(j^r_xs) = \partial_{\lambda_k}\cdots\partial_{\lambda_1}s^i(x), \qquad 0 \leq k \leq r.$$

The transformation law of these coordinates reads

$$y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial' x^\lambda} d_\mu y'^i_\Lambda, \tag{2.12}$$

where $\lambda + \Lambda = (\lambda\lambda_k \ldots \lambda_1)$ and

$$d_\lambda = \partial_\lambda + \sum_{|\Lambda|<r} y^i_{\lambda+\Lambda}\partial_i^\Lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\lambda\mu}\partial_i^\mu + \cdots$$

are higher order total derivatives. These derivatives act on exterior forms on J^rY and obey the relations

$$d_\lambda(\phi \wedge \sigma) = d_\lambda(\phi) \wedge \sigma + \phi \wedge d_\lambda(\sigma), \qquad d_\lambda(d\phi) = d(d_\lambda(\phi)).$$

For instance,

$$d_\lambda(dx^\mu) = 0, \qquad d_\lambda(dy^i_\Lambda) = dy^i_{\lambda+\Lambda}.$$

Let us also mention the following two operations: the horizontal projection h_0 given by the relations

$$h_0(dx^\lambda) = dx^\lambda, \qquad h_0(dy^i_{\lambda_k\cdots\lambda_1}) = y^i_{\mu\lambda_k\ldots\lambda_1} dx^\mu, \tag{2.13}$$

and the horizontal differential

$$\begin{aligned} &d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \\ &d_H \circ d_H = 0, \qquad h_0 \circ d = d_H \circ h_0. \end{aligned} \tag{2.14}$$

D. Differential equations and differential operators

Let us now formulate the notions of a (non-linear) differential equation and a differential operator in terms of jets.

DEFINITION 2.2. A k-order differential equation on a fibre bundle $Y \to X$ is defined as a closed subbundle $\mathfrak{E}$ of the jet bundle $J^kY \to X$. Its classical solution is a (local) section s of $Y \to X$ whose k-order jet prolongation J^ks lives in $\mathfrak{E}$. $\Box$

One usually considers differential equations associated to differential operators.

DEFINITION 2.3. Let $E \to X$ be a vector bundle coordinated by (x^λ, v^A), $A = 1, \ldots, m$. A bundle morphism

$$\mathcal{E} : J^kY \underset{X}{\to} E, \qquad v^A \circ \mathcal{E} = \mathcal{E}^A(x^\lambda, y^i, y^i_\lambda, \ldots, y^i_{\lambda_k\cdots\lambda_1}), \tag{2.15}$$

is called a k-order differential operator on a fibre bundle $Y \to X$. It sends each section s of $Y \to X$ onto the section

$$(\mathcal{E} \circ J^k s)^A(x) = \mathcal{E}^A(x^\lambda, s^i(x), \partial_\lambda s^i(x), \ldots, \partial_{\lambda_k} \cdots \partial_{\lambda_1} s^i(x))$$

of the vector bundle $E \to X$. $\Box$

Let us suppose that the canonical zero section $\widehat{0}(X)$ of the vector bundle $E \to X$ belongs to the image $\mathcal{E}(J^k Y)$. Then the kernel operator of a differential operator $\mathcal{E}$ is defined as

$$\operatorname{Ker} \mathcal{E} = \mathcal{E}^{-1}(\widehat{0}(X)) \subset J^k Y. \tag{2.16}$$

If $\operatorname{Ker} \mathcal{E}$ (2.16) is a closed subbundle of the jet bundle $J^k Y \to X$, it is a k-order differential equation, associated to the differential operator $\mathcal{E}$. It is written in the coordinate form

$$\mathcal{E}^A(x^\lambda, y^i, y^i_\lambda, \ldots, y^i_{\lambda_k \cdots \lambda_1}) = 0, \qquad A = 1, \ldots, m.$$

3 Connections on fibre bundles

Connections play a prominent role in classical field theory because they enable one to deal with invariantly defined objects. Partial derivatives of sections of fibre bundles (i.e., of classical fields) are ill defined. One need connections in order to replace them with covariant derivatives. Gauge theory shows clearly that this is a basic physical principle.

We start from the traditional geometric notion of a connection as a horizontal lift, but then follow its equivalent definition as a jet field [17, 24, 30, 42]. It enables us to include connections in an natural way in field dynamics.

A. Connections as tangent-valued forms

A connection on a fibre bundle $Y \to X$ is customarily defined as a linear bundle monomorphism

$$\Gamma : Y \underset{X}{\times} TX \underset{Y}{\to} TY, \qquad \Gamma : \dot{x}^\lambda \partial_\lambda \mapsto \dot{x}^\lambda (\partial_\lambda + \Gamma^i_\lambda(y) \partial_i), \tag{3.1}$$

which splits the exact sequence (1.17a), i.e.,

$$\pi_T \circ \Gamma = \operatorname{Id}(Y \underset{X}{\times} TX).$$

The image HY of $Y \underset{X}{\times} TX$ by a connection Γ is called the horizontal distribution. It splits the tangent bundle TY as

$$\begin{aligned} TY &= HY \underset{Y}{\oplus} VY, \\ \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i &= \dot{x}^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i) + (\dot{y}^i - \dot{x}^\lambda \Gamma^i_\lambda) \partial_i. \end{aligned} \tag{3.2}$$

By virtue of Theorem 1.5, a connection on a fibre bundle always exists.

A connection Γ (3.1) defines the horizontal tangent-valued one-form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{3.3}$$

on Y such that $\Gamma(\partial_\lambda) = \partial_\lambda \rfloor \Gamma$. Conversely, every horizontal tangent-valued one-form on a fibre bundle $Y \to X$ which projects onto the canonical tangent-valued form θ_X (1.33) on X defines a connection on $Y \to X$.

In an equivalent way, the horizontal splitting (3.2) is given by the vertical-valued form

$$\Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i, \tag{3.4}$$

which determines the epimorphism

$$\Gamma : TY \ni \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i \to (\dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i) \rfloor \Gamma = (\dot{y}^i - \dot{x}^\lambda \Gamma^i_\lambda)\partial_i \in VY.$$

Given a connection Γ, a vector field u on a fibre bundle $Y \to X$ is called horizontal if it lives in the horizontal distribution HY, i.e., takes the form

$$u = u^\lambda(y)(\partial_\lambda + \Gamma^i_\lambda(y)\partial_i). \tag{3.5}$$

Any vector field τ on the base X of a fibre bundle $Y \to X$ admits the horizontal lift

$$\Gamma\tau = \tau \rfloor \Gamma = \tau^\lambda(\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{3.6}$$

onto Y by means of a connection Γ (3.3) on $Y \to X$.

Given the splitting (3.1), the dual splitting of the exact sequence (1.17b) is

$$\Gamma : V^*Y \ni \overline{d}y^i \mapsto \Gamma \rfloor \overline{d}y^i = dy^i - \Gamma^i_\lambda dx^\lambda \in T^*Y, \tag{3.7}$$

where Γ is the vertical-valued form (3.4).

B. Connections as jet fields

There is one-to-one correspondence between the connections on a fibre bundle $Y \to X$ and the jet fields, i.e., global sections of the affine jet bundle $J^1Y \to Y$ [17, 42]. Indeed, given a global section Γ of $J^1Y \to Y$, the tangent-valued form

$$\lambda_1 \circ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

provides the horizontal splitting (3.2) of TY. Accordingly, the vertical-valued form

$$\theta_1 \circ \Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i$$

leads to the dual splitting (3.7).

It follows immediately from this definition that connections on a fibre bundle $Y \to X$ constitute an affine space modelled over the vector space of soldering forms σ (1.39). One also deduces from (2.1) the coordinate transformation law of connections

$$\Gamma'^i_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda}(\partial_\mu + \Gamma^j_\mu \partial_j) y'^i.$$

The following are two standard constructions of new connections from old ones.

- Let Y and Y' be fibre bundles over the same base X. Given a connection Γ on $Y \to X$ and a connection Γ' on $Y' \to X$, the fibred product $Y \underset{X}{\times} Y'$ is provided with the product connection

$$\Gamma \times \Gamma' = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \frac{\partial}{\partial y^i} + \Gamma'^j_\lambda \frac{\partial}{\partial y'^j}). \tag{3.8}$$

- Given a fibre bundle $Y \to X$, let $f : X' \to X$ be a manifold morphism and f^*Y the pull-back of Y over X'. Any connection Γ (3.4) on $Y \to X$ yields the pull-back connection

$$f^*\Gamma = (dy^i - \Gamma^i_\lambda(f^\mu(x'^\nu), y^j) \frac{\partial f^\lambda}{\partial x'^\mu} dx'^\mu) \otimes \partial_i \tag{3.9}$$

on the pull-back fibre bundle $f^*Y \to X'$.

The key point for physical applications lies in the fact that every connection Γ on a fibre bundle $Y \to X$ yields the first order differential operator

$$D^\Gamma : J^1Y \underset{Y}{\to} T^*X \underset{Y}{\otimes} VY, \tag{3.10}$$
$$D^\Gamma = \lambda_1 - \Gamma \circ \pi^1_0 = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i,$$

called the covariant differential relative to the connection Γ. If $s : X \to Y$ is a section, one defines its covariant differential

$$\nabla^\Gamma s \overset{\text{def}}{=} D_\Gamma \circ J^1 s = (\partial_\lambda s^i - \Gamma^i_\lambda \circ s) dx^\lambda \otimes \partial_i \tag{3.11}$$

and its covariant derivative

$$\nabla^\Gamma_\tau s = \tau \rfloor \nabla^\Gamma s \tag{3.12}$$

along a vector field τ on X. A (local) section s of $Y \to X$ is said to be an integral section of a connection Γ (or parallel with respect to Γ) if s obeys the equivalent conditions

$$\nabla^\Gamma s = 0 \quad \text{or} \quad J^1 s = \Gamma \circ s. \tag{3.13}$$

Furthermore, if $s : X \to Y$ is a global section, there exists a connection Γ such that s is an integral section of Γ. This connection is defined as an extension of the local section $s(x) \mapsto J^1 s(x)$ of the affine jet bundle $J^1Y \to Y$ over the closed imbedded submanifold $s(X) \subset Y$ in accordance with Theorem 1.4.

C. Curvature and torsion

Let Γ be a connection on a fibre bundle $Y \to X$. Given vector fields τ, τ' on X and their horizontal lifts $\Gamma\tau$ and $\Gamma\tau'$ (3.6) on Y, let us compute the vertical vector field

$$R(\tau,\tau') = \Gamma[\tau,\tau'] - [\Gamma\tau,\Gamma\tau'] = \tau^\lambda\tau'^\mu R^i_{\lambda\mu}\partial_i, \tag{3.14}$$
$$R^i_{\lambda\mu} = \partial_\lambda\Gamma^i_\mu - \partial_\mu\Gamma^i_\lambda + \Gamma^j_\lambda\partial_j\Gamma^i_\mu - \Gamma^j_\mu\partial_j\Gamma^i_\lambda. \tag{3.15}$$

It can be seen as the contraction of vector fields τ and τ' with the vertical-valued horizontal two-form

$$R = \frac{1}{2}R^i_{\lambda\mu}dx^\lambda \wedge dx^\mu \otimes \partial_i \tag{3.16}$$

on Y, called the curvature of the connection Γ. In an equivalent way, the curvature (3.16) is defined as the Nijenhuis differential

$$R = \frac{1}{2}d_\Gamma\Gamma = \frac{1}{2}[\Gamma,\Gamma]_{\rm FN} : Y \to \overset{2}{\wedge} T^*X \otimes VY. \tag{3.17}$$

Then we at once obtain from (1.35) – (1.36) the identities

$$[R,R]_{\rm FN} = 0, \tag{3.18}$$
$$d_\Gamma R = [\Gamma,R]_{\rm FN} = 0. \tag{3.19}$$

Given a soldering form σ (1.39) on $Y \to X$, one defines the soldered curvature

$$\rho = \frac{1}{2}d_\sigma\sigma = \frac{1}{2}[\sigma,\sigma]_{\rm FN} : Y \to \overset{2}{\wedge} T^*X \otimes VY, \tag{3.20}$$
$$\rho = \frac{1}{2}\rho^i_{\lambda\mu}dx^\lambda \wedge dx^\mu \otimes \partial_i,$$
$$\rho^i_{\lambda\mu} = \sigma^j_\lambda\partial_j\sigma^i_\mu - \sigma^j_\mu\partial_j\sigma^i_\lambda,$$

which fulfils the identities

$$[\rho,\rho]_{\rm FN} = 0, \qquad d_\sigma\rho = [\sigma,\rho]_{\rm FN} = 0.$$

Given a connection Γ and a soldering form σ, the torsion of Γ with respect to σ is defined as

$$T = d_\Gamma\sigma = d_\sigma\Gamma : Y \to \overset{2}{\wedge} T^*X \otimes VY,$$
$$T = (\partial_\lambda\sigma^i_\mu + \Gamma^j_\lambda\partial_j\sigma^i_\mu - \partial_j\Gamma^i_\lambda\sigma^j_\mu)dx^\lambda \wedge dx^\mu \otimes \partial_i. \tag{3.21}$$

In particular, if $\Gamma' = \Gamma + \sigma$, we have the important relations

$$T' = T + 2\rho, \tag{3.22}$$
$$R' = R + \rho + T. \tag{3.23}$$

D. Linear connections

Any vector bundle $Y \to X$ admits a linear connection. This is defined as a section of the affine jet bundle $J^Y \to Y$ which is a linear morphism of vector bundles over X. A linear connection is given by the tangent-valued form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i{}_j(x) y^j \partial_i). \tag{3.24}$$

There are the following standard constructions of new linear connections from old ones.

- Let $Y \to X$ be a vector bundle, coordinated by (x^λ, y^i), and $Y^* \to X$ its dual, coordinated by (x^λ, y_i). Any linear connection Γ (3.24) on the vector bundle $Y \to X$ defines the dual linear connection

  $$\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma_\lambda{}^j{}_i(x) y_j \partial^i) \tag{3.25}$$

  on $Y^* \to X$.

- Let Γ and Γ' be, respectively, linear connections on vector bundles $Y \to X$ and $Y' \to X$ over the same base X. The direct sum connection $\Gamma \oplus \Gamma'$ on the Whitney sum $Y \oplus Y'$ of these vector bundles is defined as the product connection (3.8).

- Let Y coordinated by (x^λ, y^i) and Y' coordinated by (x^λ, y^a) be vector bundles over the same base X. Their tensor product $Y \otimes Y'$ is endowed with the bundle coordinates (x^λ, y^{ia}). Any linear connections Γ and Γ' on $Y \to X$ and $Y' \to X$ define the linear tensor product connection

  $$\Gamma \otimes \Gamma' = dx^\lambda \otimes [\partial_\lambda + (\Gamma_\lambda{}^i{}_j y^{ja} + \Gamma'_\lambda{}^a{}_b y^{ib}) \frac{\partial}{\partial y^{ia}}] \tag{3.26}$$

  on $Y \otimes Y' \to X$.

The curvature of a linear connection Γ (3.24) on a vector bundle $Y \to X$ is usually written as a Y-valued two-form

$$\begin{aligned} &R = \frac{1}{2} R_{\lambda\mu}{}^i{}_j(x) y^j dx^\lambda \wedge dx^\mu \otimes e_i, \\ &R_{\lambda\mu}{}^i{}_j = \partial_\lambda \Gamma_\mu{}^i{}_j - \partial_\mu \Gamma_\lambda{}^i{}_j + \Gamma_\lambda{}^h{}_j \Gamma_\mu{}^i{}_h - \Gamma_\mu{}^h{}_j \Gamma_\lambda{}^i{}_h, \end{aligned} \tag{3.27}$$

due to the canonical vertical splitting (1.15), where $\{\partial_i\} = \{e_i\}$. For any two vector fields τ and τ' on X, this curvature yields the 0-order differential operator

$$R(\tau, \tau') \circ s = (\nabla^\Gamma_{[\tau,\tau']} - [\nabla^\Gamma_\tau, \nabla^\Gamma_{\tau'}]) s \tag{3.28}$$

on section s of the vector bundle $Y \to X$.

E. World connections

An important example of linear connections is a connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu) \tag{3.29}$$

on the tangent bundle TX of a manifold X. It is called a world connection or, simply, a connection on a manifold X. The dual connection (3.25) on the cotangent bundle T^*X is

$$K^* = dx^\lambda \otimes (\partial_\lambda - K_\lambda{}^\mu{}_\nu \dot{x}_\mu \dot{\partial}^\nu). \tag{3.30}$$

Then, using the tensor product connection (3.26), one can introduce the corresponding linear connection on an arbitrary tensor bundle (1.13).

A world connection (3.29) is called symmetric if $K_\mu{}^\nu{}_\lambda = K_\lambda{}^\nu{}_\mu$. Of course, this property is coordinate-independent. Let us note that, given a world connection K (3.29), the tangent-valued form

$$K_r = dx^\lambda \otimes (\partial_\lambda + (rK_\lambda{}^\mu{}_\nu + (1-r)K_\nu{}^\mu{}_\lambda)\dot{x}^\nu \dot{\partial}_\mu), \qquad 0 \leq r \leq 1, \tag{3.31}$$

is also a world connection. For instance, $K_{1/2}$ is a symmetric connection, called the symmetric part of the connection K.

Remark 3.1. It should be emphasized that the expressions (3.29) – (3.30) for a world connection differ in a minus sign from those usually used in the physics literature. •

Due to the canonical vertical splitting

$$VTX \cong TX \times TX, \tag{3.32}$$

the curvature of a world connection K (3.29) on the tangent bundle TX can be written as the TX-valued two-form (3.27) on X:

$$\begin{aligned} &R = \frac{1}{2} R_{\lambda\mu}{}^\alpha{}_\beta \dot{x}^\beta dx^\lambda \wedge dx^\mu \otimes \partial_\alpha, \\ &R_{\lambda\mu}{}^\alpha{}_\beta = \partial_\lambda K_\mu{}^\alpha{}_\beta - \partial_\mu K_\lambda{}^\alpha{}_\beta + K_\lambda{}^\gamma{}_\beta K_\mu{}^\alpha{}_\gamma - K_\mu{}^\gamma{}_\beta K_\lambda{}^\alpha{}_\gamma. \end{aligned} \tag{3.33}$$

Its Ricci tensor $R_{\lambda\beta} = R_{\lambda\mu}{}^\mu{}_\beta$ is introduced.

A torsion of a world connection is defined as the torsion (3.21) of the connection Γ (3.29) on the tangent bundle TX with respect to the canonical vertical-valued form $\dot{\theta}_X$ (1.40). Due to the vertical splitting (3.32), it is also written as a tangent-valued two-form

$$\begin{aligned} &T = \frac{1}{2} T_\mu{}^\nu{}_\lambda dx^\lambda \wedge dx^\mu \otimes \partial_\nu, \\ &T_\mu{}^\nu{}_\lambda = K_\mu{}^\nu{}_\lambda - K_\lambda{}^\nu{}_\mu, \end{aligned} \tag{3.34}$$

on X. A world connection is symmetric if and only if its torsion (3.34) vanishes.

For instance, every manifold X can be provided with a non-degenerate fibre metric

$$g \in \overset{2}{\vee} \mathcal{O}^1(X), \qquad g = g_{\lambda\mu} dx^\lambda \otimes dx^\mu,$$

in the tangent bundle TX, and with the dual metric

$$g \in \overset{2}{\vee} \mathcal{T}^1(X), \qquad g = g^{\lambda\mu}\partial_\lambda \otimes \partial_\mu$$

in the cotangent bundle T^*X. It is called a world metric on X. For any world metric g, there exists a unique symmetric world connection

$$K_\lambda{}^\nu{}_\mu = \{_\lambda{}^\nu{}_\mu\} = -\frac{1}{2} g^{\nu\rho}(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\lambda\mu}) \tag{3.35}$$

such that g is an integral section of K, i.e.

$$\nabla_\lambda g^{\alpha\beta} = \partial_\lambda \, g^{\alpha\beta} - g^{\alpha\gamma}\{_\lambda{}^\beta{}_\gamma\} - g^{\beta\gamma}\{_\lambda{}^\alpha{}_\gamma\} = 0.$$

This is the Levi–Civita connection, and its components (3.35) are called Christoffel symbols.

F. Affine connections

Any affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ admits an affine connection. This is defined as a section of the affine jet bundle $J^1Y \to Y$ which is an affine morphism of affine bundles over X. An affine connection is given by the tangent-valued form

$$\Gamma^i_\lambda = \Gamma_\lambda{}^i{}_j(x)y^j + \sigma^i_\lambda(x). \tag{3.36}$$

For any affine connection $\Gamma : Y \to J^1Y$ (3.36), the corresponding linear derivative $\overline{\Gamma} : \overline{Y} \to J^1\overline{Y}$ (1.11) defines a unique linear connection

$$\overline{\Gamma}^i_\lambda = \Gamma_\lambda{}^i{}_j(x)\overline{y}^j, \tag{3.37}$$

on the vector bundle $\overline{Y} \to X$, where $(x^\lambda, \overline{y}^i)$ are the associated linear bundle coordinates on $\overline{Y}$.

Of course, since every vector bundle has a natural structure of an affine bundle, any linear connection on a vector bundle is also an affine connection.

Affine connections on an affine bundle $Y \to X$ constitute an affine space modelled over basic soldering forms on $Y \to X$. In view of the vertical splitting (1.14), these soldering forms can be seen as global sections of the vector bundle $T^*X \otimes \overline{Y} \to X$. If $Y \to X$ is a vector bundle, both the affine connection Γ (3.36) and the associated linear connection $\overline{\Gamma}$ are connections on the same vector bundle $Y \to X$, and their difference is also a basic soldering form on Y. Thus, every affine connection on a vector bundle $Y \to X$ is the sum $\Gamma = \overline{\Gamma} + \sigma$ of a linear connection $\overline{G}$ and a basic soldering form σ on $Y \to X$. Furthermore, let R and $\overline{R}$ be the curvatures of an affine connection Γ and the associated linear connection $\overline{\Gamma}$, respectively. It is readily observed that $R = \overline{R} + T$, where the VY-valued two-form

$$\begin{aligned}
&T = d_\Gamma \sigma = d_\sigma \Gamma : X \to \overset{2}{\wedge} T^*X \underset{X}{\otimes} VY, \\
&T = \frac{1}{2} T^i_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \partial_i, \\
&T^i_{\lambda\mu} = \partial_\lambda \sigma^i_\mu - \partial_\mu \sigma^i_\lambda + \sigma^h_\lambda \Gamma_\mu{}^i{}_h - \sigma^h_\mu \Gamma_\lambda{}^i{}_h,
\end{aligned} \tag{3.38}$$

is the torsion (3.21) of the connection Γ with respect to the basic soldering form σ.

In particular, let us consider the tangent bundle TX of a manifold X and the canonical soldering form $\sigma = \dot\theta_X$ (1.40) on TX. Given an arbitrary world connection Γ (3.29) on TX, the corresponding affine connection

$$A = \Gamma + \theta_X, \qquad A^\mu_\lambda = \Gamma_\lambda{}^\mu{}_\nu \dot x^\nu + \delta^\mu_\lambda, \tag{3.39}$$

on TX is called the Cartan connection. Since the soldered curvature ρ (3.20) of $\dot\theta_X$ equals to zero, the torsion (3.22) of the Cartan connection coincides with the torsion T (3.34) of the world connection Γ, while its curvature (3.23) is the sum $R+T$ of the curvature and the torsion of Γ.

G. Composite connections

Let us consider a composite fibre bundle $Y \to \Sigma \to X$ (1.18), coordinated by $(x^\lambda, \sigma^m, y^i)$. We aim studying the relations between connections on fibre bundles $Y \to X$, $Y \to \Sigma$ and $\Sigma \to X$. These connections are given respectively by the tangent-valued forms

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^m_\lambda \partial_m + \gamma^i_\lambda \partial_i), \tag{3.40}$$
$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i), \tag{3.41}$$
$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m). \tag{3.42}$$

A connection γ (3.40) on the fibre bundle $Y \to X$ is said to be projectable over a connection Γ (3.42) on the fibre bundle $\Sigma \to X$ if, for any vector field τ on X, its horizontal lift $\gamma\tau$ on Y by means of the connection γ is a projectable vector field over the horizontal lift $\Gamma\tau$ of τ on Σ by means of the connection Γ. This property takes place if and only if $\gamma^m_\lambda = \Gamma^m_\lambda$, i.e., components γ^m_λ of the connection γ (3.40) must be independent of the fibre coordinates y^i.

A connection A_Σ (3.41) on the fibre bundle $Y \to \Sigma$ and a connection Γ (3.42) on the fibre bundle $\Sigma \to X$ define a connection on the composite fibre bundle $Y \to X$ as the composition of bundle morphisms

$$\gamma : Y \underset{X}{\times} TX \xrightarrow{(\mathrm{Id}\,,\Gamma)} Y \underset{\Sigma}{\times} T\Sigma \xrightarrow{A_\Sigma} TY.$$

This composite connection reads

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_\lambda + A^i_m \Gamma^m_\lambda)\partial_i). \tag{3.43}$$

It is projectable over Γ. Moreover, this is a unique connection such that the horizontal lift $\gamma\tau$ on Y of a vector field τ on X by means of the composite connection γ (3.43) coincides with the composition $A_\Sigma(\Gamma\tau)$ of horizontal lifts of τ on Σ by means of the connection Γ and then on Y by means of the connection A_Σ. For the sake of brevity, let us write $\gamma = A_\Sigma \circ \Gamma$.

Given a composite fibre bundle Y (1.18), there are the exact sequences

$$0 \to V_\Sigma Y \hookrightarrow VY \to Y \underset{\Sigma}{\times} V\Sigma \to 0, \tag{3.44}$$

$$0 \to Y \underset{\Sigma}{\times} V^*\Sigma \hookrightarrow V^*Y \to V_\Sigma^* Y \to 0 \tag{3.45}$$

of vector bundles over Y, where $V_\Sigma Y$ and $V_\Sigma^* Y$ are the vertical tangent and the vertical cotangent bundles of the fibre bundle $Y \to \Sigma$ which are coordinated by $(x^\lambda, \sigma^m, y^i, \dot y^i)$ and $(x^\lambda, \sigma^m, y^i, \dot y_i)$, respectively. Let us consider a splitting of these exact sequences

$$B : VY \ni \dot y^i \partial_i + \dot\sigma^m \partial_m \to (\dot y^i \partial_i + \dot\sigma^m \partial_m) \rfloor B = \tag{3.46}$$
$$(\dot y^i - \dot\sigma^m B^i_m)\partial_i \in V_\Sigma Y,$$

$$B : V_\Sigma^* Y \ni \overline{d} y^i \to B \rfloor \overline{d} y^i = \overline{d} y^i - B^i_m \overline{d}\sigma^m \in V^* Y, \tag{3.47}$$

given by the form

$$B = (\overline{d} y^i - B^i_m \overline{d}\sigma^m) \otimes \partial_i. \tag{3.48}$$

Then a connection γ (3.40) on $Y \to X$ and a splitting B (3.46) define the connection

$$A_\Sigma = B \circ \gamma : TY \to VY \to V_\Sigma Y,$$
$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + (\gamma^i_\lambda - B^i_m \gamma^m_\lambda)\partial_i) + d\sigma^m \otimes (\partial_m + B^i_m \partial_i), \tag{3.49}$$

on the fibre bundle $Y \to \Sigma$.

Conversely, every connection A_Σ (3.41) on the fibre bundle $Y \to \Sigma$ provides the splitting

$$A_\Sigma : TY \supset VY \ni \dot y^i \partial_i + \dot\sigma^m \partial_m \to (\dot y^i - A^i_m \dot\sigma^m)\partial_i \tag{3.50}$$

of the exact sequence (3.44). Using this splitting, one can construct the first order differential operator

$$\widetilde{D} : J^1 Y \to T^*X \underset{Y}{\otimes} V_\Sigma Y, \qquad \widetilde{D} = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \sigma^m_\lambda)\partial_i, \tag{3.51}$$

called the vertical covariant differential, on the composite fibre bundle $Y \to X$. It can also be defined as the composition

$$\widetilde{D} = \mathrm{pr}_1 \circ D^\gamma : J^1 Y \to T^*X \underset{Y}{\otimes} VY \to T^*X \underset{Y}{\otimes} VY_\Sigma,$$

where D^γ is the covariant differential (3.10) relative to some composite connection $A_\Sigma \circ \Gamma$ (3.43), but $\widetilde{D}$ does not depend on the choice of the connection Γ on the fibre bundle $\Sigma \to X$.

The vertical covariant differential (3.51) possesses the following important property. Let h be a section of the fibre bundle $\Sigma \to X$ and $Y_h \to X$ the restriction (1.21) of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$. This is a subbundle $i_h : Y_h \hookrightarrow Y$ of the fibre bundle $Y \to X$. Every connection A_Σ (3.41) induces the pull-back connection

$$A_h = i_h^* A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h)\partial_\lambda h^m + (A \circ h)^i_\lambda)\partial_i] \tag{3.52}$$

on $Y_h \to X$. Then the restriction of the vertical covariant differential $\widetilde{D}$ (3.51) to $J^1 i_h(J^1 Y_h) \subset J^1 Y$ coincides with the familiar covariant differential D^{A_h} (3.10) on Y_h relative to the pull-back connection A_h (3.52).

4 Lagrangian field theory

Let us apply the above mathematical formalism to formulation of Lagrangian field theory on fibre bundles. Here, we restrict our consideration to first order Lagrangian formalism since the most contemporary field models are of this type.

The configuration space of first order Lagrangian field theory on a fibre bundle $Y \to X$, coordinated by $(x^\lambda, y^i, y^i_\lambda)$, is the first order jet manifold J^1Y of $Y \to X$, coordinated by $(x^\lambda, y^i, y^i_\lambda)$. Accordingly, a first order Lagrangian is defined as a density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega : J^1Y \to \overset{n}{\wedge} T^*X, \qquad n = \dim X, \tag{4.1}$$

on J^1Y (see the notation (1.25)). Let us follow the standard formulation of the variational problem on fibre bundles where deformations of sections of a fibre bundle $Y \to X$ are induced by local one-parameter groups of automorphisms of $Y \to X$ over X (i.e., vertical gauge transformations). Here, we will not study the calculus of variations in depth, but apply in a straightforward manner the first variational formula.

Since a projectable vector field u on a fibre bundle $Y \to X$ is an infinitesimal generator of a local one-parameter group of gauge transformations of $Y \to X$, one can think of its jet prolongation J^1u (2.8) as being the infinitesimal generator of gauge transformations of the configuration space J^1Y. Let

$$\mathbf{L}_{J^1u}L = [\partial_\lambda u^\lambda \mathcal{L} + (u^\lambda\partial_\lambda + u^i\partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i)\mathcal{L}]\omega \tag{4.2}$$

be the Lie derivative of a Lagrangian L along J^1u. The first variational formula provides its canonical decomposition in accordance with the variational problem. This decomposition reads

$$\begin{aligned} \mathbf{L}_{J^1u}L &= u_V \rfloor \mathcal{E}_L + d_H h_0(u\rfloor H_L) \\ &= (u^i - y^i_\mu u^\mu)(\partial_i - d_\lambda \partial^\lambda_i)\mathcal{L}\omega - d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda \mathcal{L}]\omega, \end{aligned} \tag{4.3}$$

where $u_V = (u\rfloor\theta^i)\partial_i$,

$$\begin{aligned} &\mathcal{E}_L : J^2Y \to T^*Y \wedge (\overset{n}{\wedge} T^*X), \\ &\mathcal{E}_L = (\partial_i\mathcal{L} - d_\lambda \pi^\lambda_i)\theta^i \wedge \omega, \qquad \pi^\lambda_i = \partial^\lambda_i \mathcal{L}, \end{aligned} \tag{4.4}$$

is the Euler–Lagrange operator associated to the Lagrangian L, and

$$H_L : J^1Y \to Z_Y = T^*Y \wedge (\overset{n-1}{\wedge} T^*X), \tag{4.5}$$

$$H_L = L + \pi^\lambda_i \theta^i \wedge \omega_\lambda = \pi^\lambda_i dy^i \wedge \omega_\lambda + (\mathcal{L} - \pi^\lambda_i y^i_\lambda)\omega, \tag{4.6}$$

is the Poincaré–Cartan form (see the notation (1.25), (2.13) and (2.14)).

The kernel of the Euler-Lagrange operator $\mathcal{E}_L$ (4.4) defines the system of second order Euler–Lagrange equations, given by the coordinate equalities

$$(\partial_i - d_\lambda \partial^\lambda_i)\mathcal{L} = 0, \tag{4.7}$$

A solution of these equations is a section s of the fibre bundle $X \to Y$, whose second order jet prolongation $J^2 s$ lives in (4.7), i.e.,

$$\partial_i \mathcal{L} \circ s - (\partial_\lambda + \partial_\lambda s^j \partial_j + \partial_\lambda \partial_\mu s^j \partial_j^\mu) \partial_i^\lambda \mathcal{L} \circ s = 0. \tag{4.8}$$

Remark 4.1. The kernel (4.7) of the Euler–Lagrange operator $\mathcal{E}_L$ need not be a closed subbundle of the second order jet bundle $J^2 Y \to X$. Therefore, it may happen that the Euler–Lagrange equations are not differential equations in a strict sense. •

Remark 4.2. Different Lagrangians L and L' can lead to the same Euler–Lagrange operator if their difference $L_0 = L - L'$ is a variationally trivial Lagrangian, whose Euler–Lagrange operator vanishes identically. A Lagrangian L_0 is variationally trivial if and only if

$$L_0 = h_0(\varphi) \tag{4.9}$$

where φ is a closed n-form on Y (see Lecture 8). We have at least locally $\varphi = d\xi$, and then

$$L_0 = h_0(d\xi) = d_H(h_0(\xi)) = d_\lambda h_0(\xi)^\lambda \omega, \qquad h_0(\xi) = h_0(\xi)^\lambda \omega_\lambda.$$

•

The Poincaré–Cartan form H_L (4.6) is a particular Lepagean equivalent of a Lagrangian L (i.e., $h_0(H_L) = L$). In contrast with other Lepagean forms, it is a horizontal form on the affine jet bundle $J^1 Y \to Y$. The fibre bundle Z_Y (4.5), called the homogeneous Legendre bundle, is endowed with holonomic coordinates $(x^\lambda, y^i, p_i^\lambda, p)$, possessing the transition functions

$$p'^\lambda_i = \det(\frac{\partial x^\varepsilon}{\partial x'^\nu}) \frac{\partial y^j}{\partial y'^i} \frac{\partial x'^\lambda}{\partial x^\mu} p_j^\mu, \qquad p' = \det(\frac{\partial x^\varepsilon}{\partial x'^\nu})(p - \frac{\partial y^j}{\partial y'^i} \frac{\partial y'^i}{\partial x^\mu} p_j^\mu). \tag{4.10}$$

Relative to these coordinates, the morphism (4.5) reads

$$(p_i^\mu, p) \circ H_L = (\pi_i^\mu, \mathcal{L} - \pi_i^\mu y_\mu^i).$$

A glance at the transition functions (4.10) shows that Z_Y is a one-dimensional affine bundle

$$\pi_{Z\Pi} : Z_Y \to \Pi \tag{4.11}$$

over the Legendre bundle

$$\Pi = \overset{n}{\wedge} T^*X \underset{Y}{\otimes} V^*Y \underset{Y}{\otimes} TX = V^*Y \underset{Y}{\wedge} (\overset{n-1}{\underset{Y}{\wedge}} T^*X), \tag{4.12}$$

endowed with holonomic coordinates $(x^\lambda, y^i, p_i^\lambda)$. Then the composition

$$\widehat{L} = \pi_{Z\Pi} \circ H_L : J^1 Y \underset{Y}{\to} \Pi, \qquad (x^\lambda, y^i, p_i^\lambda) \circ \widehat{L} = (x^\lambda, y^i, \pi_i^\lambda), \tag{4.13}$$

is the well-known Legendre map. One can think of p_i^λ as being the covariant momenta of field functions, and the Legendre bundle Π (4.12) plays the role of a finite-dimensional momentum phase space of fields in the covariant Hamiltonian field theory [17, 18, 38].

The first variational formula (4.3) provides the standard procedure for the study of differential conservation laws in Lagrangian field theory as follows.

Let u be a projectable vector field on a fibre bundle $Y \to X$ treated as the infinitesimal generator of a local one-parameter group G_u of gauge transformations. On-shell, i.e., on the kernel (4.7) of the Euler–Lagrange operator $\mathcal{E}_L$, the first variational formula (4.3) leads to the weak identity

$$\mathbf{L}_{J^1u}L \approx -d_\lambda \mathfrak{T}^\lambda \omega, \tag{4.14}$$

where

$$\mathfrak{T} = \mathfrak{T}^\lambda \omega_\lambda, \qquad \mathfrak{T}^\lambda = \pi_i^\lambda(u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}, \tag{4.15}$$

is the symmetry current along the vector field u. Let a Lagrangian L be invariant under the gauge group G_u. This implies that the Lie derivative $\mathbf{L}_{J^1u}L$ (4.2) vanishes. Then we obtain the weak conservation law

$$d_\lambda \mathfrak{T}^\lambda \approx 0 \tag{4.16}$$

of the symmetry current $\mathfrak{T}$ (4.15).

Remark 4.3. It should be emphasized that, the first variational formula defines the symmetry current (4.15) modulo the terms $d_\mu(c_i^{\mu\lambda}(y_\nu^i u^\nu - u^i))$, where $c_i^{\mu\lambda}$ are arbitrary skew-symmetric functions on Y [17]. Here, we set aside these boundary terms which are independent of a Lagrangian. •

The weak conservation law (4.16) leads to the differential conservation law

$$\partial_\lambda(\mathfrak{T}^\lambda \circ s) = 0 \tag{4.17}$$

on solutions of the Euler–Lagrange equations (4.8). It implies the integral conservation law

$$\int_{\partial N} s^*\mathfrak{T} = 0, \tag{4.18}$$

where N is a compact n-dimensional submanifold of X with the boundary ∂N.

Remark 4.4. In gauge theory, a symmetry current $\mathfrak{T}$ (4.15) takes the form

$$\mathfrak{T} = W + d_H U = (W^\lambda + d_\mu U^{\mu\lambda})\omega_\lambda, \tag{4.19}$$

where the term W depends only on the variational derivatives

$$\delta_i \mathcal{L} = (\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L}, \tag{4.20}$$

i.e., $W \approx 0$ and

$$U = U^{\mu\lambda}\omega_{\mu\lambda} : J^1Y \to \overset{n-2}{\wedge} T^*X$$

is a horizontal $(n-2)$-form on $J^1Y \to X$. Then one says that $\mathfrak{T}$ reduces to the superpotential U [14, 17, 39]. On-shell, such a symmetry current reduces to a d_H-exact form (4.19). Then the differential conservation law (4.17) and the integral conservation law (4.18) become tautological. At the same time, the superpotential form (4.19) of $\mathfrak{T}$ implies the following integral relation

$$\int_{N^{n-1}} s^*\mathfrak{T} = \int_{\partial N^{n-1}} s^*U, \tag{4.21}$$

where N^{n-1} is a compact oriented $(n-1)$-dimensional submanifold of X with the boundary ∂N^{n-1}. One can think of this relation as being a part of the Euler–Lagrange equations written in an integral form. •

Remark 4.5. Let us consider conservation laws in the case of gauge transformations which preserve the Euler–Lagrange operator $\mathcal{E}_L$, but not necessarily a Lagrangian L. Let u be a projectable vector field on $Y \to X$, which is the infinitesimal generator of a local one-parameter group of such transformations, i.e.,

$$\mathbf{L}_{J^2u}\mathcal{E}_L = 0,$$

where J^2u is the second order jet prolongation of the vector field u. There is the useful relation

$$\mathbf{L}_{J^2u}\mathcal{E}_L = \mathcal{E}_{\mathbf{L}_{J^1u}L} \tag{4.22}$$

[17]. Then, in accordance with (4.9), we have locally

$$\mathbf{L}_{J^1u}L = d_\lambda h_0(\xi)^\lambda\omega. \tag{4.23}$$

In this case, the weak identity (4.14) reads

$$0 \approx d_\lambda(h_0(\xi)^\lambda - \mathfrak{T}^\lambda), \tag{4.24}$$

where $\mathfrak{T}$ is the symmetry current (4.15) along the vector field u. •

Remark 4.6. Background fields, which do not live in the dynamic shell (4.7), violate conservation laws as follows. Let us consider the product

$$Y_{\rm tot} = Y \underset{X}{\times} Y' \tag{4.25}$$

of a fibre bundle Y, coordinated by (x^λ, y^i), whose sections are dynamic fields and of a fibre bundle Y', coordinated by (x^λ, y^A), whose sections are background fields which take the background values

$$y^B = \phi^B(x), \qquad y^B_\lambda = \partial_\lambda\phi^B(x).$$

A Lagrangian L of dynamic and background fields is defined on the total configuration space $J^1Y_{\rm tot}$. Let u be a projectable vector field on $Y_{\rm tot}$ which also projects onto Y' because gauge transformations of background fields do not depend on dynamic fields. This vector field takes the coordinate form

$$u = u^\lambda(x^\mu)\partial_\lambda + u^A(x^\mu, y^B)\partial_A + u^i(x^\mu, y^B, y^j)\partial_i. \tag{4.26}$$

Substitution of u (4.26) in the formula (4.3) leads to the first variational formula in the presence of background fields:

$$\begin{aligned}&\partial_\lambda u^\lambda \mathcal{L} + [u^\lambda\partial_\lambda + u^A\partial_A + u^i\partial_i + (d_\lambda u^A - y^A_\mu\partial_\lambda u^\mu)\partial^\lambda_A + \\ &\qquad (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i]\mathcal{L} = (u^A - y^A_\lambda u^\lambda)\partial_A\mathcal{L} + \pi^\lambda_A d_\lambda(u^A - y^A_\mu u^\mu) + \\ &\qquad (u^i - y^i_\lambda u^\lambda)\delta_i\mathcal{L} - d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda\mathcal{L}].\end{aligned} \tag{4.27}$$

Then the weak identity

$$\begin{aligned}&\partial_\lambda u^\lambda \mathcal{L} + [u^\lambda\partial_\lambda + u^A\partial_A + u^i\partial_i + (d_\lambda u^A - y^A_\mu\partial_\lambda u^\mu)\partial^\lambda_A + \\ &\qquad (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i]\mathcal{L} \approx (u^A - y^A_\lambda u^\lambda)\partial_A\mathcal{L} + \pi^\lambda_A d_\lambda(u^A - y^A_\mu u^\mu) - \\ &\qquad d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda\mathcal{L}]\end{aligned}$$

holds on the shell (4.7). If a total Lagrangian L is invariant under gauge transformations of $Y_{\rm tot}$, we obtain the weak identity

$$(u^A - y^A_\mu u^\mu)\partial_A\mathcal{L} + \pi^\lambda_A d_\lambda(u^A - y^A_\mu u^\mu) \approx d_\lambda\mathfrak{T}^\lambda, \tag{4.28}$$

which is the transformation law of the symmetry current $\mathfrak{T}$ in the presence of background fields. •

5 Gauge theory of principal connections

The reader is referred, e.g., to [23] for the standard exposition of geometry of principal bundles and to [31] for the customary geometric formulation of gauge theory. Here, gauge theory of principal connections is phrased in terms of jet manifolds on the same footing as other Lagrangian field theories on fibre bundles [17, 30].

A. Principal bundles

Let $\pi_P : P \to X$ be a principal bundle with a real structure Lie group G of finite non-zero dimension. For the sake of brevity, we call P a principal G-bundle. By definition, a principal bundle $P \to X$ is provided with the free transitive right action

$$\begin{aligned}&R_G : P \underset{X}{\times} G \to P, \\ &R_g : p \mapsto pg, \quad p \in P, \quad g \in G,\end{aligned} \tag{5.1}$$

of its structure group G on P. It follows that the typical fibre of a principal G-bundle is isomorphic to the group space of G, and that $P/G = X$. The structure group G acts on the typical fibre by left multiplications which do not preserve the group structure of G. Therefore, the typical fibre of a principal bundle is only a group space, but not a group (cf. the group bundle P^G (5.47) below). Since the left action of transition functions on the typical fibre G commutes with its right multiplications, a principal bundle admits the global right action (5.1) of the structure group.

A principal G-bundle P is equipped with a bundle atlas

$$\Psi_P = \{(U_\alpha, \psi^P_\alpha, \rho_{\alpha\beta})\}, \tag{5.2}$$

whose trivialization morphisms

$$\psi^P_\alpha : \pi_P^{-1}(U_\alpha) \to U_\alpha \times G$$

obey the equivariance condition

$$(\mathrm{pr}_2 \circ \psi^P_\alpha)(pg) = (\mathrm{pr}_2 \circ \psi^P_\alpha)(p)g, \qquad \forall g \in G, \qquad \forall p \in \pi_P^{-1}(U_\alpha). \tag{5.3}$$

Due to this property, every trivialization morphism ψ^P_α determines a unique local section z_α of P over U_α such that

$$\mathrm{pr}_2 \circ \psi^P_\alpha \circ z_\alpha = \mathbf{1},$$

where $\mathbf{1}$ is the unit element of G. The transformation rules for z_α read

$$z_\beta(x) = z_\alpha(x)\rho_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta, \tag{5.4}$$

where $\rho_{\alpha\beta}(x)$ are G-valued transition functions (1.6) of the atlas Ψ_P. Conversely, the family $\{(U_\alpha, z_\alpha)\}$ of local sections of P with the transition functions (5.4) determines a unique bundle atlas of P.

In particular, it follows that only trivial principal bundles have global sections.

Let us note that the pull-back of a principal bundle is also a principal bundle with the same structure group.

The quotient of the tangent bundle $TP \to P$ and that of the vertical tangent bundle VP of P by the tangent prolongation TR_G of the canonical action R_G (5.1) are vector bundles

$$T_GP = TP/G, \qquad V_GP = VP/G \tag{5.5}$$

over X. Sections of $T_GP \to X$ are naturally identified with G-invariant vector fields on P, while those of $V_GP \to X$ are G-invariant vertical vector fields on P. Accordingly, the Lie bracket of G-invariant vector fields on P goes to the quotients (5.5), and induces the Lie brackets of their sections. Let us write these brackets in an explicit form.

Owing to the equivariance condition (5.3), any bundle atlas (5.2) of P yields the associated bundle atlases $\{U_\alpha, T\psi^P_\alpha/G)\}$ of T_GP and $\{U_\alpha, V\psi^P_\alpha/G)\}$ of V_GP. Given a basis

$\{\varepsilon_p\}$ for the right Lie algebra $\mathfrak{g}_r$, let $\{\partial_\lambda, e_p\}$ and $\{e_p\}$, where $e_p = (\psi_\alpha^P/G)^{-1}(\varepsilon_p)$, be the corresponding local fibre bases for the vector bundles T_GP and V_GP, respectively. Relative to these bases, the Lie bracket of sections

$$\xi = \xi^\lambda\partial_\lambda + \xi^p e_p, \qquad \eta = \eta^\mu\partial_\mu + \eta^q e_q$$

of the vector bundle $T_GP \to X$ reads

$$[\xi, \eta] = (\xi^\mu\partial_\mu\eta^\lambda - \eta^\mu\partial_\mu\xi^\lambda)\partial_\lambda + (\xi^\lambda\partial_\lambda\eta^r - \eta^\lambda\partial_\lambda\xi^r + c^r_{pq}\xi^p\eta^q)e_r. \tag{5.6}$$

Putting $\xi^\lambda = 0$ and $\eta^\mu = 0$, we obtain the Lie bracket

$$[\xi, \eta] = c^r_{pq}\xi^p\eta^q e_r \tag{5.7}$$

of sections of the vector bundle $V_G \to P$.

A glance at the expression (5.7) shows that $V_GP \to X$ is a finite-dimensional Lie algebra bundle, whose typical fibre is the right Lie algebra $\mathfrak{g}_r$ of the group G. The structure group G acts on this typical fibre by the adjoint representation. In the physical literature, V_GP is often called the gauge algebra bundle because, if the base X is compact, a suitable Sobolev completion of the space of sections of $V_GP \to X$ is the Lie algebra of the gauge Lie group.

Let J^1P be the first order jet manifold of a principal G-bundle $P \to X$. Its quotient

$$C = J^1P/G \tag{5.8}$$

by the jet prolongation of the canonical action R_G (5.1) is a fibre bundle over X. Bearing in mind the canonical imbedding

$$\lambda_1 : J^1P \to T^*X \otimes TP$$

(2.5) and passing to the quotient by G, we obtain the corresponding canonical imbedding

$$\lambda_C : C \to T^*X \otimes T_GP \tag{5.9}$$

of the fibre bundle C (5.8). It follows that C is an affine bundle modelled over the vector bundle

$$\overline{C} = T^*X \underset{X}{\otimes} V_GP \to X. \tag{5.10}$$

Given a bundle atlas of P and the associated bundle atlas of V_GP, the affine bundle C is provided with affine bundle coordinates (x^λ, a^q_λ), and its elements are represented by T_GP-valued one-forms

$$a = dx^\lambda \otimes (\partial_\lambda + a^q_\lambda e_q) \tag{5.11}$$

on X. One calls C (5.8) the connection bundle because its sections are naturally identified with principal connections on the principal bundle $P \to X$ as follows.

B. Principal connections

In the case of a principal bundle $P \to X$, the exact sequence (1.17a) reduces to the exact sequence

$$0 \to V_G P \underset{X}{\hookrightarrow} T_G P \to TX \to 0 \tag{5.12}$$

over X by taking the quotient with respect to the right action of the group G. The exact sequence of vector bundles (5.12) yields the exact sequence of the structure modules of their sections

$$0 \to V_G P(X) \longrightarrow T_G P(X) \longrightarrow \mathcal{T}_1(X) \to 0. \tag{5.13}$$

Principal connections split these exact sequences as follows.

A principal connection A on a principal bundle $P \to X$ is defined as a global section A of the affine jet bundle $J^1 P \to P$ which is equivariant under the right action (5.1) of the group G on P, i.e.,

$$J^1 R_g \circ A = A \circ R_g, \qquad \forall g \in G. \tag{5.14}$$

Due to this equivariance condition, there is one-to-one correspondence between the principal connections on a principal bundle $P \to X$ and the global sections of the affine bundle $C \to X$ (5.8). In view of the imbedding (5.9), a principal connection splits the exact sequence (5.12), and is represented by a $T_G P$-valued form

$$A = dx^\lambda \otimes (\partial_\lambda + A^q_\lambda(x) e_q) \tag{5.15}$$

on X. Since the connection bundle $C \to X$ is affine, principal connections on a principal bundle always exist.

Hereafter, we agree to identify gauge potentials in gauge theory on a principal G-bundle P to global sections of the connection bundle $C \to X$ (5.8).

Remark 5.1. Let us relate the $T_G P$-valued connection form (5.15) with the familiar connection form on P and the local connection form on X, associated to a principal connection in [23]. Let us first recall that, since the tangent bundle of a Lie group admits the canonical trivialization along left-invariant vector fields, the vertical tangent bundle $VP \to P$ of a principal bundle $P \to X$ also possesses the canonical trivialization

$$\alpha : VP \cong P \times \mathfrak{g}_l$$

such that $\alpha^{-1}(\epsilon_m)$ are the familiar fundamental vector fields on P corresponding to the basis elements ϵ_m of the left Lie algebra $\mathfrak{g}_l$ of the Lie group G. Let a principal connection on a principal bundle $P \to X$ be represented by the vertical-valued form A (3.4). Then

$$\widetilde{A} : P \xrightarrow{A} T^*P \underset{P}{\otimes} VP \xrightarrow{\mathrm{Id}\otimes\alpha} T^*P \otimes \mathfrak{g}_l \tag{5.16}$$

is the above mentioned $\mathfrak{g}_l$-valued connection form on the principal bundle P. Given a trivialization chart $(U_\zeta, \psi^P_\zeta, z_\zeta)$ of P, this form reads

$$\widetilde{A} = \psi^{P*}_\zeta(\theta_l - \widetilde{A}^q_\lambda dx^\lambda \otimes \epsilon_q), \tag{5.17}$$

where θ_l is the canonical $\mathfrak{g}_l$-valued one-form on G and $\widetilde{A}^p_\lambda$ are equivariant functions on P such that

$$\widetilde{A}^q_\lambda(pg)\epsilon_q = \widetilde{A}^q_\lambda(p)\mathrm{Ad}\, g^{-1}(\epsilon_q).$$

The pull-back $A_\zeta = z^*_\zeta \widetilde{A}$ of the connection form $\widetilde{A}$ onto U_ζ is the above-mentioned $\mathfrak{g}_l$-valued local connection one-form

$$A_\zeta = -A^q_\lambda dx^\lambda \otimes \epsilon_q = A^q_\lambda dx^\lambda \otimes \varepsilon_q \tag{5.18}$$

on X, where $A^q_\lambda = \widetilde{A}^q_\lambda \circ z_\zeta$ are coefficients of the form (5.15). We have $A_\zeta = \psi^P_\zeta \mathbf{A}$, where

$$\mathbf{A} = A - \theta_X = A^q_\lambda dx^\lambda \otimes e_q \tag{5.19}$$

is the local $V_G P$-valued part of the form A (5.15). We will refer to $\mathbf{A}$ (5.19) as a local connection form. •

The following theorems [23] state the pull-back and push-forward operations of principal connections.

THEOREM 5.1. Let $P \to X$ be a principal fibre bundle and $f^*P \to X'$ (1.8) the pull-back principal bundle with the same structure group. If A is a principal connection on $P \to X$, then the pull-back connection f^*A (3.9) on $f^*P \to X'$ is a principal connection. □

THEOREM 5.2. Let $P' \to X$ and $P \to X$ be principle bundles with structure groups G' and G, respectively. Let $\Phi : P' \to P$ be a principal bundle morphism over X with the corresponding homomorphism $G' \to G$. For every principal connection A' on P', there exists a unique principal connection A on P such that the tangent map $T\Phi$ to Φ sends the horizontal subspaces relative to A' into those relative to A. □

C. The strength of a principal connection

Given a principal G-bundle $P \to X$, the Frölicher–Nijenhuis bracket (1.34) on the space $\mathcal{O}^*(P)\otimes\mathcal{T}_1(P)$ of tangent-valued forms on P is compatible with the canonical action R_G (5.1) of G on P, and yields the induced Frölicher–Nijenhuis bracket on the space $\mathcal{O}^*(X)\otimes T_G P(X)$ of $T_G P$-valued forms on X. Its coordinate form issues from the Lie bracket (5.6).

Then any principal connection $A \in \mathcal{O}^1(X) \otimes T_G P(X)$ (5.15) sets the Nijenhuis differential

$$d_A : \mathcal{O}^r(X) \otimes T_G P(X) \to \mathcal{O}^{r+1}(X) \otimes V_G P(X),$$
$$d_A \phi = [A, \phi]_{\rm FN}, \quad \phi \in \mathcal{O}^r(X) \otimes T_G P(X), \tag{5.20}$$

on the space $\mathcal{O}^*(X) \otimes T_G P(X)$. Similarly to the curvature R (3.17), let us put

$$F_A = \frac{1}{2} d_A A = \frac{1}{2}[A, A]_{\rm FN} \in \mathcal{O}^2(X) \otimes V_G P(X), \tag{5.21}$$
$$F_A = \frac{1}{2} F^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r,$$
$$F^r_{\lambda\mu} = \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq} A^p_\lambda A^q_\mu, \tag{5.22}$$

It is called the strength of a principal connection A, and is given locally by the expression

$$F_A = d\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}, \tag{5.23}$$

where $\mathbf{A}$ is the local connection form (5.19). By definition, the strength (5.21) of a principal connection obeys the second Bianchi identity

$$d_A F_A = [A, F_A]_{\rm FN} = 0. \tag{5.24}$$

It should be emphasized that the strength F_A (5.21) is not the standard curvature (3.16) of a connection on P, but there are the local relations $\psi^P_\zeta F_A = z^*_\zeta \Theta$, where

$$\Theta = d\widetilde{A} + \frac{1}{2}[\widetilde{A}, \widetilde{A}] \tag{5.25}$$

is the $\mathfrak{g}_l$-valued curvature form on P (see the expression (5.29) below). In particular, a principal connection is flat if and only if its strength vanishes.

D. Associated bundles

Given a principal G-bundle $\pi_P : P \to X$, let V be a manifold provided with an effective left action

$$G \times V \ni (g, v) \mapsto gv \in V$$

of the Lie group G. Let us consider the quotient

$$Y = (P \times V)/G \tag{5.26}$$

of the product $P \times V$ by identification of elements (p, v) and $(pg, g^{-1}v)$ for all $g \in G$. We will use the notation $(pG, G^{-1}v)$ for its points. Let $[p]$ denote the restriction of the canonical surjection

$$P \times V \to (P \times V)/G \tag{5.27}$$

to the subset $\{p\} \times V$ so that

$$[p](v) = [pg](g^{-1}v).$$

Then the map

$$Y \ni [p](V) \mapsto \pi_P(p) \in X,$$

makes the quotient Y (5.26) to a fibre bundle over X.

Let us note that, for any G-bundle, there exists an associated principal G-bundle [45]. The peculiarity of the G-bundle Y (5.26) is that it appears canonically associated to a principal bundle P. Indeed, every bundle atlas $\Psi_P = \{(U_\alpha, z_\alpha)\}$ of P determines a unique associated bundle atlas

$$\Psi = \{(U_\alpha, \psi_\alpha(x) = [z_\alpha(x)]^{-1})\}$$

of the quotient Y (5.26), and each automorphism of P also yields the corresponding automorphism (5.43) of Y.

Therefore, unless otherwise stated, by a fibre bundle associated to a principal bundle $P \to X$ (or, simply, a P-associated fibre bundle) we will mean the quotient (5.26).

Every principal connection A on a principal bundle $P \to X$ induces a unique connection on the associated fibre bundle Y (5.26). Given the horizontal splitting of the tangent bundle TP relative to A, the tangent map to the canonical morphism (5.27) defines the horizontal splitting of the tangent bundle TY of Y and, consequently, a connection on $Y \to X$ [23]. This is called the associated principal connection or, simply, a principal connection on a P-associated bundle $Y \to X$. If Y is a vector bundle, this connection takes the form

$$A = dx^\lambda \otimes (\partial_\lambda - A^p_\lambda I_p{}^i{}_j y^j \partial_i), \tag{5.28}$$

where I_p are generators of the linear representation of the Lie algebra $\mathfrak{g}_r$ in the vector space V. The curvature (3.16) of this connection reads

$$R = -\frac{1}{2} F^p_{\lambda\mu} I_p{}^i{}_j y^j dx^\lambda \wedge dx^\mu \otimes \partial_i, \tag{5.29}$$

where $F^p_{\lambda\mu}$ are coefficients (5.22) of the strength of a principal connection A.

In particular, any principal connection A yields the associated linear connection on the gauge algebra bundle $V_G P \to X$. The corresponding covariant differential $\nabla^A \xi$ (3.11) of its sections $\xi = \xi^p e_p$ reads

$$\nabla^A \xi : X \to T^*X \otimes V_G P,$$
$$\nabla^A \xi = (\partial_\lambda \xi^r + c^r_{pq} A^p_\lambda \xi^q) dx^\lambda \otimes e_r. \tag{5.30}$$

It coincides with the Nijenhuis differential

$$d_A \xi = [A, \xi]_{\rm FN} = \nabla^A \xi \tag{5.31}$$

(5.20) of ξ seen as a V_GP-valued 0-form, and is given by the local expression

$$\nabla^A\xi = d\xi + [\mathbf{A}, \xi], \tag{5.32}$$

where $\mathbf{A}$ is the local connection form (5.19).

E. The configuration space of classical gauge theory

Since gauge potentials are represented by global sections of the connection bundle $C \to X$ (5.8), its first order jet manifold J^1C plays the role of a configuration space of classical gauge theory. The key point is that the jet manifold J^1C admits the canonical splitting over C which leads to a unique canonical Yang–Mills Lagrangian of gauge theory on J^1C.

Let us describe this splitting. One can show that the principal G-bundle

$$J^1P \to J^1P/G = C \tag{5.33}$$

is canonically isomorphic to the trivial pull-back bundle

$$P_C = C \underset{X}{\times} P \to C, \tag{5.34}$$

and that the latter admits the canonical principal connection

$$\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a^p_\lambda e_p) + da^r_\lambda \otimes \partial^\lambda_r \in \mathcal{O}^1(C) \otimes T_G(P_C)(C) \tag{5.35}$$

[15, 17, 30]. Since C (5.8) is an affine bundle modelled over the vector bundle $\overline{C}$ (5.10), the vertical tangent bundle of C possesses the canonical trivialization

$$VC = C \underset{X}{\times} T^*X \otimes V_GP, \tag{5.36}$$

while

$$V_GP_C = V_G(C \underset{X}{\times} P) = C \underset{X}{\times} V_GP.$$

Then the strength $F_\mathcal{A}$ of the connection (5.35) is the V_GP-valued horizontal two-form

$$F_\mathcal{A} = \frac{1}{2} d_\mathcal{A}\mathcal{A} = \frac{1}{2}[\mathcal{A}, \mathcal{A}]_{\mathrm{FN}} \in \mathcal{O}^2(C) \otimes V_GP(X),$$
$$F_\mathcal{A} = (da^r_\mu \wedge dx^\mu + \frac{1}{2} c^r_{pq} a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu) \otimes e_r, \tag{5.37}$$

on C. It is readily observed that, given a global section connection A of the connection bundle $C \to X$, the pull-back $A^*F_\mathcal{A} = F_A$ is the strength (5.21) of the principal connection A.

Let us take the pull-back of the form $F_\mathcal{A}$ onto J^1C with respect to the fibration (5.33), and consider the V_GP-valued horizontal two-form

$$\mathcal{F} = h_0(F_\mathcal{A}) = \frac{1}{2}\mathcal{F}^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r,$$
$$\mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu, \tag{5.38}$$

where h_0 is the horizontal projection (2.13). It is readily observed that

$$\mathcal{F}/2 : J^1C \to C \underset{X}{\times} \overset{2}{\wedge} T^*X \otimes V_G P \tag{5.39}$$

is an affine morphism over C of constant rank. Hence, its kernel $C_+ = \operatorname{Ker} \mathcal{F}$ is the affine subbundle of $J^1C \to C$, and we have a desired canonical splitting

$$J^1C = C_+ \underset{C}{\oplus} C_- = C_+ \oplus (C \underset{X}{\times} \overset{2}{\wedge} T^*X \otimes V_G P), \tag{5.40}$$

$$a^r_{\lambda\mu} = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq} a^p_\lambda a^q_\mu) + \frac{1}{2}(a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu), \tag{5.41}$$

over C of the jet manifold J^1C. The corresponding canonical projections are

$$\mathrm{pr}_1 = \mathcal{S} : J^1C \to C_+, \qquad \mathcal{S}^r_{\lambda\mu} = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq} a^p_\lambda a^q_\mu), \tag{5.42}$$

and $\mathrm{pr}_2 = \mathcal{F}/2$ (5.39).

F. Gauge transformations

In classical gauge theory, several classes of gauge transformations are examined [17, 31, 43]. A most general gauge transformation is defined as an automorphism Φ_P of a principal G-bundle P which is equivariant under the canonical action (5.1) of the group G on G, i.e.,

$$R_g \circ \Phi_P = \Phi_P \circ R_g, \qquad \forall g \in G.$$

Such an automorphism of P yields the corresponding automorphism

$$\Phi_Y : (pG, G^{-1}v) \to (\Phi_P(p)G, G^{-1}v) \tag{5.43}$$

of the P-associated bundle Y (5.26) and the corresponding automorphism

$$\Phi_C : J^1P/G \to J^1\Phi_P(J^1P)/G \tag{5.44}$$

of the connection bundle C.

Hereafter, we deal with only vertical automorphisms of the principal bundle P, and agree to call them gauge transformations in gauge theory. Accordingly, the group $\mathfrak{G}(P)$ of vertical automorphisms of a principal G-bundle P is called the gauge group.

Every vertical automorphism of a principal bundle P is represented as

$$\Phi_P(p) = pf(p), \qquad p \in P, \tag{5.45}$$

where f is a G-valued equivariant function on P, i.e.,

$$f(pg) = g^{-1}f(p)g, \qquad \forall g \in G. \tag{5.46}$$

There is one-to-one correspondence between these functions and the global sections s of the group bundle

$$P^G = (P \times G)/G, \tag{5.47}$$

whose typical fibre is the group G, subject to the adjoint representation of the structure group G. Therefore, P^G (5.47) is also called the adjoint bundle. There is the canonical fibre-to-fibre action of the group bundle P^G on any P-associated bundle Y by the law

$$\begin{aligned} &P^G \underset{X}{\times} Y \to Y, \\ &((pG, G^{-1}gG), (pG, G^{-1}v)) \mapsto (pG, G^{-1}gv). \end{aligned}$$

Then the above-mentioned correspondence is given by the relation

$$P^G \underset{X}{\times} P \ni (s(\pi_P(p)), p) \mapsto pf(p) \in P,$$

where P is considered as a G-bundle associated to itself. Hence, the gauge group $\mathfrak{G}(P)$ of vertical automorphisms of a principal G-bundle $P \to X$ is isomorphic to the group of global sections of the P-associated group bundle (5.47).

In order to study the gauge invariance of one or another object in gauge theory , it suffices to examine its invariance under an arbitrary one-parameter subgroup $[\Phi_P]$ of the gauge group. Its infinitesimal generator is a G-invariant vertical vector field ξ on a principal bundle P or, equivalently, a section

$$\xi = \xi^p(x) e_p \tag{5.48}$$

of the gauge algebra bundle $V_G P \to X$ (5.5). We will call it a gauge vector field. One can think of its components $\xi^p(x)$ as being gauge parameters. Gauge vector fields (5.48) are transformed under the infinitesimal generators of gauge transformations (i.e., other gauge vector fields) ξ' by the adjoint representation

$$\mathbf{L}_{\xi'} \xi = [\xi', \xi] = c^p_{rq} \xi'^r \xi^q e_p, \qquad \xi, \xi' \in V_G P(X).$$

Accordingly, gauge parameters are subject to the coadjoint representation

$$\xi' : \xi^p \mapsto -c^p_{rq} \xi'^r \xi^q. \tag{5.49}$$

Given a gauge vector field ξ (5.48) seen as the infinitesimal generator of a one-parameter gauge group $[\Phi_P]$, let us obtain the gauge vector fields on a P-associated bundle Y and the connection bundle C.

The corresponding gauge vector field on the P-associated vector bundle $Y \to X$ issues from the relation (5.43), and reads

$$\xi_Y = \xi^p I^i_p \partial_i,$$

where I_p are generators of the group G, acting on the typical fibre V of Y.

The gauge vector field ξ (5.48) acts on elements a (5.11) of the connection bundle by the law

$$\mathbf{L}_\xi a = [\xi, a]_{\rm FN} = (-\partial_\lambda \xi^r + c^r_{pq}\xi^p a^q_\lambda) dx^\lambda \otimes e_r.$$

In view of the vertical splitting (5.36), this quantity can be regarded as the vertical vector field

$$\xi_C = (-\partial_\lambda \xi^r + c^r_{pq}\xi^p a^q_\lambda)\partial^\lambda_r \tag{5.50}$$

on the connection bundle C, and is the infinitesimal generator of the one-parameter group $[\Phi_C]$ of vertical automorphisms (5.44) of C, i.e., a desired gauge vector field on C.

G. Lagrangian gauge theory

Classical gauge theory of unbroken symmetries on a principal G-bundle $P \to X$ deals with two types of fields. These are gauge potentials identified to global sections of the connection bundle $C \to X$ (5.8) and matter fields represented by global sections of a P-associated vector bundle Y (5.26), called a matter bundle. Therefore, the total configuration space of classical gauge theory is the product of jet bundles

$$J^1Y_{\rm tot} = J^1Y \underset{X}{\times} J^1C. \tag{5.51}$$

Let us study a gauge invariant Lagrangian on this configuration space.

A total gauge vector field on the product $C \underset{X}{\times} Y$ reads

$$\xi_{YC} = (-\partial_\lambda \xi^r + c^r_{pq}\xi^p a^q_\lambda)\partial^\lambda_r + \xi^p I^i_p \partial_i = (u^{A\lambda}_p \partial_\lambda \xi^p + u^A_p \xi^p)\partial_A, \tag{5.52}$$

where we utilize the collective index A, and put the notation

$$u^{A\lambda}_p \partial_A = -\delta^r_p \partial^\lambda_r, \qquad u^A_p \partial_A = c^r_{qp} a^q_\lambda \partial^\lambda_r + I^i_p \partial_i.$$

A Lagrangian L on the configuration space (5.51) is said to be gauge-invariant if its Lie derivative $\mathbf{L}_{J^1\xi_{YC}} L$ along any gauge vector field ξ (5.48) vanishes. Then the first variational formula (4.3) leads to the strong equality

$$0 = (u^A_p \xi^p + u^{A\mu}_p \partial_\mu \xi^p)\delta_A \mathcal{L} + d_\lambda[(u^A_p \xi^p + u^{A\mu}_p \partial_\mu \xi^p)\pi^\lambda_A], \tag{5.53}$$

where $\delta_A \mathcal{L}$ are the variational derivatives (4.20) of L and the total derivative reads

$$d_\lambda = \partial_\lambda + a^p_{\lambda\mu}\partial^\mu_p + y^i_\lambda \partial_i.$$

Due to the arbitrariness of gauge parameters ξ^p, this equality falls into the system of strong equalities

$$u_p^A \delta_A \mathcal{L} + d_\mu (u_p^A \pi_A^\mu) = 0, \tag{5.54a}$$
$$u_p^{A\mu} \delta_A \mathcal{L} + d_\lambda (u_p^{A\mu} \pi_A^\lambda) + u_p^A \pi_A^\mu = 0, \tag{5.54b}$$
$$u_p^{A\lambda} \pi_A^\mu + u_p^{A\mu} \pi_A^\lambda = 0. \tag{5.54c}$$

Substituting (5.54b) and (5.54c) in (5.54a), we obtain the well-known constraints

$$u_p^A \delta_A \mathcal{L} - d_\mu (u_p^{A\mu} \delta_A \mathcal{L}) = 0$$

for the variational derivatives of a gauge-invariant Lagrangian L.

Treating the equalities (5.54a) – (5.54c) as the equations for a gauge-invariant Lagrangian, let us solve these equations for a Lagrangian

$$L = \mathcal{L}(x^\lambda, a_\mu^r, a_{\lambda\mu}^r)\omega : J^1C \to \overset{n}{\wedge} T^*X \tag{5.55}$$

without matter fields. In this case, the equations (5.54a) – (5.54c) read

$$c_{pq}^r (a_\mu^p \partial_r^\mu \mathcal{L} + a_{\lambda\mu}^p \partial_r^{\lambda\mu} \mathcal{L}) = 0, \tag{5.56a}$$
$$\partial_q^\mu \mathcal{L} + c_{pq}^r a_\lambda^p \partial_r^{\mu\lambda} \mathcal{L} = 0, \tag{5.56b}$$
$$\partial_p^{\mu\lambda} \mathcal{L} + \partial_p^{\lambda\mu} \mathcal{L} = 0. \tag{5.56c}$$

Let rewrite them relative to the coordinates $(a_\mu^q, \mathcal{S}_{\mu\lambda}^r, \mathcal{F}_{\mu\lambda}^r)$ (5.38) and (5.42), associated to the canonical splitting (5.40) of the jet manifold J^1C. The equation (5.56c) reads

$$\frac{\partial \mathcal{L}}{\partial \mathcal{S}_{\mu\lambda}^r} = 0. \tag{5.57}$$

Then a simple computation brings the equation (5.56b) into the form

$$\partial_q^\mu \mathcal{L} = 0. \tag{5.58}$$

A glance at the equations (5.57) and (5.58) shows that the gauge-invariant Lagrangian (5.55) factorizes through the strength $\mathcal{F}$ (5.38) of gauge potentials. As a consequence, the equation (5.56a) takes the form

$$c_{pq}^r \mathcal{F}_{\lambda\mu}^p \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\lambda\mu}^r} = 0.$$

It admits a unique solution in the class of quadratic Lagrangians which is the conventional Yang-Mills Lagrangian $L_{\rm YM}$ of gauge potentials on the configuration space J^1C. In the presence of a background world metric g on the base X, it reads

$$L_{\rm YM} = \frac{1}{4\varepsilon^2} a_{pq}^G g^{\lambda\mu} g^{\beta\nu} \mathcal{F}_{\lambda\beta}^p \mathcal{F}_{\mu\nu}^q \sqrt{\mid g \mid}\omega, \qquad g = \det(g_{\mu\nu}), \tag{5.59}$$

where a^G is a G-invariant bilinear form on the Lie algebra of $\mathfrak{g}_r$ and ε is a coupling constant.

H. Gauge conservation laws

On-shell, the strong equality (5.53) becomes the weak Noether conservation law

$$0 \approx d_\lambda[(u_p^A \xi^p + u_p^{A\mu}\partial_\mu \xi^p)\pi_A^\lambda] \tag{5.60}$$

of the Noether current

$$\mathfrak{T}^\lambda = -(u_p^A \xi^p + u_p^{A\mu}\partial_\mu \xi^p)\pi_A^\lambda. \tag{5.61}$$

Accordingly, the equalities (5.54a) – (5.54c) on-shell lead to the familiar Noether identities

$$d_\mu(u_p^A \pi_A^\mu) \approx 0, \tag{5.62a}$$
$$d_\lambda(u_p^{A\mu}\pi_A^\lambda) + u_p^A \pi_A^\mu \approx 0, \tag{5.62b}$$
$$u_p^{A\lambda}\pi_A^\mu + u_p^{A\mu}\pi_A^\lambda = 0 \tag{5.62c}$$

for a gauge-invariant Lagrangian L. They are equivalent to the weak equality (5.60) due to the arbitrariness of the gauge parameters $\xi^p(x)$.

A glance at the expressions (5.60) and (5.61) shows that both the Noether conservation law and the Noether current depend on gauge parameters. The weak identities (5.62a) – (5.62c) play the role of the necessary and sufficient conditions in order that the Noether conservation law (5.60) is maintained under changes of gauge parameters. This means that, if the equality (5.60) holds for gauge parameters ξ, it does so for arbitrary deviations $\xi + \delta\xi$ of ξ. In particular, the Noether conservation law (5.60) is maintained under gauge transformations, when gauge parameters are transformed by the coadjoint representation (5.49).

It is easily seen that the equalities (5.62a) – (5.62c) are not mutually independent, but (5.62a) is a corollary of (5.62b) and (5.62c). This property reflects the fact that, in accordance with the strong equalities (5.54b) and (5.54c), the Noether current (5.61) is brought into the superpotential form

$$\mathfrak{T}^\lambda = \xi^p u_p^{A\lambda}\delta_A \mathcal{L} - d_\mu(\xi^p u_p^{A\mu}\pi_A^\lambda), \qquad U^{\mu\lambda} = -\xi^p u_p^{A\mu}\pi_A^\lambda,$$

(4.19). Since a matter field Lagrangian is independent of the jet coordinates $a^p_{\lambda\mu}$, the Noether superpotential

$$U^{\mu\lambda} = \xi^p \pi_p^{\mu\lambda}$$

depends only on gauge potentials. The corresponding integral relation (4.21) reads

$$\int\limits_{N^{n-1}} s^*\mathfrak{T}^\lambda \omega_\lambda = \int\limits_{\partial N^{n-1}} s^*(\xi^p \pi_p^{\mu\lambda})\omega_{\mu\lambda}, \tag{5.63}$$

where N^{n-1} is a compact oriented $(n-1)$-dimensional submanifold of X with the boundary ∂N^{n-1}. One can think of (5.63) as being the integral relation between the Noether current (5.61) and the gauge field, generated by this current. In electromagnetic theory seen as a $U(1)$ gauge theory, the similar relation between an electric current and the electromagnetic field generated by this current is well known. However, it is free from gauge parameters due to the peculiarity of Abelian gauge models.

It should be emphasized that the Noether current (5.61) in gauge theory takes the superpotential form (4.19) o because the infinitesimal generators of gauge transformations (5.52) depend on derivatives of gauge parameters.

6 Higher order jets

As was mentioned in Lecture 2, there is a natural higher order generalization of the first and second order jet manifolds [17, 24, 29, 42]. Recall the notation. Given bundle coordinates (x^λ, y^i) of a fibre bundle $Y \to X$, by Λ, $\mid \Lambda \mid= r$, is meant a collection of numbers $(\lambda_r ... \lambda_1)$ modulo permutations. By $\Lambda + \Sigma$ we denote the collection

$$\Lambda + \Sigma = (\lambda_r \cdots \lambda_1 \sigma_k \cdots \sigma_1)$$

modulo permutations, while $\Lambda\Sigma$ is the union of collections

$$\Lambda\Sigma = (\lambda_r \cdots \lambda_1 \sigma_k \cdots \sigma_1),$$

where the indices λ_i and σ_j are not permuted. Recall the symbol of the total derivative

$$d_\lambda = \partial_\lambda + \sum_{|\Lambda|=0}^{k} y^i_{\Lambda+\lambda} \partial_i^\Lambda. \tag{6.1}$$

We will use the notation

$$\partial_\Lambda = \partial_{\lambda_r} \circ \cdots \circ \partial_{\lambda_1}, \qquad d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}, \qquad \Lambda = (\lambda_r ... \lambda_1).$$

The r-order jet manifold J^rY of a fibre bundle $Y \to X$ is defined as the disjoint union

$$J^rY = \bigcup_{x\in X} j^r_x s \tag{6.2}$$

of the equivalence classes $j^r_x s$ of sections s of Y so that sections s and s' belong to the same equivalence class $j^r_x s$ if and only if

$$s^i(x) = {s'}^i(x), \qquad \partial_\Lambda s^i(x) = \partial_\Lambda {s'}^i(x), \qquad 0 <\mid \Lambda \mid\leq r.$$

In brief, one can say that sections of $Y \to X$ are identified by the $r+1$ terms of their Taylor series at points of X. The particular choice of a coordinate atlas does not matter for this definition. Given an atlas of bundle coordinates (x^λ, y^i) of a fibre bundle $Y \to X$, the set (6.2) is endowed with an atlas of the adapted coordinates

$$\begin{aligned} &(x^\lambda, y^i_\Lambda), \qquad 0 \leq \mid \Lambda \mid \leq r, \\ &(x^\lambda, y^i_\Lambda) \circ s = (x^\lambda, \partial_\Lambda s^i(x)), \end{aligned} \tag{6.3}$$

together with transition functions (2.12). The coordinates (6.3) bring the set J^rY into a smooth manifold of finite dimension

$$\dim J^rY = n + m \sum_{i=0}^{r} \frac{(n+i-1)!}{i!(n-1)!}.$$

The coordinates (6.3) are compatible with the natural surjections

$$\pi^r_l : J^rY \to J^lY, \quad r > l,$$

which form the composite bundle

$$\pi^r : J^rY \xrightarrow{\pi^r_{r-1}} J^{r-1}Y \xrightarrow{\pi^{r-1}_{r-2}} \cdots \xrightarrow{\pi^1_0} Y \xrightarrow{\pi} X$$

with the properties

$$\pi^k_h \circ \pi^r_k = \pi^r_h, \qquad \pi^h \circ \pi^r_h = \pi^r.$$

A glance at the transition functions (2.12) when $\mid \Lambda \mid = r$ shows that the fibration

$$\pi^r_{r-1} : J^rY \to J^{r-1}Y$$

is an affine bundle modelled over the vector bundle

$$\overset{r}{\vee} T^*X \underset{J^{r-1}Y}{\otimes} VY \to J^{r-1}Y. \tag{6.4}$$

Remark 6.1. To introduce higher order jet manifolds, one can use the construction of the repeated jet manifolds. Let us consider the r-order jet manifold J^rJ^kY of the jet bundle $J^kY \to X$. It is coordinated by

$$(x^\mu, y^i_{\Sigma\Lambda}), \qquad \mid \Lambda \mid \leq k, \qquad \mid \Sigma \mid \leq r.$$

There is the canonical monomorphism

$$\sigma_{rk} : J^{r+k}Y \hookrightarrow J^rJ^kY$$

given by the coordinate relations

$$y^i_{\Sigma\Lambda} \circ \sigma_{rk} = y^i_{\Sigma+\Lambda}.$$

•

In the calculus in r-order jets, we have the r-order jet prolongation functor such that, given fibre bundles Y and Y' over X, every bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f of X admits the r-order jet prolongation to the morphism

$$J^r\Phi : J^rY \ni j^r_x s \mapsto j^r_{f(x)}(\Phi \circ s \circ f^{-1}) \in J^rY' \tag{6.5}$$

of the r-order jet manifolds. The jet prolongation functor is exact. If Φ is an injection [surjection], so is $J^r\Phi$. It also preserves an algebraic structure. In particular, if $Y \to X$ is a vector bundle, so is $J^rY \to X$. If $Y \to X$ is an affine bundle modelled over the vector bundle $\overline{Y} \to X$, then $J^rY \to X$ is an affine bundle modelled over the vector bundle $J^r\overline{Y} \to X$.

Every section s of a fibre bundle $Y \to X$ admits the r-order jet prolongation to the section

$$(J^rs)(x) = j^r_x s$$

of the jet bundle $J^rY \to X$. Such a section of $J^rY \to X$ is called holonomic.

Every exterior form ϕ on the jet manifold J^kY gives rise to the pull-back form $\pi^{k+i*}_k\phi$ on the jet manifold $J^{k+i}Y$. Let $\mathcal{O}^*_k = \mathcal{O}^*(J^kY)$ be the algebra of exterior forms on the jet manifold J^kY. We have the direct system of $\mathbb{R}$-algebras

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^{1*}_0} \mathcal{O}^*_1 \xrightarrow{\pi^{2*}_1} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots. \tag{6.6}$$

The subsystem of (6.6) is the direct system

$$C^\infty(X) \xrightarrow{\pi^*} C^\infty(Y) \xrightarrow{\pi^{1*}_0} \mathcal{O}^0_1 \xrightarrow{\pi^{2*}_1} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^0_r \longrightarrow \cdots \tag{6.7}$$

of the $\mathbb{R}$-rings of real smooth functions $\mathcal{O}^0_k = C^\infty(J^kY)$ on the jet manifolds J^kY. Therefore, one can think of (6.6) and (6.7) as being the direct systems of $C^\infty(X)$-modules.

Given the k-order jet manifold J^kY of $Y \to X$, there exists the canonical bundle morphism

$$r_{(k)} : J^kTY \to TJ^kY$$

over $J^kY \underset{X}{\times} J^kTX \to J^kY \underset{X}{\times} TX$ whose coordinate expression is

$$(x^\lambda, y^i_\Lambda, \dot x^\lambda, \dot y^i_\Lambda) \circ r_{(k)} = (x^\lambda, y^i_\Lambda, \dot x^\lambda, (\dot y^i)_\Lambda - \sum (\dot y^i)_{\mu+\Sigma}(\dot x^\mu)_\Xi), \qquad 0 \leq \mid \Lambda \mid \leq k,$$

where the sum is taken over all partitions $\Sigma + \Xi = \Lambda$ and $0 < \mid \Xi \mid$. In particular, we have the canonical isomorphism over J^kY

$$r_{(k)} : J^kVY \to VJ^kY, \qquad (\dot y^i)_\Lambda = \dot y^i_\Lambda \circ r_{(k)}. \tag{6.8}$$

As a consequence, every projectable vector field $u = u^\mu\partial_\mu + u^i\partial_i$ on a fibre bundle $Y \to X$ has the following k-order jet prolongation to the vector field on J^kY:

$$J^ku = r_{(k)} \circ J^ku : J^kY \to TJ^kY, \tag{6.9}$$
$$J^ku = u^\lambda\partial_\lambda + u^i\partial_i + u^i_\Lambda\partial_i^\Lambda, \qquad 0 <\mid \Lambda \mid\leq k,$$
$$u^i_{\lambda+\Lambda} = d_\lambda u^i_\Lambda - y^i_{\mu+\Lambda}\partial_\lambda u^\mu, \qquad 0 <\mid \Lambda \mid< k,$$

(cf. (2.8) for $k = 1$). In particular, the k-order jet lift (6.9) of a vertical vector field on $Y \to X$ is a vertical vector field on $J^kY \to X$ due to the isomorphism (6.8).

A vector field u_r on an r-order jet manifold J^rY is called projectable if, for any $k < r$, there exists a projectable vector field u_k on J^kY such that

$$u_k \circ \pi^r_k = T\pi^r_k \circ u_r.$$

A projectable vector field υ on J^rY has the coordinate expression

$$\upsilon = u_\lambda\partial_\lambda + u^i_\Lambda\partial_i^\Lambda, \qquad 0 \leq |\Lambda| \leq r,$$

such that u_λ depends only on the coordinates x^μ and every component u^i_Λ is independent of the coordinates y^i_Ξ, $|\Xi| > |\Lambda|$.

Let us denote by $\mathcal{P}^k$ the vector space of projectable vector fields on the jet manifold J^kY. It is easily seen that $\mathcal{P}^r$ is a Lie algebra over $\mathbb{R}$ and that the morphisms $T\pi^r_k$, $k < r$, constitute the inverse system

$$\mathcal{P}^0 \xleftarrow{T\pi^1_0} \mathcal{P}^1 \xleftarrow{T\pi^2_1} \cdots \xleftarrow{T\pi^{r-1}_{r-2}} \mathcal{P}^{r-1} \xleftarrow{T\pi^r_{r-1}} \mathcal{P}^r \longleftarrow \cdots \tag{6.10}$$

of these Lie algebras.

PROPOSITION 6.1. [4, 46]. The k-order jet lift (6.9) is the Lie algebra monomorphism of the Lie algebra $\mathcal{P}^0$ of projectable vector fields on $Y \to X$ to the Lie algebra $\mathcal{P}^k$ of projectable vector fields on J^kY such that

$$T\pi^r_k(J^ru) = J^ku \circ \pi^r_k. \tag{6.11}$$

□

The jet lift J^ku (6.9) is said to be an integrable vector field on J^kY. Every projectable vector field u_k on J^kY is decomposed into the sum

$$u_k = J^k(T\pi^k_0(u_k)) + \upsilon_k \tag{6.12}$$

of the integrable vector field $J^k(T\pi^k_0(u_k))$ and the projectable vector field υ_r which is vertical with respect to some fibration $J^kY \to Y$.

Similarly to the exact sequences (1.17a) – (1.17b) over $J^0Y = Y$, we have the exact sequences

$$0 \to VJ^kY \hookrightarrow TJ^kY \to TX \underset{X}{\times} J^kY \to 0, \tag{6.13}$$

$$0 \to J^kY \underset{X}{\times} T^*X \hookrightarrow TJ^kY \to V^*J^kY \to 0, \tag{6.14}$$

of vector bundles over J^kY. They do not admit a canonical splitting. Nevertheless, their pull-backs onto $J^{k+1}Y$ are split canonically due to the following canonical bundle monomorphisms over J^kY:

$$\lambda_{(k)} : J^{k+1}Y \hookrightarrow T^*X \underset{J^kY}{\otimes} TJ^kY,$$
$$\lambda_{(k)} = dx^\lambda \otimes d^{(k)}_\lambda, \tag{6.15}$$

$$\theta_{(k)} : J^{k+1}Y \hookrightarrow T^*J^kY \underset{J^kY}{\otimes} VJ^kY,$$
$$\theta_{(k)} = \sum (dy^i_\Lambda - y^i_{\lambda+\Lambda} dx^\lambda) \otimes \partial^\Lambda_i, \tag{6.16}$$

where the sum is over all multi-indices Λ, $\mid \Lambda \mid \leq k$. The forms

$$\theta^i_\Lambda = dy^i_\Lambda - y^i_{\Lambda+\lambda} dx^\lambda \tag{6.17}$$

are also called the contact forms. The monomorphisms (6.15) and (6.16) yield the bundle monomorphisms over $J^{k+1}Y$

$$\widehat{\lambda}_{(k)} : TX \underset{X}{\times} J^{k+1}Y \hookrightarrow TJ^kY \underset{J^kY}{\times} J^{k+1}Y, \tag{6.18}$$

$$\widehat{\theta}_{(k)} : V^*J^kY \underset{J^kY}{\times} \hookrightarrow T^*J^kY \underset{J^kY}{\times} J^{k+1}Y. \tag{6.19}$$

These monomorphisms split the exact sequences (6.13) and (6.14) over $J^{k+1}Y$ and define the canonical horizontal splittings of the pull-backs

$$\pi^{k+1*}_k TJ^kY = \widehat{\lambda}_{(k)}(TX \underset{X}{\times} J^{k+1}Y) \underset{J^{k+1}Y}{\oplus} VJ^kY, \tag{6.20}$$
$$\dot{x}^\lambda \partial_\lambda + \sum \dot{y}^i_\Lambda \partial^\Lambda_i = \dot{x}^\lambda d^{(k)}_\lambda + \sum (\dot{y}^i_\Lambda - \dot{x}^\lambda y^i_{\lambda+\Lambda}) \partial^\Lambda_i,$$

$$\pi^{k+1*}_k T^*J^kY = T^*X \underset{J^{k+1}Y}{\oplus} \widehat{\theta}_{(k)}(V^*J^kY \underset{J^kY}{\times} J^{k+1}Y), \tag{6.21}$$
$$\dot{x}_\lambda dx^\lambda + \sum \dot{y}^\Lambda_i dy^i_\Lambda = (\dot{x}_\lambda + \sum \dot{y}^\Lambda_i y^i_{\lambda+\Lambda}) dx^\lambda + \sum \dot{y}^\Lambda_i \theta^i_\Lambda,$$

where summation are over all multi-indices $\mid \Lambda \mid \leq k$.

In accordance with the canonical horizontal splitting (6.20), the pull-back

$$\overline{u}_k : J^{k+1}Y \xrightarrow{\pi^{k+1}_k \times \mathrm{Id}} J^kY \times J^{k+1} \xrightarrow{u_k \times \mathrm{Id}} TJ^kY \underset{J^kY}{\times} J^{k+1}$$

onto $J^{k+1}Y$ of any vector field u_k on J^kY admits the canonical horizontal splitting

$$\overline{u} = u_H + u_V = (u^\lambda d_\lambda^{(k)} + \sum y^i_{\lambda+\Lambda}\partial_i^\Lambda) + \sum(u^i_\Lambda - u^\lambda y^i_{\lambda+\Lambda})\partial_i^\Lambda, \tag{6.22}$$

where the sums are over all multi-indices $|\Lambda| \leq k$. By virtue of the canonical horizontal splitting (6.21), every exterior 1-form ϕ on J^kY admits the canonical splitting of its pull-back

$$\pi_k^{k+1*}\phi = h_0\phi + (\phi - h_0(\phi)), \tag{6.23}$$

where h_0 is the horizontal projection (2.13).

7 Infinite order jets

The direct system (6.6) of $\mathbb{R}$-algebras of exterior forms and the inverse system (6.10) of the real Lie algebras of projectable vector fields on jet manifolds admit the limits for $r \to \infty$ in the category of modules and that of Lie algebras, respectively. Intuitively, one can think of elements of these limits as being the objects defined on the projective limit of the inverse system

$$X \overset{\pi}{\longleftarrow} Y \overset{\pi_0^1}{\longleftarrow} \cdots \longleftarrow J^{r-1}Y \overset{\pi^r_{r-1}}{\longleftarrow} J^rY \longleftarrow \cdots \tag{7.1}$$

of finite order jet manifolds J^rY.

Remark 7.1. Recall that, by a projective limit of the inverse system (7.1) is meant a set $J^\infty Y$ such that, for any k, there exist surjections

$$\pi^\infty : J^\infty Y \to X, \quad \pi_0^\infty : J^\infty Y \to Y, \qquad \pi_k^\infty : J^\infty Y \to J^kY, \tag{7.2}$$

which make up the commutative diagrams

$$\begin{tikzcd} & J^\infty Y \arrow[dl, "\pi_k^\infty"'] \arrow[dr, "\pi_r^\infty"] & \\ J^kY \arrow[rr, "\pi_r^k"'] & & J^rY \end{tikzcd}$$

for any admissible k and $r < k$ [32]. •

The projective limit of the inverse system (7.1) exists. It is called the infinite order jet space. This space consists of those elements

$$(\ldots, q_i, \ldots, q_j, \ldots), \qquad q_i \in J^iY, \qquad q_j \in J^jY,$$

of the Cartesian product $\prod\limits_k J^kY$ which satisfy the relations $q_i = \pi_i^j(q_j)$ for all $j > i$. Thus, elements of the infinite order jet space $J^\infty Y$ really represent ∞-jets $j_x^\infty s$ of local sections of $Y \to X$. These sections belong to the same jet $j^\infty_x s$ if and only if their Taylor series at a point $x \in X$ coincide with each other.

Remark 7.2. The space $J^\infty Y$ is also the projective limit of the inverse subsystem of (7.1) which starts from any finite order $J^r Y$. •

The infinite order jet space $J^\infty Y$ is provided with the weakest topology such that the surjections (7.2) are continuous. The base of open sets of this topology in $J^\infty Y$ consists of the inverse images of open subsets of $J^k Y$, $k = 0, \ldots$, under the mappings (7.2). This topology makes $J^\infty Y$ a paracompact Fréchet manifold [47]. A bundle coordinate atlas $\{U, (x^\lambda, y^i)\}$ of $Y \to X$ yields the manifold coordinate atlas

$$\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}, \qquad 0 \leq |\Lambda|, \tag{7.3}$$

of $J^\infty Y$, together with the transition functions

$$y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \tag{7.4}$$

where $\Lambda = (\lambda_k \ldots \lambda_1)$, $\lambda + \Lambda = (\lambda\lambda_k \ldots \lambda_1)$ are multi-indices and d_λ denotes the total derivative

$$d_\lambda = \partial_\lambda + \sum_{|\Lambda| \geq 0} y^i_{\lambda+\Lambda} \partial^\Lambda_i. \tag{7.5}$$

Though $J^\infty Y$ fails to be a smooth manifold, one can introduce the differential calculus on $J^\infty Y$ as follows.

Let us consider the direct system (6.6) of $\mathbb{R}$-modules $\mathcal{O}^*_k$ of exterior forms on finite order jet manifolds $J^k Y$. The limit $\mathcal{O}^*_\infty$ of this direct system, by definition, obeys the following conditions [32]:

- for any r, there exists an injection $\mathcal{O}^*_r \to \mathcal{O}^*_\infty$;
- the diagrams

$$\begin{tikzcd} \mathcal{O}^*_k \arrow[rr, "\pi^{r*}_k"] \arrow[dr, "\pi^{\infty*}_k"'] & & \mathcal{O}^*_r \arrow[dl, "\pi^{\infty*}_r"] \\ & \mathcal{O}^*_\infty & \end{tikzcd}$$

  are commutative for any r and $k < r$.

Such a direct limit exists. This is the quotient of the direct sum $\bigoplus\limits_k \mathcal{O}^*_k$ with respect to identification of the pull-back forms

$$\pi^{\infty*}_r \phi = \pi^{\infty*}_k \sigma, \qquad \phi \in \mathcal{O}^*_r, \qquad \sigma \in \mathcal{O}^*_k,$$

if $\phi = \pi_k^{r*}\sigma$. In other words, $\mathcal{O}_\infty^*$ consists of all the exterior forms on finite order jet manifolds module pull-back identification. Therefore, we will denote the image of $\mathcal{O}_r^*$ in $\mathcal{O}_\infty^*$ by $\mathcal{O}_r^*$ and the elements $\pi_r^{\infty *}\phi$ of $\mathcal{O}_\infty^*$ simply by ϕ.

Remark 7.3. Obviously, $\mathcal{O}_\infty^*$ is the direct limit of the direct subsystem of (6.6) which starts from any finite order r. •

The $\mathbb{R}$-module $\mathcal{O}_\infty^*$ possesses the structure of the filtered module as follows [26]. Let us consider the direct system (6.7) of the commutative $\mathbb{R}$-rings of smooth functions on the jet manifolds J^rY. Its direct limit $\mathcal{O}_\infty^0$ consists of functions on finite order jet manifolds modulo pull-back identification. This is the $\mathbb{R}$-ring filtered by the $\mathbb{R}$-rings $\mathcal{O}_k^0 \subset \mathcal{O}_{k+i}^0$. Then $\mathcal{O}_\infty^*$ has the filtered $\mathcal{O}^0$-module structure given by the $\mathcal{O}_k^0$-submodules $\mathcal{O}_k^*$ of $\mathcal{O}_\infty^*$.

An endomorphism Δ of $\mathcal{O}_\infty^*$ is called a filtered morphism if there exists $i \in \mathbb{N}$ such that the restriction of Δ to $\mathcal{O}_k^*$ is the homomorphism of $\mathcal{O}_k^*$ into $\mathcal{O}_{k+i}^*$ over the injection $\mathcal{O}_k^0 \hookrightarrow \mathcal{O}_{k+i}^0$ for all k.

THEOREM 7.1. [32]. Every direct system of endomorphisms $\{\gamma_k\}$ of $\mathcal{O}_k$ such that

$$\pi_i^{j*} \circ \gamma_i = \gamma_j \circ \pi_i^{j*}$$

for all $j > i$ has the direct limit γ_∞ in filtered endomorphisms of $\mathcal{O}_\infty^*$. If all γ_k are monomorphisms (resp. epimorphisms), then γ_∞ is also a monomorphism (resp. epimorphism). This result also remains true for the general case of morphism between two different direct systems. □

COROLLARY 7.2. [32]. The operation of taking homology groups of chain and cochain complexes commutes with the passage to the direct limit. □

The operation of multiplication

$$\phi \to f\phi, \qquad f \in C^\infty(X), \qquad \phi \in \mathcal{O}_r^*$$

has the direct limit, and $\mathcal{O}_\infty^*$ possesses the structure of $C^\infty(X)$-algebra. The operations of the exterior product $\wedge$ and the exterior differential d also have the direct limits on $\mathcal{O}_\infty^*$. We will denote them by the same symbols $\wedge$ and d, respectively. They provide $\mathcal{O}_\infty^*$ with the structure of a $\mathbb{Z}$-graded exterior algebra:

$$\mathcal{O}_\infty^* = \bigoplus_{m=0}^{\infty} \mathcal{O}_\infty^m,$$

where $\mathcal{O}_\infty^m$ are the direct limits of the direct systems

$$\mathcal{O}^m(X) \xrightarrow{\pi^*} \mathcal{O}_0^m \xrightarrow{\pi_0^{1*}} \mathcal{O}_1^m \longrightarrow \cdots \mathcal{O}_r^m \xrightarrow{\pi_r^{r+1*}} \mathcal{O}_{r+1}^m \longrightarrow \cdots$$

of $\mathbb{R}$-modules $\mathcal{O}^m_r$ of exterior m-forms on r-order jet manifolds J^rY. We agree to call lements of $\mathcal{O}^m_\infty$ the exterior m-forms on the infinite order jet space. The familiar relations of an exterior algebra take place:

$$\mathcal{O}^i_\infty \wedge \mathcal{O}^j_\infty \subset \mathcal{O}^{i+j}_\infty,$$
$$d : \mathcal{O}^i_\infty \to \mathcal{O}^{i+1}_\infty,$$
$$d \circ d = 0.$$

As a consequence, we have the following De Rham complex of exterior forms on the infinite order jet space

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_\infty \xrightarrow{d} \mathcal{O}^1_\infty \xrightarrow{d} \cdots . \tag{7.6}$$

Let us consider the cohomology group $H^m(\mathcal{O}^*_\infty)$ of this complex. By virtue of Corollary 7.2, this is isomorphic to the direct limit of the direct system of homomorphisms

$$H^m(\mathcal{O}^*_r) \longrightarrow H^m(\mathcal{O}^*_{r+1}) \longrightarrow \cdots$$

of the cohomology groups $H^m(\mathcal{O}^*_r)$ of the cochain complexes

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_r \xrightarrow{d} \mathcal{O}^1_r \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{O}^l_r \to 0, \qquad l = \dim J^rY,$$

i.e., of the De Rham cohomology groups $H^m(\mathcal{O}^*_r) = H^m(J^rY)$ of jet manifolds J^rY. The following assertion completes our consideration of cohomology of the complex (7.6).

PROPOSITION 7.3. The De Rham cohomology $H^*(J^rY)$ of jet manifolds J^rY coincide with the De Rham cohomology $H^*(Y)$ of the fibre bundle $Y \to X$ [3, 4]. □

The proof is based on the fact that the fibre bundle $J^rY \to J^{r-1}Y$ is affine, and it has the same De Rham cohomology than its base. It follows that the cohomology groups $H^m(\mathcal{O}^*_\infty)$, $m > 0$, of the cochain complex (7.6) coincide with the De Rham cohomology groups $H^m(Y)$ of $Y \to X$.

Given a manifold coordinate atlas (7.3) of $J^\infty Y$, the elements of the direct limit $\mathcal{O}^*_\infty$ can be written in the coordinate form as exterior forms on finite order jets. In particular, the basic 1-forms dx^λ and the contact 1-forms θ^i_Λ (6.17) constitute the set of local generating elements of the $\mathcal{O}^0_\infty$-module $\mathcal{O}^1_\infty$ of 1-forms on $J^\infty Y$. Moreover, the basic 1-forms dx^λ and the contact 1-forms θ^i_Λ have independent coordinate transformation laws. It follows that there is the canonical splitting

$$\mathcal{O}^1_\infty = \mathcal{O}^{0,1}_\infty \oplus \mathcal{O}^{1,0}_\infty \tag{7.7}$$

of the module $\mathcal{O}^1_\infty$ in the $\mathcal{O}^0_\infty$-submodules $\mathcal{O}^{0,1}_\infty$ and $\mathcal{O}^{1,0}_\infty$ generated separately by basic and contact forms. The splitting (7.7) provides the canonical decomposition

$$\mathcal{O}^*_\infty = \mathop{\oplus}_{k,s} \mathcal{O}^{k,s}_\infty, \qquad 0 \le k, \qquad 0 \le s \le n,$$

of $\mathcal{O}^*_\infty$ into $\mathcal{O}^0_\infty$-modules $\mathcal{O}^{k,s}_\infty$ of k-contact and s-horizontal forms, together with the corresponding projections

$$h_k : \mathcal{O}^*_\infty \to \mathcal{O}^{k,*}_\infty, \quad 0 \leq k, \qquad h^s : \mathcal{O}^*_\infty \to \mathcal{O}^{*,s}_\infty, \quad 0 \leq s \leq n.$$

Accordingly, the exterior differential on $\mathcal{O}^*_\infty$ is split into the sum $d = d_H + d_V$ of horizontal and vertical differentials such that

$$\begin{array}{lll} d_H \circ h_k = h_k \circ d \circ h_k, & d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), & \\ d_V \circ h^s = h^s \circ d \circ h^s, & d_V(\phi) = \theta^i_\Lambda \wedge \partial^\Lambda_i \phi, & \phi \in \mathcal{O}^*_\infty. \end{array}$$

The operators d_H and d_V obey the familiar relations

$$\begin{array}{ll} d_H(\phi \wedge \sigma) = d_H(\phi) \wedge \sigma + (-1)^{|\phi|}\phi \wedge d_H(\sigma), & \phi, \sigma \in \mathcal{O}^*_\infty, \\ d_V(\phi \wedge \sigma) = d_V(\phi) \wedge \sigma + (-1)^{|\phi|}\phi \wedge d_V(\sigma), & \end{array}$$

and possess the nilpotency property

$$d_H \circ d_H = 0, \qquad d_V \circ d_V = 0, \qquad d_V \circ d_H + d_H \circ d_V = 0. \tag{7.8}$$

Recall also the relation

$$h_0 \circ d = d_H \circ h_0.$$

The horizontal differential can be written in the form

$$d_H \phi = dx^\lambda \wedge d_\lambda(\phi), \tag{7.9}$$

where d_λ (7.5) are the total derivatives in infinite order jets. It should be emphasized that, though the sum in the expression (7.5) is taken with respect to an infinite number of collections Λ, the operator (7.5) is well defined since, given any form $\phi \in \mathcal{O}^*_\infty$, the expression $d_\lambda(\phi)$ involves only a finite number of the terms ∂^Λ_i. The total derivatives satisfy the relations

$$\begin{array}{l} d_\lambda(\phi \wedge \sigma) = d_\lambda(\phi) \wedge \sigma + \phi \wedge d_\lambda(\sigma), \\ d_\lambda(d\phi) = d(d_\lambda(\phi)), \\ [d_\lambda, d_\alpha] = 0, \end{array}$$

and, in contrast with partial derivatives ∂_λ, they have the coordinate transformation law

$$d'_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu.$$

The operators d_H and d_V take the following coordinate form:

$$\begin{array}{lll} d_H f = d_\lambda f dx^\lambda, & d_V f = \partial^\Lambda_i f \theta^i_\Lambda, & f \in \mathcal{O}^0_\infty, \\ d_H(dx^\mu) = 0, & d_V(dx^i) = 0, & \\ d_H(\theta^i_\Lambda) = dx^\lambda \wedge \theta^i_{\lambda+\Lambda}, & d_V(\theta^i_\Lambda) = 0 & 0 \leq |\,\Lambda\,|\,. \end{array}$$

One can think of the splitting (7.7) as being the canonical horizontal splitting. It is similar both to the horizontal splitting (3.7) of the cotangent bundle of a fibre bundle by means of a connection and the canonical horizontal splittings (6.23) of 1-forms on finite order jet manifolds. Therefore, one can say that the splitting (7.7) defines the canonical connection on the infinite order jet space $J^\infty Y$.

Given a vector field τ on X, let us consider the map

$$\nabla_\tau : \mathcal{O}^0_\infty \ni f \to \tau\rfloor(d_H f) \in \mathcal{O}^0_\infty. \tag{7.10}$$

This is a derivation of the ring $\mathcal{O}^0_\infty$. Moreover, if $\mathcal{O}^0_\infty$ is regarded as a $C^\infty(X)$-ring, the map (7.10) satisfies the Leibniz rule. Hence, the assignment

$$\nabla : \tau \mapsto \tau\rfloor d_H = \tau^\lambda d_\lambda \tag{7.11}$$

is the canonical connection on the $C^\infty(X)$-ring $\mathcal{O}^0_\infty$ [30]. One can think of ∇_τ (7.11) as being the horizontal lift $\tau^\lambda\partial_\lambda \mapsto \tau^\lambda d_\lambda$ onto $J^\infty Y$ of a vector field τ on X by means of a canonical connection on the (topological) fibre bundle $J^\infty Y \to X$.

However, ∇_τ (7.11) on $J^\infty Y$ is not a projectable vector field on $J^\infty Y$, though they projected over vector fields on X. Projectable vector fields on $J^\infty Y$ (their definition is a repetition of that for finite order jet manifolds) are elements of the projective limit $\mathcal{P}^\infty$ of the inverse system (6.10). This projective limit exists. Its definition is a repetition of that of $J^\infty Y$. This is a Lie algebra such that the surjections

$$T\pi^\infty_k : \mathcal{P}^\infty \to \mathcal{P}^k$$

are Lie algebra morphisms which constitute the commutative diagrams

$$\begin{tikzcd} & \mathcal{P}^\infty \arrow[dl, "T\pi^\infty_k"'] \arrow[dr, "T\pi^\infty_r"] & \\ \mathcal{P}^k \arrow[rr, "T\pi^k_r"'] & & \mathcal{P}^r \end{tikzcd}$$

for any k and $r < k$. In brief, we will say that elements of $\mathcal{P}^\infty$ are vector fields on the infinite order jet space $J^\infty Y$.

In particular, let u be a projectable vector field on Y. There exists an element $J^\infty u \in \mathcal{P}^\infty$ such that

$$T\pi^\infty_k(J^\infty u) = J^k u, \qquad \forall k \geq 0.$$

One can think of $J^\infty u$ as being the ∞-order jet prolongation of the vector field u on Y. It is given by the recurrence formula (6.9) where $0 \leq |\Lambda|$. Then any element of $\mathcal{P}^\infty$ is decomposed into the sum similar to (6.12) where $k = \infty$. Of course, it is not the horizontal decomposition. Given a vector field υ on $J^\infty Y$, projected onto a vector field τ on X, we have its horizontal splitting

$$\upsilon = \upsilon_H + \upsilon_V = \tau^\lambda d_\lambda + (\upsilon - \tau^\lambda d_\lambda)$$

by means of the canonical connection ∇ (7.11) (cf. (6.22)). Note that the component υ_V of this splitting is not a projectable vector field on $J^\infty Y$, but is a vertical vector field with respect to the fibration $J^\infty Y \to X$.

Though ∇_τ (7.11) on $J^\infty Y$ is not an element of the projective limit $\mathcal{P}^\infty$, it is also a vector field on $J^\infty Y$ as follows.

A real function $f : J^\infty Y \to \mathbb{R}$ is said to be smooth if, for every $q \in J^\infty Y$, there exists a neighbourhood U of q and a smooth function $f^{(k)}$ on $J^k Y$ for some k such that

$$f|_U = f^{(k)} \circ \pi^\infty_k \,|_U \,.$$

Then the same equality takes place for any $r > k$. Smooth functions on $J^\infty Y$ constitute an $\mathbb{R}$-ring $C^\infty(J^\infty Y)$. In particular, the pull-back $\pi^{\infty *}_r f$ of any smooth function on $J^r Y$ is a smooth function on $J^\infty Y$, and there is a monomorphism $\mathcal{O}^0_\infty \to C^\infty(J^\infty Y)$. The key point is that the paracompact space $J^\infty Y$ admits partition of unity by elements of the ring $C^\infty(J^\infty Y)$ [47].

Vector fields on $J^\infty Y$ can be defined as derivations of the ring $C^\infty(J^\infty Y$. Since a derivation of $\mathcal{O}^0_\infty$ is a local operation and $J^\infty Y$ admits a smooth partition of unity, the derivations (7.10) can be extended to the ring $C^\infty(J^\infty Y)$ of smooth functions on the infinite order jet space $J^\infty Y$. Accordingly, the connection ∇ (7.11) is extended to the canonical connection on the $C^\infty(X)$-ring $C^\infty(J^\infty Y)$. Extended to $C^\infty(J^\infty Y)$, the derivations (7.10), by definition, are vector fields on the infinite order jet space $J^\infty Y$.

8 The variational bicomplex

The algebra $\mathcal{O}^*_\infty$ together with the horizontal differential d_H and the variational operator δ constitute the variational bicomplex of exterior forms on $J^\infty Y$. Cohomology of this bicomplex provide solution of the global inverse problem of the calculus of variations in field theory. Moreover, extended to the jet space of ghosts and antifields, the algebra $\mathcal{O}^*_\infty$ is the main ingredient in the field-antfield BRST theory for studying BRST cohomology modulo d_H. Passing to the direct limit of the de Rham complexes of exterior forms on finite order jet manifolds, the de Rham cohomology has been found in Proposition 7.3. However, this is not a way of studying other cohomology of the algebra $\mathcal{O}^*_\infty$.

To solve this problem, we enlarge $\mathcal{O}^*_\infty$ to the graded differential algebra $\mathcal{Q}^*_\infty$ of exterior forms which are locally the pull-back of exterior forms on finite order jet manifolds. The de Rham cohomology, d_H- and δ-cohomology of $\mathcal{Q}^*_\infty$ have been investigated in [2, 47]. Then one can show that the graded differential algebra $\mathcal{O}^*_\infty$ has the same d_H- and δ-cohomology as $\mathcal{Q}^*_\infty$ [18, 19].

We follow the terminology of [12, 22], where a sheaf S is a particular topological bundle, $\overline{S}$ denotes the canonical presheaf of sections of the sheaf S, and $\Gamma(S)$ is the group of global sections of S.

A. The variational bicomplex

Let $\mathfrak{O}^*_r$ be a sheaf of germs of exterior forms on the r-order jet manifold J^rY and $\overline{\mathfrak{O}}^*_r$ its canonical presheaf. There is the direct system of canonical presheaves

$$\overline{\mathfrak{O}}^*_X \xrightarrow{\pi^*} \overline{\mathfrak{O}}^*_0 \xrightarrow{\pi^1_0{}^*} \overline{\mathfrak{O}}^*_1 \xrightarrow{\pi^2_1{}^*} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \overline{\mathfrak{O}}^*_r \longrightarrow \cdots,$$

where $\pi^r_{r-1}{}^*$ are the pull-back monomorphisms. Its direct limit $\overline{\mathfrak{O}}^*_\infty$ is a presheaf of graded differential $\mathbb{R}$-algebras on $J^\infty Y$. Let $\mathfrak{Q}^*_\infty$ be a sheaf constructed from $\overline{\mathfrak{O}}^*_\infty$, $\overline{\mathfrak{Q}}^*_\infty$ its canonical presheaf, and $\mathcal{Q}^*_\infty = \Gamma(\mathfrak{Q}^*_\infty)$ the structure algebra of sections of the sheaf $\mathfrak{Q}^*_\infty$. In particular, $\mathcal{Q}^0_\infty = C^\infty(J^\infty Y)$. There are $\mathbb{R}$-algebra monomorphisms $\overline{\mathfrak{O}}^*_\infty \to \overline{\mathfrak{Q}}^*_\infty$ and $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$. The key point is that, since the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\mathcal{Q}^0_\infty$, the sheaves of $\mathcal{Q}^0_\infty$-modules on $J^\infty Y$ are fine and, consequently, acyclic. Therefore, the abstract de Rham theorem on cohomology of a sheaf resolution [22] can be called into play in order to obtain cohomology of the graded differential algebra $\mathcal{Q}^*_\infty$.

For short, we agree to call elements of $\mathcal{Q}^*_\infty$ the exterior forms on $J^\infty Y$, too. Restricted to a coordinate chart $(\pi^\infty_0)^{-1}(U)$ of $J^\infty Y$, they as like as elements of $\mathcal{O}^*_\infty$ can be written in a coordinate form, where horizontal forms $\{dx^\lambda\}$ and contact 1-forms $\{\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda\}$ provide local generators of the algebra $\mathcal{Q}^*_\infty$. There is the canonical decomposition

$$\mathcal{Q}^*_\infty = \underset{k,s}{\oplus} \mathcal{Q}^{k,s}_\infty, \qquad 0 \le k, \qquad 0 \le s \le n,$$

of $\mathcal{Q}^*_\infty$ into $\mathcal{Q}^0_\infty$-modules $\mathcal{Q}^{k,s}_\infty$ of k-contact and s-horizontal forms. Accordingly, the exterior differential on $\mathcal{Q}^*_\infty$ is split into the sum $d = d_H + d_V$ of horizontal and vertical differentials.

Being nilpotent, the differentials d_V and d_H provide the natural bicomplex $\{\mathfrak{Q}^{k,m}_\infty\}$ of the sheaf $\mathfrak{Q}^*_\infty$ on $J^\infty Y$. To complete it to the variational bicomplex, one defines the projection $\mathbb{R}$-module endomorphism

$$\tau = \sum_{k>0} \frac{1}{k}\overline{\tau} \circ h_k \circ h^n,$$

$$\overline{\tau}(\phi) = (-1)^{|\Lambda|}\theta^i \wedge [d_\Lambda(\partial^\Lambda_i \rfloor \phi)], \qquad 0 \le \mid \Lambda \mid, \qquad \phi \in \overline{\mathfrak{O}}^{>0,n}_\infty,$$

of $\overline{\mathfrak{O}}^*_\infty$ such that

$$\tau \circ d_H = 0, \qquad \tau \circ d \circ \tau - \tau \circ d = 0.$$

Introduced on elements of the presheaf $\overline{\mathfrak{O}}^*_\infty$ (see, e.g., [4, 17, 48]), this endomorphism is induced on the sheaf $\mathfrak{Q}^*_\infty$ and its structure algebra $\mathcal{Q}^*_\infty$. Put

$$\mathfrak{E}_k = \tau(\mathfrak{Q}^{k,n}_\infty), \qquad E_k = \tau(\mathcal{Q}^{k,n}_\infty), \qquad k > 0.$$

Since τ is a projection operator, we have isomorphisms

$$\overline{\mathfrak{E}}_k = \tau(\overline{\mathfrak{Q}}^{k,n}_\infty), \qquad E_k = \Gamma(\mathfrak{E}_k).$$

The variational operator on $\mathfrak{O}^{*,n}_\infty$ is defined as the morphism $\delta = \tau \circ d$. It is nilpotent, and obeys the relation

$$\delta \circ \tau - \tau \circ d = 0. \tag{8.1}$$

Let $\mathbb{R}$ and $\mathfrak{O}^*_X$ denote the constant sheaf on $J^\infty Y$ and the sheaf of exterior forms on X, respectively. The operators d_V, d_H, τ and δ give the following variational bicomplex of sheaves of differential forms on $J^\infty Y$:

$$\begin{array}{ccccccccccc}
 & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & -\delta\uparrow \\
 & 0\to & \mathfrak{O}^{k,0}_\infty & \xrightarrow{d_H} & \mathfrak{O}^{k,1}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{k,m}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{k,n}_\infty & \xrightarrow{\tau} & \mathfrak{E}_k \to 0 \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & -\delta\uparrow \\
 & 0\to & \mathfrak{O}^{1,0}_\infty & \xrightarrow{d_H} & \mathfrak{O}^{1,1}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{1,m}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{1,n}_\infty & \xrightarrow{\tau} & \mathfrak{E}_1 \to 0 \\
 & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & d_V\uparrow & & -\delta\uparrow \\
0\to\mathbb{R}\to & & \mathfrak{O}^{0}_\infty & \xrightarrow{d_H} & \mathfrak{O}^{0,1}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{0,m}_\infty & \xrightarrow{d_H}\cdots & \mathfrak{O}^{0,n}_\infty & \equiv & \mathfrak{O}^{0,n}_\infty \\
 & & \pi^{\infty *}\uparrow & & \pi^{\infty *}\uparrow & & \pi^{\infty *}\uparrow & & \pi^{\infty *}\uparrow & & \\
0\to\mathbb{R}\to & & \mathfrak{O}^{0}_X & \xrightarrow{d} & \mathfrak{O}^{1}_X & \xrightarrow{d}\cdots & \mathfrak{O}^{m}_X & \xrightarrow{d}\cdots & \mathfrak{O}^{n}_X & \xrightarrow{d} & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{8.2}$$

The second row and the last column of this bicomplex form the variational complex

$$0 \to \mathbb{R} \to \mathfrak{O}^0_\infty \xrightarrow{d_H} \mathfrak{O}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{O}^{0,n}_\infty \xrightarrow{\delta} \mathfrak{E}_1 \xrightarrow{\delta} \mathfrak{E}_2 \longrightarrow \cdots . \tag{8.3}$$

The corresponding variational bicomplexes and variational complexes of graded differential algebras $\mathcal{Q}^*_\infty$ and $\mathcal{O}^*_\infty$ take place.

There are the well-known statements summarized usually as the algebraic Poincaré lemma (see, e.g., [33, 48]).

LEMMA 8.1. If Y is a contactible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, the variational bicomplex of the graded differential algebra $\mathcal{O}^*_\infty$ is exact. $\square$

It follows that the variational bicomplex (8.2) and, consequently, the variational complex (8.3) are exact for any smooth bundle $Y \to X$. Moreover, the sheaves $\mathfrak{O}^{k,m}_\infty$ in this bicomplex are fine, and so are the sheaves $\mathfrak{E}_k$ in accordance with the following lemma.

LEMMA 8.2. Sheaves $\mathfrak{E}_k$ are fine. $\square$

Proof. Though the $\mathbb{R}$-modules $E_{k>1}$ fail to be $\mathcal{Q}^0_\infty$-modules [48], one can use the fact that the sheaves $\mathfrak{E}_{k>0}$ are projections $\tau(\mathfrak{O}^{k,n}_\infty)$ of sheaves of $\mathcal{Q}^0_\infty$-modules. Let $\{U_i\}_{i\in I}$ be a locally finite open covering of $J^\infty Y$ and $\{f_i \in \mathcal{Q}^0_\infty\}$ the associated partition of unity. For any open subset

$U \subset J^\infty Y$ and any section φ of the sheaf $\mathfrak{O}^{k,n}_\infty$ over U, let us put $h_i(\varphi) = f_i\varphi$. The endomorphisms h_i of $\mathfrak{O}^{k,n}_\infty$ yield the $\mathbb{R}$-module endomorphisms

$$\overline{h}_i = \tau \circ h_i : \mathfrak{E}_k \xrightarrow{\text{in}} \mathfrak{O}^{k,n}_\infty \xrightarrow{h_i} \mathfrak{O}^{k,n}_\infty \xrightarrow{\tau} \mathfrak{E}_k$$

of the sheaves $\mathfrak{E}_k$. They possess the properties required for $\mathfrak{E}_k$ to be a fine sheaf. Indeed, for each $i \in I$, supp $f_i \subset U_i$ provides a closed set such that $\overline{h}_i$ is zero outside this set, while the sum $\sum\limits_{i\in I} \overline{h}_i$ is the identity morphism. **QED**

Thus, the columns and rows of the bicomplex (8.2) as like as the variational complex (8.3) are sheaf resolutions, and the abstract de Rham theorem can be applied to them. Here, we restrict our consideration to the variational complex.

B. Cohomology of $\mathcal{Q}^*_\infty$

The variational complex (8.3) is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$. Let us start from the following lemma.

LEMMA 8.3. There is an isomorphism

$$H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R}) = H^*(Y) \tag{8.4}$$

between cohomology $H^*(J^\infty Y, \mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$, that $H^*(Y, \mathbb{R})$ of Y, and the de Rham cohomology $H^*(Y)$ of Y. □

Proof. Since Y is a strong deformation retract of $J^\infty Y$ [3], the first isomorphism in (8.4) follows from the Vietoris–Begle theorem [12], while the second one results from the familiar de Rham theorem. **QED**

Let us consider the de Rham complex of sheaves

$$0 \to \mathbb{R} \to \mathfrak{O}^0_\infty \xrightarrow{d} \mathfrak{O}^1_\infty \xrightarrow{d} \cdots \tag{8.5}$$

on $J^\infty Y$ and the corresponding de Rham complex of their structure algebras

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d} \mathcal{Q}^1_\infty \xrightarrow{d} \cdots . \tag{8.6}$$

The complex (8.5) is exact due to the Poincaré lemma, and is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$ since sheaves $\mathfrak{O}^r_\infty$ are fine. Then, the abstract de Rham theorem and Lemma 8.3 lead to the following.

PROPOSITION 8.4. The de Rham cohomology $H^*(\mathcal{Q}^*_\infty)$ of the graded differential algebra $\mathcal{Q}^*_\infty$ is isomorphic to that $H^*(Y)$ of the bundle Y. □

It follows that every closed form $\phi \in \mathcal{Q}^*_\infty$ is split into the sum

$$\phi = \varphi + d\xi, \qquad \xi \in \mathcal{Q}^*_\infty, \tag{8.7}$$

where φ is a closed form on the fiber bundle Y.

Similarly, from the abstract de Rham theorem and Lemma 8.3, we obtain the following.

PROPOSITION 8.5. There is an isomorphism between d_H- and δ-cohomology of the variational complex

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d_H} \mathcal{Q}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}^{0,n}_\infty \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \longrightarrow \cdots \tag{8.8}$$

and the de Rham cohomology of the fiber bundle Y, namely,

$$H^{k<n}(d_H; \mathcal{Q}^*_\infty) = H^{k<n}(Y), \qquad H^{k-n}(\delta; \mathcal{Q}^*_\infty) = H^{k\geq n}(Y).$$

□

This isomorphism recovers the results of [2, 47], but notes also the following. The relation (8.1) for τ and the relation $h_0 d = d_H h_0$ for h_0 define a homomorphism of the de Rham complex (8.6) of the algebra $\mathcal{Q}^*_\infty$ to its variational complex (8.8). The corresponding homomorphism of their cohomology groups is an isomorphism by virtue of Proposition 8.4 and Proposition 8.5. Then, the splitting (8.7) leads to the following decompositions.

PROPOSITION 8.6. Any d_H-closed form $\sigma \in \mathcal{Q}^{0,m}$, $m < n$, is represented by a sum

$$\sigma = h_0\varphi + d_H\xi, \qquad \xi \in \mathcal{Q}^{m-1}_\infty, \tag{8.9}$$

where φ is a closed m-form on Y. Any δ-closed form $\sigma \in \mathcal{Q}^{k,n}$, $k \geq 0$, is split into

$$\sigma = h_0\varphi + d_H\xi, \qquad k = 0, \qquad \xi \in \mathcal{Q}^{0,n-1}_\infty, \tag{8.10}$$

$$\sigma = \tau(\varphi) + \delta(\xi), \qquad k = 1, \qquad \xi \in \mathcal{Q}^{0,n}_\infty, \tag{8.11}$$

$$\sigma = \tau(\varphi) + \delta(\xi), \qquad k > 1, \qquad \xi \in E_{k-1}, \tag{8.12}$$

where φ is a closed $(n+k)$-form on Y. □

C. Cohomology of $\mathcal{O}^*_\infty$

THEOREM 8.7. Graded differential algebra $\mathcal{O}^*_\infty$ has the same d_H- and δ-cohomology as $\mathcal{Q}^*_\infty$. □

Proof. Let the common symbol D stand for d_H and δ. Bearing in mind decompositions (8.9) – (8.12), it suffices to show that, if an element $\phi \in \mathcal{O}^*_\infty$ is D-exact in the algebra $\mathcal{Q}^*_\infty$, then it is so in the algebra $\mathcal{O}^*_\infty$. Lemma 8.1 states that, if Y is a contractible bundle and a D-exact form ϕ on

$J^\infty Y$ is of finite jet order $[\phi]$ (i.e., $\phi\in\mathcal{O}^*_\infty$), there exists an exterior form $\varphi\in\mathcal{O}^*_\infty$ on $J^\infty Y$ such that $\phi=D\varphi$. Moreover, a glance at the homotopy operators for d_H and δ [33] shows that the jet order $[\varphi]$ of φ is bounded by an integer $N([\phi])$, depending only on the jet order of ϕ. Let us call this fact the finite exactness of the operator D. Given an arbitrary bundle Y, the finite exactness takes place on $J^\infty Y|_U$ over any domain (i.e., a contractible open subset) $U\subset Y$. Let us prove the following.

(i) Given a family $\{U_\alpha\}$ of disjoint open subsets of Y, let us suppose that the finite exactness takes place on $J^\infty Y|_{U_\alpha}$ over every subset U_α from this family. Then, it is true on $J^\infty Y$ over the union $\underset{\alpha}{\cup} U_\alpha$ of these subsets.

(ii) Suppose that the finite exactness of the operator D takes place on $J^\infty Y$ over open subsets U, V of Y and their non-empty overlap $U\cap V$. Then, it is also true on $J^\infty Y|_{U\cup V}$.

Proof of (i). Let $\phi\in\mathcal{O}^*_\infty$ be a D-exact form on $J^\infty Y$. The finite exactness on $(\pi^\infty_0)^{-1}(\cup U_\alpha)$ holds since $\phi=D\varphi_\alpha$ on every $(\pi^\infty_0)^{-1}(U_\alpha)$ and $[\varphi_\alpha]<N([\phi])$.

Proof of (ii). Let $\phi=D\varphi\in\mathcal{O}^*_\infty$ be a D-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi_V$ on $(\pi^\infty_0)^{-1}(V)$, where φ_U and φ_V are exterior forms of bounded jet order. Let us consider their difference $\varphi_U-\varphi_V$ on $(\pi^\infty_0)^{-1}(U\cap V)$. It is a D-exact form of bounded jet order $[\varphi_U-\varphi_V]<N([\phi])$ which, by assumption, can be written as $\varphi_U-\varphi_V=D\sigma$ where σ is also of bounded jet order $[\sigma]<N(N([\phi]))$. Lemma 8.8 below shows that $\sigma=\sigma_U+\sigma_V$ where σ_U and σ_V are exterior forms of bounded jet order on $(\pi^\infty_0)^{-1}(U)$ and $(\pi^\infty_0)^{-1}(V)$, respectively. Then, putting

$$\varphi'|_U=\varphi_U-D\sigma_U,\qquad \varphi'|_V=\varphi_V+D\sigma_V,$$

we have the form ϕ, equal to $D\varphi'_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi'_V$ on $(\pi^\infty_0)^{-1}(V)$, respectively. Since the difference $\varphi'_U-\varphi'_V$ on $(\pi^\infty_0)^{-1}(U\cap V)$ vanishes, we obtain $\phi=D\varphi'$ on $(\pi^\infty_0)^{-1}(U\cup V)$ where

$$\varphi'\overset{\rm def}{=}\begin{cases}\varphi'|_U=\varphi'_U,\\ \varphi'|_V=\varphi'_V\end{cases}$$

is of bounded jet order $[\varphi']<N(N([\phi]))$.

To prove the finite exactness of D on $J^\infty Y$, it remains to choose an appropriate cover of Y. A smooth manifold Y admits a countable cover $\{U_\xi\}$ by domains U_ξ, $\xi\in\mathbf{N}$, and its refinement $\{U_{ij}\}$, where $j\in\mathbf{N}$ and i runs through a finite set, such that $U_{ij}\cap U_{ik}=\emptyset$, $j\neq k$ [21]. Then Y has a finite cover $\{U_i=\cup_j U_{ij}\}$. Since the finite exactness of the operator D takes place over any domain U_ξ, it also holds over any member U_{ij} of the refinement $\{U_{ij}\}$ of $\{U_\xi\}$ and, in accordance with item (i) above, over any member of the finite cover $\{U_i\}$ of Y. Then by virtue of item (ii) above, the finite exactness of D takes place over Y. **QED**

LEMMA 8.8. Let U and V be open subsets of a bundle Y and $\sigma\in\mathfrak{O}^*_\infty$ an exterior form of bounded jet order on $(\pi^\infty_0)^{-1}(U\cap V)\subset J^\infty Y$. Then, σ is split into a sum $\sigma_U+\sigma_V$ of exterior forms σ_U and σ_V of bounded jet order on $(\pi^\infty_0)^{-1}(U)$ and $(\pi^\infty_0)^{-1}(V)$, respectively. □

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$ and passing to the function with support in V, one gets a smooth real function f on $U \cup V$ which is 0 on a neighborhood of $U - V$ and 1 on a neighborhood of $V - U$ in $U \cup V$. Let $(\pi_0^\infty)^* f$ be the pull-back of f onto $(\pi_0^\infty)^{-1}(U \cup V)$. The exterior form $((\pi_0^\infty)^* f)\sigma$ is 0 on a neighborhood of $(\pi_0^\infty)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi_0^\infty)^{-1}(U)$. Let us denote it σ_U. Accordingly, the exterior form $(1 - (\pi_0^\infty)^* f)\sigma$ has an extension σ_V by 0 to $(\pi_0^\infty)^{-1}(V)$. Then, $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because σ_U and σ_V are of the jet order which does not exceed that of σ. **QED**

D. The global inverse problem

The expressions (8.10) – (8.11) in Proposition 8.6 provide a solution of the global inverse problem of the calculus of variations on fiber bundles in the class of Lagrangians $L \in \mathcal{Q}_\infty^{0,n}$ of locally finite order [2, 47] (which is not so interesting for physical applications). These expressions together with Theorem 8.7 give a solution of the global inverse problem of the finite order calculus of variations.

COROLLARY 8.9. (i) A finite order Lagrangian $L \in \mathcal{O}_\infty^{0,n}$ is variationally trivial, i.e., $\delta(L) = 0$ iff

$$L = h_0\varphi + d_H\xi, \qquad \xi \in \mathcal{O}_\infty^{0,n-1}, \tag{8.13}$$

where φ is a closed n-form on Y. (ii) A finite order Euler–Lagrange-type operator satisfies the Helmholtz condition $\delta(\mathcal{E}) = 0$ iff

$$\mathcal{E} = \delta(L) + \tau(\phi), \qquad L \in \mathcal{O}_\infty^{0,n},$$

where ϕ is a closed $(n+1)$-form on Y. □

Note that item (i) in Corollary 8.9 contains the particular result of [49].

A solution of the global inverse problem of the calculus of variations in the class of exterior forms of bounded jet order has been suggested in [2] by a computation of cohomology of a fixed order variational sequence. However, this computation requires rather sophisticated *ad hoc* technique in order to be reproduced (see [27, 28, 50] for a different variational sequence). The theses of Corollary 8.9 also agree with those of [2], but the proof of Theorem 8.7 does not give a sharp bound on the order of a Lagrangian.

9 Geometry of simple graded manifolds

The most of odd fields in quantum field theory can be described in terms of graded manifolds. These are fermions and odd ghosts and antifields. It should be emphasized that graded manifolds are not supermanifolds, though every graded manifold determines a DeWitt H^∞-supermanifold, and *vice versa*. Referring the reader to [7, 25, 44] for a general theory of graded manifolds, we here focus on the most physically relevant case of simple graded manifolds. We do not restrict a class of graded manifolds in question, but an arbitrary graded manifold can be brought into a certain explicit form (see Batchelor's Theorem 9.1 below) which, of course, narrows the class of automorphisms of a graded manifold. n physical applications, graded manifolds are usually given in this form from the beginning.

By a graded manifold of dimension (n, m) is meant a locally ringed space $(Z, \mathfrak{A})$ where Z is an n-dimensional smooth manifold Z and $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of graded commutative algebras of rank m such that [7]:

- there is the exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathfrak{A} \xrightarrow{\sigma} C^\infty_Z \to 0, \qquad \mathcal{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2, \tag{9.1}$$

  where C^∞_Z is the sheaf of smooth functions on Z;

- $\mathcal{R}/\mathcal{R}^2$ is a locally free C^∞_Z-module of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the exterior algebra (or the exterior bundle) $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

The sheaf $\mathfrak{A}$ is called a structure sheaf of the graded manifold $(Z, \mathfrak{A})$, while the manifold Z is said to be a body of $(Z, \mathfrak{A})$. Global sections of the sheaf $\mathfrak{A}$ are called graded functions. They constitute the structure module $\mathfrak{A}(Z)$ of the sheaf $\mathfrak{A}$.

A graded manifold $(Z, \mathfrak{A})$, by definition, has the following local structure. Given a point $z \in Z$, there exists its open neighbourhood U, called a splitting domain, such that

$$\mathfrak{A}(U) \cong C^\infty(U) \otimes \wedge \mathbb{R}^m. \tag{9.2}$$

It means that the restriction $\mathfrak{A}\,|_U$ of the structure sheaf $\mathfrak{A}$ to U is isomorphic to the sheaf $C^\infty_U \otimes \wedge \mathbb{R}^m$ of sections of some exterior bundle $\wedge E^*_U = U \times \wedge \mathbb{R}^m \to U$.

The well-known Batchelor's theorem [7, 8] states that such a structure of graded manifolds is global.

THEOREM 9.1. Let $(Z, \mathfrak{A})$ be a graded manifold. There exists a vector bundle $E \to Z$ with an m-dimensional typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z, \mathfrak{A})$ is isomorphic to the structure sheaf $\mathfrak{A}_E = C^\infty_Z \otimes \wedge V^*$ of sections of the exterior bundle $\wedge E^*$, whose typical fibre is the Grassmann algebra $\wedge V^*$. $\Box$

It should be emphasized that Batchelor's isomorphism in Theorem 9.1 fails to be canonical. At the same time, there are many physical models where a vector bundle E is introduced from the beginning. In this case, it suffices to consider the structure sheaf $\mathfrak{A}_E$ of the exterior bundle $\wedge E^*$ [16, 30, 40]. We agree to call the pair $(Z, \mathfrak{A}_E)$ a simple graded manifold. Its automorphisms are restricted to those, induced by automorphisms of the vector bundle $E \to Z$. This is called the characteristic vector bundle of the simple graded manifold $(Z, \mathfrak{A}_E)$. Accordingly, the structure module $\mathfrak{A}_E(Z) = \wedge E^*(Z)$ of the sheaf $\mathfrak{A}_E$ (and of the exterior bundle $\wedge E^*$) is said to be the structure module of the simple graded manifold $(Z, \mathfrak{A}_E)$.

Given a simple graded manifold $(Z, \mathfrak{A}_E)$, every trivialization chart $(U; z^A, y^a)$ of the vector bundle $E \to Z$ is a splitting domain of $(Z, \mathfrak{A}_E)$. Graded functions on such a chart are Λ-valued function

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \ldots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{9.3}$$

where $f_{a_1 \cdots a_k}(z)$ are smooth functions on U, $\{c^a\}$ is the fibre basis for E^*, and we omit the symbol of the exterior product of elements c. In particular, the sheaf epimorphism σ in (9.1) is induced by the body morphism of Λ. We agree to call $\{z^A, c^a\}$ the local basis for the graded manifold $(Z, \mathfrak{A}_E)$. Transition functions $y'^a = \rho^a_b(z^A) y^b$ of bundle coordinates on $E \to Z$ induce the corresponding transformation

$$c'^a = \rho^a_b(z^A) c^b \tag{9.4}$$

of the associated local basis for the graded manifold $(Z, \mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (9.3).

Let us note that general transformations of a graded manifold take the form

$$c'^a = \rho^a(z^A, c^b), \tag{9.5}$$

where $\rho^a(z^A, c^b)$ are local graded functions. Considering only simple graded manifolds, we actually restrict the class of graded manifold transformations (9.5) to the linear ones (9.4), compatible with a given Batchelor's isomorphism.

Remark 9.1. Although graded functions are locally represented by Λ-valued functions (9.3), they are not Λ-valued functions on a manifold Z because of the transformation law (9.4) (or (9.5)). •

Given a graded manifold $(Z, \mathfrak{A})$, by the sheaf $\mathfrak{d}\mathfrak{A}$ of graded derivations of $\mathfrak{A}$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}$ such that any section u of $\mathfrak{d}\mathfrak{A}$ over an open subset $U \subset Z$ is a graded derivation of the graded algebra $\mathfrak{A}(U)$, i.e.

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f') \tag{9.6}$$

for all homogeneous elements $u \in \mathfrak{d}\mathfrak{A}(U)$ and $f, f' \in \mathfrak{A}(U)$. Conversely, one can show that, given open sets $U' \subset U$, there is a surjection of the derivation modules $\mathfrak{d}(\mathfrak{A}(U)) \to \mathfrak{d}(\mathfrak{A}(U'))$ [7]. It follows that any graded derivation of the local graded algebra $\mathfrak{A}(U)$ is also a local section over U of the sheaf $\mathfrak{d}\mathfrak{A}$. Sections of $\mathfrak{d}\mathfrak{A}$ are called graded vector fields on the graded manifold $(Z, \mathfrak{A})$. The graded derivation sheaf $\mathfrak{d}\mathfrak{A}$ is a sheaf of Lie superalgebras with respect to the bracket

$$[u, u'] = uu' + (-1)^{[u][u']+1} u'u. \tag{9.7}$$

In comparison with general theory of graded manifolds, an essential simplification is that graded vector fields on a simple graded manifold $(Z, \mathfrak{A}_E)$ can be seen as sections of a vector bundle as follows [30, 40].

Due to the vertical splitting $VE \cong E \times E$ (1.15), the vertical tangent bundle VE of $E \to Z$ can be provided with the fibre bases $\{\partial/\partial c^a\}$, which are the duals of the bases $\{c^a\}$. These are the fibre bases for $\mathrm{pr}_2 VE \cong E$. Then graded vector fields on a trivialization chart $(U; z^A, y^a)$ of E read

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \tag{9.8}$$

where u^λ, u^a are local graded functions on U [7, 30]. In particular,

$$\frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} = -\frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \qquad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A.$$

The derivations (9.8) act on graded functions $f \in \mathfrak{A}_E(U)$ (9.3) by the rule

$$u(f_{a\ldots b} c^a \cdots c^b) = u^A \partial_A (f_{a\ldots b}) c^a \cdots c^b + u^k f_{a\ldots b} \frac{\partial}{\partial c^k} \rfloor (c^a \cdots c^b). \tag{9.9}$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \qquad u'^a = \rho^a_j u^j + u^A \partial_A (\rho^a_j) c^j$$

of graded vector fields. It follows that graded vector fields (9.8) can be represented by sections of the vector bundle $\mathcal{V}_E \to Z$ which is locally isomorphic to the vector bundle

$$\mathcal{V}_E|_U \approx \wedge E^* \underset{Z}{\otimes} (E \underset{Z}{\oplus} TZ)|_U,$$

and is characterized by an atlas of bundle coordinates $(z^A, z^A_{a_1\ldots a_k}, v^i_{b_1\ldots b_k})$, $k = 0, \ldots, m$, possessing the transition functions

$$\begin{aligned} z'^A_{i_1\ldots i_k} &= \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} z^A_{a_1\ldots a_k}, \\ v'^i_{j_1\ldots j_k} &= \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\ldots b_k} + \frac{k!}{(k-1)!} z^A_{b_1\ldots b_{k-1}} \partial_A \rho^i_{b_k} \right], \end{aligned} \tag{9.10}$$

which fulfil the cocycle condition (1.4).

Remark 9.2. One tries to construct a graded tangent bundle over a graded manifold $(Z, \mathfrak{A})$ whose sheaf of sections is the graded derivation sheaf $\mathfrak{d}\mathfrak{A}$. Nevertheless, the transformation law (9.10) shows that the projection

$$\mathcal{V}_E\,|_U \to \mathrm{pr}_2 VE \underset{Z}{\oplus} TZ$$

is not global, i.e., $\mathcal{V}_E$ is not an exterior bundle. It means that the sheaf of derivations $\mathfrak{d}\mathfrak{A}$ is not a structure sheaf of a graded manifold. •

There is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} E \to \mathcal{V}_E \to \wedge E^* \underset{Z}{\otimes} TZ \to 0 \tag{9.11}$$

of vector bundles over Z. Its splitting

$$\widetilde{\gamma} : \dot{z}^A \partial_A \mapsto \dot{z}^A (\partial_A + \widetilde{\gamma}_A^a \frac{\partial}{\partial c^a}) \tag{9.12}$$

transforms every vector field τ on Z into the graded vector field

$$\tau = \tau^A \partial_A \mapsto \nabla_\tau = \tau^A (\partial_A + \widetilde{\gamma}_A^a \frac{\partial}{\partial c^a}), \tag{9.13}$$

which is a graded derivation of the sheaf $\mathfrak{A}_E$ satisfying the Leibniz rule

$$\nabla_\tau (sf) = (\tau \rfloor ds) f + s \nabla_\tau (f), \quad f \in \mathfrak{A}_E(Z), \quad s \in C^\infty(Z).$$

Therefore, one can think of the splitting (9.12) of the exact sequence (9.11) as being a graded connection on the simple graded manifold $(Z, \mathfrak{A}_E)$ [30, 40]. In particular, this connection provides the corresponding horizontal splitting

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a} = u^A (\partial_A + \widetilde{\gamma}_A^a \frac{\partial}{\partial c^a}) + (u^a - u^A \widetilde{\gamma}_A^a) \frac{\partial}{\partial c^a}$$

of graded vector fields.

In accordance with Theorem 1.5, a graded connection (9.12) always exists.

Remark 9.3. By virtue of the isomorphism (9.2), any connection $\widetilde{\gamma}$ on a graded manifold $(Z, \mathfrak{A})$, restricted to a splitting domain U, takes the form (9.12). Given two splitting domains U and U' of $(Z, \mathfrak{A})$ with the transition functions (9.5), the connection components $\widetilde{\gamma}_A^a$ obey the transformation law

$$\widetilde{\gamma}'^a_A = \widetilde{\gamma}^b_A \frac{\partial}{\partial c^b} \rho^a + \partial_A \rho^a. \tag{9.14}$$

If U and U' are the trivialization charts of the same vector bundle E in Theorem 9.1 together with the transition functions (9.4), the transformation law (9.14) takes the form

$$\widetilde{\gamma}'^a_A = \rho^a_b(z) \widetilde{\gamma}^b + \partial_A \rho^a_b(z) c^b. \tag{9.15}$$

•

Remark 9.4. It should be emphasized that the above notion of a graded connection differs from that of a connection on a graded fibre bundle $(Z, \mathfrak{A}) \to (X, \mathcal{B})$ in [1]. The latter is a section of the jet graded bundle $J^1(Z/X) \to (Z, \mathfrak{A})$ of sections of the graded fibre bundle $(Z, \mathfrak{A}) \to (X, \mathcal{B})$ (see [35] for formalism of jets of graded manifolds). •

Example 9.5. Every linear connection

$$\gamma = dz^A \otimes (\partial_A + \gamma_A{}^a{}_b y^b \partial_a)$$

on the vector bundle $E \to Z$ yields the graded connection

$$\gamma_S = dz^A \otimes (\partial_A + \gamma_A{}^a{}_b c^b \frac{\partial}{\partial c^a}) \tag{9.16}$$

on the simple graded manifold $(Z, \mathfrak{A}_E)$. In view of Remark 9.3, γ_S is also a graded connection on the graded manifold $(Z, \mathfrak{A}) \cong (Z, \mathfrak{A}_E)$, but its linear form (9.16) is not maintained under the transformation law (9.14). •

The curvature of the graded connection ∇_τ (9.13) is defined by the expression:

$$\begin{aligned}
&R(\tau, \tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau,\tau']}, \\
&R(\tau, \tau') = \tau^A \tau'^B R^a_{AB} \frac{\partial}{\partial c^a} : \mathfrak{A}_E \to \mathfrak{A}_E, \\
&R^a_{AB} = \partial_A \widetilde{\gamma}^a_B - \partial_B \widetilde{\gamma}^a_A + \widetilde{\gamma}^k_A \frac{\partial}{\partial c^k} \widetilde{\gamma}^a_B - \widetilde{\gamma}^k_B \frac{\partial}{\partial c^k} \widetilde{\gamma}^a_A.
\end{aligned} \tag{9.17}$$

It can also be written in the form

$$R = \frac{1}{2} R^a_{AB} dz^A \wedge dz^B \otimes \frac{\partial}{\partial c^a}. \tag{9.18}$$

Let now $\mathcal{V}^*_E \to Z$ be a vector bundle which is the pointwise $\wedge E^*$-dual of the vector bundle $\mathcal{V}_E \to Z$. It is locally isomorphic to the vector bundle

$$\mathcal{V}^*_E|_U \approx \wedge E^* \underset{Z}{\otimes} (E^* \underset{Z}{\oplus} T^*Z)|_U.$$

With respect to the dual bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for $\mathrm{pr}_2 V^*E \cong E^*$, sections of the vector bundle $\mathcal{V}^*_E$ take the coordinate form

$$\phi = \phi_A dz^A + \phi_a dc^a,$$

together with transition functions

$$\phi'_a = \rho^{-1b}{}_a \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1b}{}_a \partial_A(\rho^a_j) \phi_b c^j.$$

They are regarded as graded exterior one-forms on the graded manifold $(Z, \mathfrak{A}_E)$.

The sheaf $\mathfrak{D}^1\mathfrak{A}_E$ of germs of sections of the vector bundle $\mathcal{V}^*_E \to Z$ is the dual of the graded derivation sheaf $\mathfrak{d}\mathfrak{A}_E$, where the duality morphism is given by the graded interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a. \tag{9.19}$$

In particular, the dual of the exact sequence (9.11) is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} T^*Z \to \mathcal{V}^*_E \to \wedge E^* \underset{Z}{\otimes} E^* \to 0. \tag{9.20}$$

Any graded connection $\widetilde{\gamma}$ (9.12) yields the splitting of the exact sequence (9.20), and determines the corresponding decomposition of graded one-forms

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \widetilde{\gamma}^a_A) dz^A + \phi_a (dc^a - \widetilde{\gamma}^a_A dz^A).$$

Graded exterior k-forms ϕ are defined as sections of the graded exterior bundle $\overset{k}{\underset{Z}{\wedge}} \mathcal{V}^*_E$ such that

$$\phi \wedge \sigma = (-1)^{|\phi||\sigma| + [\phi][\sigma]} \sigma \wedge \phi, \tag{9.21}$$

where $|.|$ denotes the form degree. For instance,

$$dz^A \wedge dc^i = -dc^i \wedge dz^A, \qquad dc^i \wedge dc^j = dc^j \wedge dc^i. \tag{9.22}$$

The graded interior product (9.19) is extended to higher graded exterior forms by the rule

$$u \rfloor (\phi \wedge \sigma) = (u \rfloor \phi) \wedge \sigma + (-1)^{|\phi| + [\phi][u]} \phi \wedge (u \rfloor \sigma). \tag{9.23}$$

The graded exterior differential d of graded functions is introduced by the condition $u \rfloor df = u(f)$ for an arbitrary graded vector field u. It is extended uniquely to graded exterior forms by the rule

$$d(\phi \wedge \sigma) = d\phi \wedge \sigma + (-1)^{|\phi|} \phi \wedge d\sigma, \qquad d \circ d = 0, \tag{9.24}$$

and is given by the coordinate expression

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \frac{\partial}{\partial c^a} \phi,$$

where the left derivatives ∂_λ, $\partial/\partial c^a$ act on coefficients of graded exterior forms by the rule (9.9), and they are graded commutative with the forms dz^A, dc^a [16, 25]. The Lie derivative of a graded exterior form ϕ along a graded vector field u is defined by the familiar formula

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi). \tag{9.25}$$

It possesses the property

$$\mathbf{L}_u (\phi \wedge \phi') = \mathbf{L}_u (\phi) \wedge \phi' + (-1)^{[u][\phi]} \phi \wedge \mathbf{L}_u (\phi').$$

With the graded exterior differential$d$, graded exterior forms constitute an $\mathbb{N}$-, $\mathbb{Z}_2$-graded differential algebra $\mathcal{O}^*\mathcal{A}_E$, where $\mathcal{O}^*\mathcal{A}_E = \mathcal{A}_E = \mathfrak{A}_E(Z)$ denotes the structure module of graded functions on Z. It is called a graded commutative differential algebra. The corresponding graded de Rham complex is

$$0 \to \mathbb{R} \to \mathcal{A}_E \xrightarrow{d} \mathcal{O}^1\mathcal{A}_E \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A}_E \xrightarrow{d} \cdots. \tag{9.26}$$

Cohomology $H^q_{GR}(Z)$ of the graded de Rham complex (9.26) is called graded de Rham cohomology of the graded manifold $(Z, \mathfrak{A}_E)$. One can compute this cohomology with the aid of the abstract de Rham theorem. Let $\mathfrak{O}^k\mathfrak{A}_E$ denote the sheaf of germs of graded k-forms on $(Z, \mathfrak{A}_E)$. Its structure module is $\mathcal{O}^k\mathcal{A}_E = \mathfrak{O}^k\mathfrak{A}_E(Z)$. These sheaves make up the complex

$$0 \to \mathbb{R} \longrightarrow \mathfrak{A}_E \xrightarrow{d} \mathfrak{O}^1\mathfrak{A}_E \xrightarrow{d} \cdots \mathfrak{O}^k\mathfrak{A}_E \xrightarrow{d} \cdots. \tag{9.27}$$

All $\mathfrak{O}^k\mathfrak{A}_E$ are sheaves of C^∞_Z-modules on Z and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded exterior forms holds [7, 25]. It follows that the complex (9.27) is a fine resolution of the constant sheaf $\mathbb{R}$ on the manifold Z. Then, by virtue of the abstract de Rham theorem, there is an isomorphism

$$H^*_{GR}(Z) = H^*(Z; \mathbb{R}) = H^*(Z) \tag{9.28}$$

of the graded de Rham cohomology $H^*_{GR}(Z)$ to the de Rham cohomology $H^*(Z)$ of the smooth manifold Z [25]. Moreover, the cohomology isomorphism (9.28) accompanies the cochain monomorphism $\mathcal{O}^*(Z) \to \mathcal{O}^*\mathcal{A}_E$ of the de Rham complex $\mathcal{O}^*(Z)$ of smooth exterior forms on Z to the graded de Rham complex (9.26). Hence, any closed graded exterior form is split into a sum $\phi = d\sigma + \varphi$ of an exact graded exterior form $d\sigma \in \mathcal{O}^*\mathcal{A}_E$ and a closed exterior form $\varphi \in \mathcal{O}^*(Z)$ on Z.

10 Jets of ghosts and antifields

In field-antifield BRST theory, the antibracket is defined by means of the variational operator. This operator can be introduced in a rigorous algebraic way as the coboundary operator of the variational complex of exterior forms on the infinite jet space of physical fields, ghosts and antifields [5, 6, 10, 11]. Herewith, the antibracket and the BRTS operator are expressed in terms of jets of ghosts and physical fields. For example, the BRST transformation of gauge potentials a^r_λ in Yang–Mills theory reads

$$\mathbf{s}a^r_\lambda = C^r_\lambda + c^r_{pq}a^p_\lambda C^q,$$

where C^r_λ are jets of ghosts C^r introduced in a heuristic way.

Furthermore, the variational complex in BRST theory on a contractible manifold $X = \mathbb{R}^n$ is exact. It follows that the kernel of the variational operator δ equals the image of the horizontal differential d_H. Therefore, several objects in field-antifield BRST theory on $\mathbb{R}^n$ are determined modulo d_H-exact forms. In particular, let us mention the iterated cohomology $H^{k,p}(\mathbf{s}|d_H)$ of the BRST bicomplex, defined with respect to the BRST operator $\mathbf{s}$ and the horizontal differential d_H, and graded by the ghost number k and the form degree p. The iterated cohomology of form degree $p = n = \dim X$ coincides with the local BRST cohomology (i.e., the $\mathbf{s}$-cohomology modulo d_H). If $X = \mathbb{R}^n$, an isomorphism of the local BRST cohomology $H^{k,n}(\mathbf{s}|d_H)$, $k \neq -n$, to the cohomology $H^{k+n}_{\rm tot}$ of the total BRST operator $\mathbf{s} + d_H$ has been proved by constructing the descent equations [10]. This result has been generalized to an arbitrary connected manifold X [19, 41]. For this purpose, we provide a (global) differential geometric definition of jets of odd ghosts and antifields, and extend the variational complex to the space of these jets.

For the sake of simplicity, we consider BRST theory of even physical fields and finitely reducible gauge transformations. Its finite classical basis consists of even physical fields of zero ghost number, even and odd ghosts (including ghosts-for-ghosts) of strictly positive ghost number, and even and odd antifields of strictly negative ghost number. For instance, this is the case of Yang–Mills theory.

A. Jets of odd ghosts

There exist different geometric models of ghosts. For instance, ghosts in Yang–Mills theory are often represented by the Maurer–Cartan form on the gauge group [9]. This representation, however, is not extended to other gauge models. We describe all odd fields as elements of simple graded manifolds [30, 40, 41].

Let $Y \to X$ be the characteristic vector bundle of a simple graded manifold $(X, \mathfrak{A}_Y)$. The r-order jet manifold J^rY of Y is also a vector bundle over X. Let us consider the simple graded manifold $(X, \mathfrak{A}_{J^rY})$, determined by the characteristic vector bundle $J^rY \to X$. Its local basis is $\{x^\lambda, c^a_\Lambda\}$, $0 \leq |\Lambda| \leq r$, where $\Lambda = (\lambda_k, \ldots, \lambda_1)$ are multi-indices. It possesses the transition functions

$$c'^a_{\lambda+\Lambda} = d_\lambda(\rho^a_j c^j_\Lambda), \qquad d_\lambda = \partial_\lambda + \sum_{|\Lambda|<r} c^a_{\lambda+\Lambda}\frac{\partial}{\partial c^a_\Lambda}, \tag{10.1}$$

where d_λ is the graded total derivative. In view of the transition functions (10.1), one can think of $(X, \mathfrak{A}_{J^rY})$ as being a graded r-order jet manifold of the simple graded manifold $(X, \mathfrak{A}_Y)$.

Let $\mathcal{O}^*\mathcal{A}_{J^rY}$ be the differential algebra of graded exterior forms on the graded jet manifold $(X, \mathfrak{A}_{J^rY})$. Since $Y \to X$ is a vector bundle, the canonical fibration $\pi^r_{r-1} : J^rY \to J^{r-1}Y$ is a linear morphism of vector bundles over X and, thereby, yields the corresponding morphism of graded jet manifolds $(X, \mathfrak{A}_{J^rY}) \to (X, \mathfrak{A}_{J^{r-1}Y})$ accompanied by the pull-back monomorphism of differential algebras $\mathcal{O}^*\mathcal{A}_{J^{r-1}Y} \to \mathcal{O}^*\mathcal{A}_{J^rY}$. Then we have the direct system of differential algebras

$$\mathcal{O}^*\mathcal{A}_Y \longrightarrow \mathcal{O}^*\mathcal{A}_{J^1Y} \longrightarrow \cdots \mathcal{O}^*\mathcal{A}_{J^rY} \stackrel{\pi^{r+1*}_r}{\longrightarrow} \cdots .$$

Its direct limit $\mathcal{O}^*_\infty \mathcal{A}_Y$ consists of graded exterior forms on all graded jet manifolds $(X, \mathfrak{A}_{J^rY})$ modulo the pull-back identification. It is a locally free graded $C^\infty(X)$-algebra generated by the elements

$$(1, c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda} dx^\lambda), \qquad 0 \leq |\Lambda|,$$

where dx^λ and θ^a_Λ are called horizontal and contact forms, respectively. In particular, $\mathcal{O}^0_\infty \mathcal{A}_Y$ is the graded commutative ring of graded functions on all graded jet manifolds $(X, \mathfrak{A}_{J^rY})$ modulo the pull-back identification.

Let us consider the sheaf $\mathfrak{Q}^0_\infty \mathcal{A}_Y$ of germs of graded functions $\phi \in \mathcal{O}^*_\infty \mathcal{A}_Y$. It is a sheaf of graded commutative algebras on X, and the pair $(X, \mathfrak{Q}^0_\infty \mathcal{A}_Y)$ is a graded manifold. This graded manifold is the projective limit of the inverse system of graded jet manifolds

$$(X, \mathfrak{A}_Y) \longleftarrow (X, \mathfrak{A}_{J^1Y}) \longleftarrow \cdots (X, \mathfrak{A}_{J^rY}) \longleftarrow \cdots,$$

and is called the graded infinite jet manifold. Then one can think of elements of the algebra $\mathcal{O}^*_\infty \mathcal{A}_Y$ as being graded exterior forms on the graded manifold $(X, \mathfrak{Q}^0_\infty \mathcal{A}_Y)$.

There is the canonical splitting of

$$\mathcal{O}^*_\infty \mathcal{A}_Y = \underset{k,s}{\oplus} \mathcal{O}^{k,s}_\infty \mathcal{A}_Y, \qquad 0 \leq k, \qquad 0 \leq s \leq n,$$

into $\mathcal{O}^0_\infty \mathcal{A}_Y$-modules $\mathcal{O}^{k,s}_\infty \mathcal{A}_Y$ of k-contact and s-horizontal graded forms. Accordingly, the graded exterior differential d on $\mathcal{O}^*_\infty \mathcal{A}_Y$ is split into the sum $d = d_H + d_V$, where d_H is the nilpotent graded horizontal differential

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi) : \mathcal{O}^{k,s}_\infty \mathcal{A}_Y \to \mathcal{O}^{k,s+1}_\infty \mathcal{A}_Y.$$

With respect to the BRST operator $\mathbf{s}$, the graded exterior forms $\phi \in \mathcal{O}^*_\infty \mathcal{A}_Y$ are characterized by the ghost number

$$\mathrm{gh}(dc^a_\Lambda) = \mathrm{gh}(c^a_\Lambda) = \mathrm{gh}(c^a),$$

and one puts $\mathbf{s} \circ d_H + d_H \circ \mathbf{s} = 0$.

B. Even physical fields and ghosts

In order to describe odd and even elements of the classical basis of field-antifield BRST theory on the same footing, we will generalize the notion of a graded manifold to graded commutative algebras generated both by odd and even elements [41].

Let $Y = Y_0 \oplus Y_1$ be the Whitney sum of vector bundles $Y_0 \to X$ and $Y_1 \to X$. We regard it as a bundle of graded vector spaces with the typical fibre $V = V_0 \oplus V_1$. Let us consider the quotient of the tensor bundle

$$\otimes Y^* = \overset{\infty}{\underset{k=0}{\oplus}} (\overset{k}{\underset{X}{\otimes}} Y^*)$$

by the elements

$$y_0y_0' - y_0'y_0, \quad y_1y_1' + y_1'y_1, \quad y_0y_1 - y_1y_0$$

for all $y_0, y_0' \in Y_{0x}^*$, $y_1, y_1' \in Y_{1x}^*$, and $x \in X$. It is an infinite-dimensional vector bundle, further denoted by $\wedge Y^*$. Global sections of $\wedge Y^*$ constitute a graded commutative algebra $\mathcal{A}_Y$, which is the product over $C^\infty(X)$ of the commutative algebra $\mathcal{A}_0$ of global sections of the symmetric bundle $\vee Y_0^* \to X$ and the graded algebra $\mathcal{A}_1$ of global sections of the exterior bundle $\wedge Y_1^* \to X$.

Let $\mathfrak{A}$, $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be the sheaves of germs of sections of the vector bundles $\wedge Y^*$, $\vee Y_0^*$ and $\wedge Y_1^*$, respectively. The pair $(X, \mathfrak{A}_1)$ is a familiar simple graded manifold. Therefore, we agree to call $(X, \mathfrak{A})$ the graded commutative manifold, determined by the characteristic graded vector bundle Y. Given a bundle coordinate chart $(U; x^\lambda, y_0^i, y_1^a)$ of Y, the local basis for $(X, \mathfrak{A})$ is $(x^\lambda, c_0^i, c_1^a)$, where $\{c_0^i\}$ and $\{c_1^a\}$ are the fibre bases for the vector bundles Y_0^* and Y_1^*, respectively. Then a straightforward repetition of all the above constructions for a simple graded manifold provides us with the differential algebra $\mathcal{O}_\infty^*\mathcal{A}$ of graded commutative exterior forms on the graded commutative infinite jet manifold $(X, \mathfrak{Q}_\infty^0\mathcal{A})$. This is a $C^\infty(X)$-algebra generated locally by the elements

$$(1, c_{0\Lambda}^i, c_{1\Lambda}^a, dx^\lambda, \theta_{0\Lambda}^i, \theta_{1\Lambda}^a), \qquad 0 \leq |\Lambda|.$$

Its $C^\infty(X)$-subalgebra $\mathcal{O}_\infty^*\mathcal{A}_1$, generated locally by the elements $(1, c_{1\Lambda}^i, dx^\lambda, \theta_{1\Lambda}^i)$, is exactly the differential algebra of graded exterior forms on the graded manifold $(X, \mathfrak{Q}^0\mathcal{A}_1)$. The $C^\infty(X)$-subalgebra $\mathcal{O}_\infty^*\mathcal{A}_0$ of $\mathcal{O}_\infty^*\mathcal{A}$ generated locally by the elements $(1, c_{0\Lambda}^i, dx^\lambda, \theta_{0\Lambda}^i)$, $0 \leq |\Lambda|$, is isomorphic to the polynomial subalgebra P_∞^* of the differential algebra $\mathcal{O}_\infty^*$ of exterior forms on the infinite jet manifold $J^\infty Y_0$ of the vector bundle $Y_0 \to X$ after its pull-back onto X [19, 20]. The algebra $\mathcal{O}_\infty^*$ provides the differential calculus in classical field theory.

C. Antifields

The jet formulation of field-antifield BRST theory enables one to introduce antifields on the same footing as physical fields and ghosts. Let Φ^A be a collective symbol for physical fields and ghosts. Let E be the characteristic graded vector bundle of the graded commutative manifold, generated by Φ^A. Treated as source coefficients of BRST transformations, antifields Φ_A^* with the ghost number

$$\mathrm{gh}\,\Phi_A^* = -\mathrm{gh}\,\Phi^A - 1$$

are represented by elements of the graded commutative manifold, determined by the characteristic graded vector bundle $\overset{n}{\wedge} T^*X \otimes E^*$. Then the total characteristic graded vector bundle of a graded commutative manifold for a classical basis of field-antifield BRST theory is

$$Y = E \oplus (\overset{n}{\wedge} T^*X \otimes E^*).$$

In particular, gauge potentials a^r_λ in Yang–Mills theory on a principal bundle $P \to X$ are represented by sections of the affine bundle $J^1P/G \to X$, modelled on the vector bundle $T^*X \otimes V_GP \to X$. Accordingly, the characteristic vector bundle for their odd antifields is

$$\overset{n}{\wedge} T^*X \otimes TX \otimes V^*_GP \to X.$$

As was mentioned above, the characteristic vector bundle for ghosts C^r in Yang–Mills theory is the Lie algebra bundle $V_GP \to X$. Then the characteristic vector bundle for their even antifields is

$$\overset{n}{\wedge} T^*X \otimes V^*_GP \to X.$$

Thus, the total characteristic graded vector bundle for BRST Yang–Mills theory is

$$Y = Y_0 \oplus Y_1 = [(T^*X \otimes V_GP) \oplus (\overset{n}{\wedge} T^*X \otimes V^*_GP)] \oplus$$
$$[V_GP \oplus (\overset{n}{\wedge} T^*X \otimes TX \otimes V^*_GP)].$$

The jets $\Phi^*_{A\Lambda}$ of antifields Φ^*_A are introduced similarly to jets Φ^A_Λ of physical fields and ghosts Φ^A.

D. The variational complex in BRST theory

The differential algebra $\mathcal{O}^*_\infty\mathcal{A}$ gives everything for global formulation of Lagrangian field-antifield BRST theory on a manifold X. We restrict our consideration to the short variational complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_\infty\mathcal{A} \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0, \tag{10.2}$$

where δ is the variational operator such that $\delta \circ d_H = 0$. It is given by the expression

$$\delta(L) = (-1)^{|\Lambda|}\theta^a \wedge d_\Lambda(\partial^\Lambda_a L), \qquad L \in \mathcal{O}^{0,n}_\infty\mathcal{A},$$

with respect to a physical basis $(\zeta^a) = (\Phi^A, \Phi^*_A)$.

The variational complex (10.2) provides the algebraic approach to the antibracket technique, where one can think of elements L of $\mathcal{O}^{0,n}_\infty\mathcal{A}$ as being Lagrangians of fields, ghosts and antifields. Note that, to be well-defined, a global BRST Lagrangian should factorize through covariant differentials of physical fields, ghosts and antifields $D_\lambda\zeta^a = \zeta^a_\lambda - \widetilde{\gamma}^a_\lambda$, where $\widetilde{\gamma}$ is a connection on the graded commutative manifold $(X, \mathfrak{A})$.

In order to obtain cohomology of the variational complex (10.2), let us consider the sheaf $\mathfrak{Q}^*_\infty\mathcal{A}$ of germs of elements $\phi \in \mathcal{O}^*_\infty\mathcal{A}$ and the graded differential algebra $P^*_\infty\mathcal{A}$ of global sections of this sheaf. Note that $P^*_\infty\mathcal{A} \neq \mathcal{O}^*_\infty\mathcal{A}$. Roughly speaking, any element of $\mathcal{O}^*_\infty\mathcal{A}$ is of bounded jet order, whereas elements of $P^*_\infty\mathcal{A}$ need not be so.

We have the short variational complex of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{Q}^0_\infty\mathcal{A} \xrightarrow{d_H} \mathfrak{Q}^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} \mathfrak{Q}^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0. \tag{10.3}$$

Graded commutative exterior forms $\phi \in \mathcal{O}^*_\infty \mathcal{A}$ are proved to satisfy the algebraic Poincaré lemma, i.e., any closed graded commutative exterior form on the graded manifold $(\mathbb{R}^n, \mathfrak{Q}^0_\infty \mathcal{A})$ is exact [10]. Consequently, the complex (10.3) is exact. Since $\mathfrak{Q}^{0,*}_\infty \mathcal{A}$ are sheaves of $C^\infty(X)$-modules on X, they are fine and acyclic. Without inspecting the acyclicity of the sheaf $\operatorname{Im}\delta$, one can apply a minor modification of the abstract de Rham theorem [20] to the complex (10.3), and obtains that cohomology of the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow P^0_\infty \mathcal{A} \xrightarrow{d_H} P^{0,1}_\infty \mathcal{A} \cdots \xrightarrow{d_H} P^{0,n}_\infty \mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{10.4}$$

is isomorphic to the de Rham cohomology of a manifold X.

Following suit of Theorem 8.7 and replacing exterior forms on $J^\infty Y$ with graded commutative forms on $(X, \mathfrak{Q}^0_\infty \mathcal{A})$, one can show that cohomology of the short variational complex (10.2) is isomorphic to that of the complex (10.4) and, consequently, to the de Rham cohomology of X. Moreover, this isomorphism is performed by the natural monomorphism of the de Rham complex $\mathcal{O}^*$ of exterior forms on X to the complex (10.4). It follows that:

(i) every d_H-closed graded form $\phi \in \mathcal{O}^{0,m<n}_\infty \mathcal{A}$ is split into the sum $\phi = \varphi + d_H \xi$, where φ is a closed exterior m-form on X;

(ii) every δ-closed graded form $\phi \in \mathcal{O}^{0,n}_\infty \mathcal{A}$ is split into the sum $\phi = \varphi + d_H \xi$, where φ is an exterior n-form on X.

One should mention the important case of BRST theory where Lagrangians are independent on coordinates x^λ. Let us consider the subsheaf $\overline{\mathfrak{Q}}^*_\infty \mathcal{A}$ of the sheaf $\mathfrak{Q}^*_\infty \mathcal{A}$ which consists of germs of x-independent graded commutative exterior forms. Then we have the subcomplex

$$0 \longrightarrow \mathbb{R} \longrightarrow \overline{\mathfrak{Q}}^0_\infty \mathcal{A} \xrightarrow{d_H} \overline{\mathfrak{Q}}^{0,1}_\infty \mathcal{A} \cdots \xrightarrow{d_H} \overline{\mathfrak{Q}}^{0,n}_\infty \mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{10.5}$$

of the complex (10.3) and the corresponding subcomplex

$$0 \longrightarrow \mathbb{R} \longrightarrow \overline{P}^0_\infty \mathcal{A} \xrightarrow{d_H} \overline{P}^{0,1}_\infty \mathcal{A} \cdots \xrightarrow{d_H} \overline{P}^{0,n}_\infty \mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{10.6}$$

of the complex (10.4). Clearly, $\overline{P}^{0,*}_\infty \mathcal{A} \subset \mathcal{O}^{0,*}_\infty \mathcal{A}$, i.e., the complex (10.6) is also a subcomplex of the short variational complex (10.2).

The key point is that the complex of sheaves (10.5) fails to be exact. The obstruction to its exactness at the term $\overline{\mathfrak{Q}}^{0,k}_\infty$ consists of the germs of constant exterior k-forms on X [6]. Let us denote their sheaf by S^k_X. We have the short exact sequences of sheaves

$$0 \to \operatorname{Im} d_H \to \operatorname{Ker} d_H \to S^k_X \to 0, \qquad 0 < k < n,$$
$$0 \to \operatorname{Im} d_H \to \operatorname{Ker}\delta \to S^n_X \to 0$$

and the corresponding sequences of modules of their global sections

$$0 \to \operatorname{Im} d_H(X) \to \operatorname{Ker} d_H(X) \to S^k_X(X) \to 0, \qquad 0 < k < n,$$
$$0 \to \operatorname{Im} d_H(X) \to \operatorname{Ker}\delta(X) \to S^n_X(X) \to 0.$$

The latter are exact because $S_X^{k<n}$ and S_X^n are subsheaves of the sheaves $\mathrm{Ker}\, d_H$ and $\mathrm{Ker}\,\delta$, respectively. Therefore, the kth cohomology group of the complex (10.6) is isomorphic to the $\mathbb{R}$-module $S_X^k(X)$ of constant exterior k-forms, $0 < k \leq n$, on the manifold X. Consequently, any d_H-closed graded commutative k-form, $0 < k < n$, and any δ-closed graded commutative n-form ϕ, constant on X, are split into the sum $\phi = \varphi + d_H\xi$ where φ is a constant exterior form on X.

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