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MAGIC SQUARES AND MATRIX MODELS OF LIE ALGEBRAS

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Abstract. This paper is concerned with the description of exceptional simple Lie algebras as octonionic analogues of the classical matrix Lie algebras. We review the Tits-Freudenthal construction of the magic square, which includes the exceptional Lie algebras as the octonionic case of a construction in terms of a Jordan algebra of hermitian 3×3 matrices (Tits) or various plane and other geometries (Freudenthal). We present alternative constructions of the magic square which explain its symmetry, and show explicitly how the use of split composition algebras leads to analogues of the matrix Lie algebras $\mathfrak{su}(3)$, $\mathfrak{sl}(3)$ and $\mathfrak{sp}(6)$. We adapt the magic square construction to include analogues of $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{sp}(4)$ for all real division algebras.

Contents

1. Introduction

Semisimple Lie groups and Lie algebras are normally discussed in terms of their root systems, which makes possible a unified treatment and emphasises the common features of their underlying structures. However, some classical investigations [20] depend on particularly simple matrix descriptions of Lie groups, which are only available for the classical groups. This creates a distinction between the classical Lie algebras and the exceptional ones, which is maintained in some more recent work (e.g. [6, 10, 11]). This paper is motivated by the desire to give matrix descriptions of the exceptional Lie algebras, assimilating them to the classical ones, with a view to extending results like the Capelli identities to the exceptional cases.

It has long been known [8] that most exceptional Lie algebras are related to the exceptional Jordan algebra of 3×3 hermitian matrices with entries from the octonions, $\mathbb{O}$. Here we show that this relation yields descriptions of certain real forms of the complex Lie algebras F_4, E_6 and E_7 which can be interpreted as octonionic versions of the Lie algebras of, respectively, antihermitian 3×3 matrices, traceless 3×3 matrices and symplectic 6×6 matrices. To be precise, we define for each alternative algebra $\mathbb{K}$ a Lie algebra $\mathfrak{sa}(3, \mathbb{K})$ such that $\mathfrak{sa}(3, \mathbb{C}) = \mathfrak{su}(3)$ and $\mathfrak{sa}(3, \mathbb{O})$ is the compact real form of F_4; a Lie algebra $\mathfrak{sl}(3, \mathbb{K})$ which is equal to $\mathfrak{sl}(3, \mathbb{C})$ for $\mathbb{K} = \mathbb{C}$ and a non-compact real form of E_6 for $\mathbb{K} = \mathbb{O}$; and a Lie algebra $\mathfrak{sp}(6, \mathbb{K})$ such that $\mathfrak{sp}(6, \mathbb{C})$ is the set of 6×6 complex matrices X satisfying $X^\dagger J = -JX$ (where J is an antisymmetric real 6×6 matrix and $X^\dagger$ denotes the hermitian conjugate of X), and such that $\mathfrak{sp}(6, \mathbb{O})$ is a non-compact real form of E_7.

Our definitions can be adapted to 2×2 matrices to yield Lie algebras $\mathfrak{sa}(2, \mathbb{K})$, $\mathfrak{sl}(2, \mathbb{K})$ and $\mathfrak{sp}(4, \mathbb{K})$ reducing to $\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C})$ when $\mathbb{K} = \mathbb{C}$. These Lie algebras are isomorphic to various pseudo-orthogonal algebras.

The constructions in this paper are all related to Tits's magic square of Lie algebras [19]. This is a construction of a Lie algebra $T(\mathbb{K}, \mathbb{J})$ for any alternative algebra $\mathbb{K}$ and Jordan algebra $\mathbb{J}$. If $\mathbb{K} = \mathbb{K}_1$ and $\mathbb{J} = H_3(\mathbb{K}_2)$ is the Jordan algebra of 3×3 hermitian matrices over another alternative algebra $\mathbb{K}_2$, the Jordan product being the anticommutator, this yields a Lie algebra $L_3(\mathbb{K}_1, \mathbb{K}_2)$ for any pair of alternative algebras. Taking $\mathbb{K}_1$ and $\mathbb{K}_2$ to be real division algebras, we obtain a 4×4 square of compact Lie algebras which (magically) is symmetric and contains the compact real forms of F_4, E_6, E_7 and E_8. We will show that if the division algebra $\mathbb{K}_1$ is replaced by its split form $\widetilde{\mathbb{K}}_1$, one obtains a non-symmetric square of Lie algebras whose first three rows are the sets of matrix Lie algebras described above:

$$\begin{aligned} L_3(\mathbb{K}, \mathbb{R}) &= \mathfrak{sa}(3, \mathbb{K}) \\ L_3(\mathbb{K}, \widetilde{\mathbb{C}}) &= \mathfrak{sl}(3, \mathbb{K}) \\ L_3(\mathbb{K}, \widetilde{\mathbb{H}}) &= \mathfrak{sp}(6, \mathbb{K}). \end{aligned} \tag{1.1}$$

We will also describe magic squares of Lie algebras based on 2×2 matrices, which have similar properties.

The organisation of the paper is as follows. In Section 2 we establish notation, recall the definitions of various kinds of algebra, and introduce our generalised definitions of the Lie algebras $\mathfrak{sa}(n,\mathbb{K})$, $\mathfrak{sl}(n,\mathbb{K})$ and $\mathfrak{sp}(2n,\mathbb{K})$. In Section 3 we present Tits's general construction of $T(\mathbb{K},\mathbb{J})$, show that appropriate (split) choices of $\mathbb{K}$ yield the derivation, structure and conformal algebras of $\mathbb{J}$, and describe the magic squares obtained when $\mathbb{J}$ is a Jordan algebra of 3×3 hermitian matrices. Section 4 presents two alternative, manifestly symmetric, constructions of the magic square; one, due to Vinberg, describes the algebras in terms of matrices over tensor products of division algebras, while the other is based on Ramond's concept [14] of the triality algebra. In Section 5 we develop the description of the rows of the non-compact magic square as unitary, special linear and symplectic Lie algebras, and briefly describe Freudenthal's geometrical interpretation. In Section 6 we discuss the extension of these results from $n=3$ to general n, which is only possible for the associative division algebras $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, and in Section 8 we discuss the case $n=2$, with the octonions again included. In an appendix we prove extensions to alternative algebras of various matrix identities that are needed throughout the paper.

2. Algebras: Notation

When dealing with a Lie algebra, we will use the notation $\dot{+}$ for the direct sum of vector spaces. As well as making formulae easier to read, this enables us to reserve the use of $\oplus$ to denote the direct sum of Lie algebras, i.e. $L = M \oplus N$ implies that $[M,N]=0$ in L. For direct sums of a number of copies of a vector space we will use a multiple (rather than power) notation, writing $nV = V \dot{+} V \dot{+} \cdots V$ (n times). For real vector spaces with no Lie algebra structure but with a pseudo-orthogonal structure (a preferred bilinear form, not necessarily positive definite), we replace $\dot{+}$ by $\oplus$ to denote the internal direct sum of orthogonal subspaces, or the external direct sum of pseudo-orthogonal spaces in which $V\oplus W$ has the inherited inner product making V and W orthogonal subspaces.

Let $\mathbb{K}$ be an algebra over $\mathbb{R}$ with a non-degenerate quadratic form $x \mapsto |x|^2$ and associated bilinear form $\langle x,y\rangle$. If the quadratic form satisfies

$$|xy|^2 = |x|^2|y|^2, \qquad \forall x,y\in\mathbb{K}, \tag{2.1}$$

then $\mathbb{K}$ is a *composition algebra*. We consider $\mathbb{R}$ to be embedded in $\mathbb{K}$ as the set of scalar multiples of the identity element, and denote by $\mathbb{K}'$ the subspace of $\mathbb{K}$ orthogonal to $\mathbb{R}$, so that $\mathbb{K}=\mathbb{R}\dot{+}\mathbb{K}'$; we write $x = \operatorname{Re}x + \operatorname{Im}x$ with $\operatorname{Re}x\in\mathbb{R}$ and $\operatorname{Im}x\in\mathbb{K}'$. It can then be shown [9] that the conjugation which fixes each element of $\mathbb{R}$ and multiplies every element of $\mathbb{K}'$ by -1, denoted $x\mapsto\overline{x}$, satisfies

$$\overline{xy} = \overline{y}\,\overline{x} \tag{2.2}$$

and

$$x\overline{x} = |x|^2 . \tag{2.3}$$

The inner product in $\mathbb{K}$ is given in terms of this conjugation as

$$\langle x, y\rangle = \mathrm{Re}(x\overline{y}) = \mathrm{Re}(\overline{x}y).$$

In a composition algebra $\mathbb{K}$ we will generally adopt the typographical convention that lower-case letters from the end of the roman alphabet $(\dots, x, y, z)$ denote general elements of $\mathbb{K}$, while elements of $\mathbb{K}'$ are denoted by letters from the beginning of the roman alphabet $(a, b, c, \dots)$. We use the notations $[x, y]$ and $[x, y, z]$ for the commutator and associator

$$[x, y] = xy - yx,$$
$$[x, y, z] = (xy)z - x(yz).$$

These change sign if any one of their arguments is conjugated:

$$[\overline{x}, y] = -[x, y], \qquad [\overline{x}, y, z] = -[x, y, z].$$

Any composition algebra $\mathbb{K}$ satisfies the *alternative law*, i.e. the associator is an alternating function of x, y and z [17].

A division algebra is an algebra in which

$$xy = 0 \quad \Longrightarrow \quad x = 0 \ \text{ or } \ y = 0.$$

This is true in a composition algebra if the quadratic form $|x|^2$ is positive definite. By Hurwitz's Theorem [17], the only such positive definite composition algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. These algebras are obtained by the Cayley-Dickson process [17]; the same process with different signs yields *split* forms of these algebras. These are so called because the familiar equation

$$i^2 + 1 = 0$$

in $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ is replaced in the split algebras $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{O}}$ by

$$i^2 - 1 = (i + 1)(i - 1) = 0$$

i.e. the equation can be *split* for at least one of the imaginary basis elements (specifically: for the one imaginary unit of $\widetilde{\mathbb{C}}$, for two of the three imaginary units of $\widetilde{\mathbb{H}}$, and for four of the seven imaginary units of $\widetilde{\mathbb{O}}$). Unlike the positive definite algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, the split forms $\widetilde{\mathbb{C}}, \widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{O}}$ are not division algebras.

In any algebra we denote by L_x and R_x the maps of left and right multiplication by x:

$$L_x(y) = xy, \qquad R_x(y) = yx.$$

Matrix notation: $\mathbf{1}$ denotes the identity matrix (of a size which will be clear from the context), X' denotes the traceless part of the $n \times n$ matrix X:

$$X' = X - \frac{\operatorname{tr} X}{n}\mathbf{1},$$

and $X^\dagger$ denotes the hermitian conjugate of the matrix X with entries in $\mathbb{K}$, defined in analogy to the complex case by

$$(X^\dagger)_{ij} = \overline{x_{ji}}.$$

Our notation for Lie algebras is that of [18]. We use $\mathfrak{su}(s,t)$ for the Lie algebra of the unimodular pseudo-unitary group,

$$\mathfrak{su}(s,t) = \{X \in \mathbb{C}^{n\times n} : X^\dagger G + GX = 0, \operatorname{tr} X = 0\}$$

where $G = \operatorname{diag}(1,\ldots,1,-1,\ldots,-1)$ with s $+$ signs and t $-$ signs; $\mathfrak{sq}(n)$ for the Lie algebra of antihermitian quaternionic matrices X,

$$\mathfrak{sq}(n) = \{X \in \mathbb{H}^{n\times n} : X^\dagger = -X\};$$

and $\mathfrak{sp}(2n,\mathbb{K})$ for the Lie algebra of the symplectic group of $2n \times 2n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\mathfrak{sp}(2n,\mathbb{K}) = \{X \in \mathbb{K}^{2n\times 2n} : X^\dagger J + JX = 0\}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. For a general composition algebra $\mathbb{K}$, however, this set of matrices is not closed under commutation. We will denote it by

$$Q_{2n}(\mathbb{K}) = \{X \in \mathbb{K}^{2n\times 2n} : X^\dagger J + JX = 0\}$$

and its traceless subspace by $Q'_{2n}(\mathbb{K})$. We will see that this can be extended to a Lie algebra $\mathfrak{sp}(2n,\mathbb{K})$. We also have $\mathfrak{so}(s,t)$, the Lie algebra of the pseudo-orthogonal group $\mathrm{SO}(s,t)$, given by

$$\mathfrak{so}(s,t) = \{X \in \mathbb{R}^{n\times n} : X^T G + GX = 0\}$$

where G is defined as before.

We will write $\mathrm{O}(V,q)$ for the group of linear maps of the vector space V preserving the non-degenerate quadratic form q, $\mathrm{SO}(V,q)$ for its unimodular (or special) subgroup, and $\mathfrak{o}(V,q)$ or $\mathfrak{so}(V,q)$ for their common Lie algebra. We omit q if it is understood from the context. Thus for any division algebra we have the group $\mathrm{SO}(\mathbb{K})$ and the Lie algebra $\mathfrak{so}(\mathbb{K})$.

A *Jordan algebra* $\mathbb{J}$ is defined to be a commutative algebra (over a field which in this paper will always be $\mathbb{R}$) in which all products satisfy the Jordan identity

$$(xy)x^2 = x(yx^2). \tag{2.4}$$

Let $M_n(\mathbb{K})$ be the set of all $n \times n$ matrices with entries in $\mathbb{K}$, and let $H_n(\mathbb{K})$ and $A_n(\mathbb{K})$ be the sets of all hermitian and antihermitian matrices with entries in $\mathbb{K}$ respectively. We denote by $H'_n(\mathbb{K})$, $A'_n(\mathbb{K})$

and $M'_n(\mathbb{K})$ the subspaces of traceless matrices of $H_n(\mathbb{K})$, $A_n(\mathbb{K})$ and $M_n(\mathbb{K})$ respectively. We thus have $M_n(\mathbb{K}) = H_n(\mathbb{K}) \dot{+} A_n(\mathbb{K})$ and $M'_n(\mathbb{K}) = H'_n(\mathbb{K}) \dot{+} A'_n(\mathbb{K})$. We will use the fact that $H_n(\mathbb{K})$ is a Jordan algebra for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ for all n and for $\mathbb{K} = \mathbb{O}$ when $n = 2, 3$ [18], with the Jordan product as the anticommutator

$$X \cdot Y = XY + YX.$$

This is a commutative but non-associative product.

We denote the associative algebra of linear endomorphisms of a vector space V by $\operatorname{End} V$. The *derivation* algebra, $\operatorname{Der}\mathcal{A}$, of any algebra $\mathcal{A}$ is the Lie algebra

$$\operatorname{Der}\mathcal{A} = \{D \in \operatorname{End}\mathcal{A} \mid D(xy) = D(x)y + xD(y),\ \forall x, y \in \mathcal{A}\} \quad (2.5)$$

with bracket given by the commutator. The derivation algebras of the four positive definite composition algebras are as follows:

$$\operatorname{Der}\mathbb{R} = \operatorname{Der}\mathbb{C} = 0; \quad (2.6)$$

$$\operatorname{Der}\mathbb{H} = C(\mathbb{H}') = \{C_a \mid a \in \mathbb{H}'\} \text{ where } C_a(q) = aq - qa \quad (2.7)$$

$$\cong \mathfrak{su}(2) \cong \mathfrak{so}(3); \quad (2.8)$$

$$\operatorname{Der}\mathbb{O} \text{ is a compact exceptional Lie algebra of type } G_2. \quad (2.9)$$

In both alternative algebras and Jordan algebras there are constructions of derivations from left and right multiplication maps. In an alternative algebra $\mathbb{K}$

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \quad (2.10)$$

is a derivation for any $x, y \in \mathbb{K}$, also given by

$$D_{x,y}(z) = [[x, y], z] - 3[x, y, z]. \quad (2.11)$$

It satisfies the Jacobi-like identity ([17], p.78)

$$D_{[x,y],z} + D_{[y,z],x} + D_{[z,x],y} = 0. \quad (2.12)$$

In Section 4, following Ramond [14], we will extend the derivation algebra to the *triality* algebra, which consists of triples of linear maps $A, B, C : \mathcal{A} \mapsto \mathcal{A}$ satisfying

$$A(xy) = (Bx)y + x(Cy), \qquad \forall x, y \in \mathcal{A}. \quad (2.13)$$

The *structure algebra* $\operatorname{Str}\mathcal{A}$ of any algebra $\mathcal{A}$ is defined to be the Lie algebra generated by left and right multiplication maps L_a and R_a for $a \in \mathcal{A}$. For a Jordan algebra with identity this can be shown to be [17]

$$\operatorname{Str}\mathbb{J} = \operatorname{Der}\mathbb{J} \dot{+} L(\mathbb{J}) \quad (2.14)$$

where $L(\mathbb{J})$ is the set of all L_x with $x \in \mathbb{J}$. The Jordan axiom (2.4) implies that the commutator $[L_x, L_y]$ is a derivation of $\mathbb{J}$ for all $x, y \in \mathbb{J}$;

thus the Lie algebra structure of $\operatorname{Str}\mathbb{J}$ is defined by the statements that $\operatorname{Der}\mathbb{J}$ is a Lie subalgebra and

$$[D, L_x] = L_{Dx} \qquad (D \in \operatorname{Der}\mathbb{J},\ x \in \mathbb{J}),$$
$$[L_x, L_y] = L_x L_y - L_y L_x \in \operatorname{Der}\mathbb{J} \qquad (x, y \in \mathbb{J}).$$

We denote by $\operatorname{Str}'\mathbb{J}$ the quotient of $\operatorname{Str}\mathbb{J}$ by the subspace of multiples of L_e where e is the identity of $\mathbb{J}$. Both $\operatorname{Str}\mathbb{J}$ and $\operatorname{Str}'\mathbb{J}$ have an involutive automorphism $T \mapsto T^*$ which leaves $\operatorname{Der}\mathbb{J}$ fixed and multiplies each element of $L(\mathbb{J})$ by -1.

We also require another Lie algebra associated with a Jordan algebra with identity, namely the conformal algebra as constructed by Kantor (1973) and Koecher (1967). This is the vector space

$$\operatorname{Con}\mathbb{J} = \operatorname{Str}\mathbb{J} \dotplus 2\mathbb{J} \tag{2.15}$$

with brackets

$$[T,\ (x,y)] = (Tx, T^*y), \tag{2.16}$$
$$[(x,0),\ (y,0)] = 0 = [(0,x),\ (0,y)], \tag{2.17}$$
$$[(x,0),\ (0,y)] = \tfrac{1}{2}L_{xy} + \tfrac{1}{2}[L_x, L_y]. \tag{2.18}$$

where * is the involution described above.

When $\mathbb{J}$ is a Jordan algebra of symmetric or hermitian matrices, the Lie algebras $\operatorname{Der}\mathbb{J}$, $\operatorname{Str}'\mathbb{J}$ and $\operatorname{Con}\mathbb{J}$ can be identified with matrix Lie algebras:

$$\operatorname{Der} H_n(\mathbb{R}) = A_n'(\mathbb{R}) = \mathfrak{so}(n), \tag{2.19}$$
$$\operatorname{Der} H_n(\mathbb{C}) = A_n'(\mathbb{C}) = \mathfrak{su}(n), \tag{2.20}$$
$$\operatorname{Str}' H_n(\mathbb{K}) = M_n'(\mathbb{K}) = \mathfrak{sl}(\mathbb{K}) \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}), \tag{2.21}$$
$$\operatorname{Con} H_n(\mathbb{K}) = Q_{2n}'(\mathbb{K}) = \mathfrak{sp}(2n, \mathbb{K}) \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}). \tag{2.22}$$

We adopt these as definitions of the following series of Lie algebras for any composition algebra:

Definition 1. If $\mathbb{K}$ is a real composition algebra and n is a natural number such that $H_n(\mathbb{K})$ is a Jordan algebra,

$$\mathfrak{sa}(n, \mathbb{K}) = \operatorname{Der} H_n(\mathbb{K}), \tag{2.23}$$
$$\mathfrak{sl}(n, \mathbb{K}) = \operatorname{Str}' H_n(\mathbb{K}), \tag{2.24}$$
$$\mathfrak{sp}(2n, \mathbb{K}) = \operatorname{Con} H_n(\mathbb{K}) \tag{2.25}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Thus $\mathfrak{sa}(n, \mathbb{R}) = \mathfrak{so}(n)$, $\mathfrak{sa}(n, \mathbb{C}) = \mathfrak{su}(n)$, and we will see in Section 5 that $\mathfrak{sa}(n, \mathbb{H}) = \mathfrak{sq}(n)$. We will also find matrix descriptions of the other quaternionic Lie algebras as follows:

$$\mathfrak{sl}(n,\mathbb{H}) = \{X \in \mathbb{H}^{n\times n} : \operatorname{Re}(\operatorname{tr} X) = 0\}, \tag{2.26}$$

$$\mathfrak{sp}(2n,\mathbb{H}) = \{X \in \mathbb{H}^{2n\times 2n} : X^\dagger J + JX = 0,\ \operatorname{Re}(\operatorname{tr} X) = 0\}. \tag{2.27}$$

We note that the standard notations [5, 4] for the quaternionic Lie algebras are

$$\begin{aligned} \mathfrak{sa}(n,\mathbb{H}) &= \mathfrak{usp}(2n) \\ \mathfrak{sl}(n,\mathbb{H}) &= \mathfrak{su}^*(2n) \\ \mathfrak{sp}(2n,\mathbb{H}) &= \mathfrak{so}^*(4n). \end{aligned}$$

3. The Tits Construction

Let $\mathbb{K}$ be a real composition algebra and $(\mathbb{J},\cdot)$ a real Jordan algebra with identity E, and suppose $\mathbb{J}$ has an inner product $\langle,\rangle$ satisfying

$$\langle X,\, Y\cdot Z\rangle = \langle X\cdot Y,\, Z\rangle. \tag{3.1}$$

Let $\mathbb{J}'$ and $\mathbb{K}'$ be the subspaces of $\mathbb{J}$ and $\mathbb{K}$ orthogonal to the identity, and let $*$ denote the product on $\mathbb{J}'$ obtained from the Jordan product by projecting back into $\mathbb{J}'$:

$$A * B = A\cdot B - \frac{4}{n}\langle A, B\rangle \mathbf{1} \qquad \text{where } \frac{n}{4} = \langle E, E\rangle$$

(the notation is chosen to fit the case $\mathbb{J} = H_n(\mathbb{K})$ with $X\cdot Y = XY + YX$ and $\langle X, Y\rangle = \frac{1}{2}\operatorname{tr}(X\cdot Y)$; then $E = \frac{1}{2}\mathbf{1}$ and $\langle E, E\rangle = n/4$). Tits defined a Lie algebra structure on the vector space

$$T(\mathbb{K},\mathbb{J}) = \operatorname{Der}\mathbb{K} \dot{+} \operatorname{Der}\mathbb{J} \dot{+} \mathbb{K}'\otimes\mathbb{J}' \tag{3.2}$$

with the usual brackets in the Lie subalgebra $\operatorname{Der}\mathbb{K}\oplus\operatorname{Der}\mathbb{J}$, brackets between this and $\mathbb{K}'\otimes\mathbb{J}'$ defined by the usual action of $\operatorname{Der}\mathbb{K}\oplus\operatorname{Der}\mathbb{J}$ on $\mathbb{K}'\otimes\mathbb{J}'$, and further brackets

$$[a\otimes A, b\otimes B] = \frac{1}{n}\langle A, B\rangle D_{a,b} - \langle a, b\rangle [L_A, L_B] + \tfrac{1}{2}[a,b]\otimes(A*B) \tag{3.3}$$

where $a, b\in\mathbb{K}'$; $A, B\in\mathbb{J}'$; the square brackets on the right-hand side denote commutators in $\mathbb{K}'$ and $\operatorname{End}\mathbb{J}$; and $D_{a,b}$ is the derivation of $\mathbb{K}'$ defined in (2.10). For future reference we sketch the proof of a slightly generalised version of Tits's theorem ([19]; see also [3], [17]).

Theorem 3.1. (Tits) *The brackets* (3.3) *define a Lie algebra structure on* $T(\mathbb{K},\mathbb{J})$ *if either* $\mathbb{K}$ *is associative or in* $\mathbb{J}$ *there is a cubic identity*

$$\frac{n}{6}X*(X\cdot X) = \langle X, X\rangle X, \qquad \text{all } X\in\mathbb{J}'. \tag{3.4}$$

Proof. The identity (3.1) guarantees that derivations of $\mathbb{J}$ are antisymmetric with respect to the inner product $\langle,\rangle$; since the inner product in $\mathbb{K}$ is constructed from the multiplication, the same applies in $\mathbb{K}$. It follows that the brackets (3.3) are equivariant under the action of

$\mathrm{Der}\,\mathbb{K} \oplus \mathrm{Der}\,\mathbb{J}$, so that all Jacobi identities involving these derivations are satisfied. Thus we need only consider the Jacobi identity between three elements of $\mathbb{K}' \otimes \mathbb{J}'$, namely the vanishing of

$$[[a \otimes A, b \otimes B], c \otimes C] + [[b \otimes B, c \otimes C], a \otimes A] + [[c \otimes C, a \otimes A], b \otimes B].$$

The component of this in $\mathrm{Der}\,\mathbb{K}$ is

$$\frac{1}{8n}\langle A * B, C\rangle \left(D_{[a,b],c} + D_{[b,c],a} + D_{[c,a],b}\right)$$

which vanishes by (2.12). The component in $\mathrm{Der}\,\mathbb{J}$ is

$$-\tfrac{1}{2}\langle [a,b], c\rangle \big([L_{A\cdot B}, L_C] + [L_{B\cdot C}, L_A] + [L_{C\cdot A}, L_B]\big)$$

which vanishes by the polarisation of the Jordan axiom $[L_{X\cdot X}, L_X] = 0$ (obtained by putting $X = \lambda A + \mu B + \nu C$ and equating coefficients of $\lambda\mu\nu$).

Finally, the component in $\mathbb{K}' \otimes \mathbb{J}'$ is

$$\begin{aligned} Q = -\frac{3}{n}&[a,b,c] \otimes \big(\langle A,B\rangle C + \langle B,C\rangle A + \langle C,A\rangle B\big) \\ &+ \big(\langle a,c\rangle b - \langle a,b\rangle c + \tfrac{1}{4}[[b,c],a]\big) \otimes A * (B \cdot C) \\ &+ \big(\langle b,a\rangle c - \langle b,c\rangle a + \tfrac{1}{4}[[c,a],b]\big) \otimes B * (C \cdot A) \\ &+ \big(\langle c,b\rangle a - \langle c,a\rangle b + \tfrac{1}{4}[[a,b],c]\big) \otimes C * (A \cdot B). \end{aligned}$$

Now $\langle a,b\rangle = \mathrm{Re}(a\overline{b}) = -\frac{1}{2}(ab + ba)$ since $b \in \mathbb{K}'$; hence

$$\begin{aligned} 4\big(\langle a,c\rangle b - \langle a,b\rangle c\big) &= -(ac+ca)b - b(ac+ca) + (ab+ba)c + c(ab+ba) \\ &= -[[b,c],a] + 2[a,b,c] \end{aligned}$$

and so

$$\begin{aligned} Q = [a,b,c] \otimes \Big\{ &-\frac{3}{n}\big(\langle A,B\rangle C + \langle B,C\rangle A + \langle C,A\rangle B\big) \\ &+ \frac{1}{2}\big(A * (B \cdot C) + B * (C \cdot A) + C * (A \cdot B)\big)\Big\}. \end{aligned}$$

If $\mathbb{K}$ is associative, the first factor vanishes; if the identity (3.4) holds in $\mathbb{J}$, then polarising it shows that the second factor vanishes. □

Taking $\mathbb{K}$ to be $\mathbb{R}$ or one of the split composition algebras $\widetilde{\mathbb{C}}$ or $\widetilde{\mathbb{H}}$ gives three of the Lie algebras associated with $\mathbb{J}$ defined in Section 2:

Theorem 3.2. *For any Jordan algebra* $\mathbb{J}$,

$$T(\mathbb{R}, \mathbb{J}) \cong \mathrm{Der}\,\mathbb{J}, \tag{3.5}$$

$$T(\widetilde{\mathbb{C}}, \mathbb{J}) \cong \mathrm{Str}'\,\mathbb{J}, \tag{3.6}$$

$$T(\widetilde{\mathbb{H}}, \mathbb{J}) \cong \mathrm{Con}\,\mathbb{J}. \tag{3.7}$$

Proof. Since $\operatorname{Der}\mathbb{R} = \mathbb{R}' = 0$, the first statement is true by definition.

For the second statement, we define $\theta : T(\widetilde{\mathbb{C}}, \mathbb{J}) \to \operatorname{Str}'\mathbb{J}$ to be the identity on $\operatorname{Der}\mathbb{J}$, and on $\widetilde{\mathbb{C}}' \otimes \mathbb{J}'$

$$\theta(\tilde{i} \otimes A) = L_A.$$

Since $\operatorname{Der}\widetilde{\mathbb{C}} = 0$ and $\widetilde{\mathbb{C}}'$ is spanned by $\tilde{i}$ which satisfies $\langle \tilde{i}, \tilde{i} \rangle = -1$, this is an isomorphism between $T(\widetilde{\mathbb{C}}, \mathbb{J})$ and the subspace of $\operatorname{Str}\mathbb{J}$ spanned by $\operatorname{Der}\mathbb{J}$ and the multiplication maps L_A with $A \in \mathbb{J}$, i.e. the subspace $\operatorname{Der}\mathbb{J} \dotplus L(\mathbb{J}') \cong \operatorname{Str}'\mathbb{J}$ by (2.14).

In the third statement we have

$$\begin{aligned} \operatorname{Con}\mathbb{J} &\cong \operatorname{Str}\mathbb{J} \dotplus 2\mathbb{J} \\ &\cong \operatorname{Der}\mathbb{J} \dotplus 3\mathbb{J} \end{aligned}$$

which is isomorphic to $T(\widetilde{\mathbb{H}}, \mathbb{J})$ as a vector space since

$$\begin{aligned} T(\widetilde{\mathbb{H}}, \mathbb{J}) &= \operatorname{Der}\mathbb{J} \dotplus \widetilde{\mathbb{H}}' \otimes \mathbb{J}' \dotplus \operatorname{Der}\widetilde{\mathbb{H}} \\ &\cong \operatorname{Der}\mathbb{J} \dotplus 3\mathbb{J}' \dotplus C(\widetilde{\mathbb{H}}') \\ &\cong \operatorname{Der}\mathbb{J} \dotplus 3\mathbb{J} \end{aligned}$$

since $C(\widetilde{\mathbb{H}}')$ is 3-dimensional. Taking the multiplication in $\widetilde{\mathbb{H}}$ to be given by

$$\tilde{i}^2 = \tilde{j}^2 = 1, \qquad \tilde{k}^2 = -1;$$
$$\tilde{i}\tilde{j} = -\tilde{j}\tilde{i} = -\tilde{k}, \quad \tilde{j}\tilde{k} = -\tilde{k}\tilde{j} = \tilde{i}, \quad \tilde{k}\tilde{i} = -\tilde{i}\tilde{k} = \tilde{j},$$

we define $\phi : T(\widetilde{\mathbb{H}}, \mathbb{J}) \to \operatorname{Con}\mathbb{J}$ by

$$\begin{aligned} \phi(D) &= D \qquad (D \in \operatorname{Der}\mathbb{J}), \\ \phi(C_{\tilde{i}}) &= 2L_1 \in \operatorname{Str}\mathbb{J}, \\ \phi(C_{\tilde{j}}) &= 2(1,1) \in 2\mathbb{J}, \\ \phi(C_{\tilde{k}}) &= -2(1,-1) \in 2\mathbb{J}, \\ \phi(\tilde{i} \otimes A) &= L_A \in \operatorname{Str}\mathbb{J}, \\ \phi(\tilde{j} \otimes A) &= (A, A) \in 2\mathbb{J}, \\ \phi(\tilde{k} \otimes A) &= (-A, A) \in 2\mathbb{J} \quad (A \in \mathbb{J}'). \end{aligned}$$

It is straightforward to check that this is a Lie algebra isomorphism. □

Tits obtained the magic square of Lie algebras by taking the Jordan algebra $\mathbb{J}$ to be the algebra of hermitian 3×3 matrices over a second composition algebra, defining

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = T(\mathbb{K}_1, H_3(\mathbb{K}_2)).$$

The inner product in $H_3(\mathbb{K}_2)$ is given by $\langle X, Y \rangle = \frac{1}{2}\operatorname{tr}(X \cdot Y)$. This yields the Lie algebras whose complexifications are

$\mathbb{K}_2$ / $\mathbb{K}_1$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

i.e. the Lie algebras with compact real forms

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	F_4
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	E_6
$\mathbb{H}$	$\mathfrak{sq}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

.

The striking properties of this square are (a) its symmetry and (b) the fact that four of the five exceptional Lie algebras occur in its last row. The fifth exceptional Lie algebra, G_2, can be included by adding an extra row corresponding to the Jordan algebra $\mathbb{R}$. The explanation of the symmetry property is the subject of the following section.

If one of the composition algebras is split, the magic square $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$, according to Theorem 3.2, contains matrix Lie algebras as follows:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \mathbb{R}) \cong \mathfrak{su}(3, \mathbb{K})$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	$F_4(52)$
$\mathrm{Str}'\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \cong \mathfrak{sl}(3, \mathbb{K})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$E_6(26)$
$\mathrm{Con}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \mathfrak{sp}(6, \mathbb{K})$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{su}(3, 3)$	$\mathfrak{sp}(6, \mathbb{H})$	$E_7(25)$
$L_3(\mathbb{K}, \widetilde{\mathbb{O}})$	$F_4(-4)$	$E_6(-2)$	$E_7(5)$	$E_8(24)$

where the real forms of the exceptional Lie algebras in the last row and column are labelled by the signatures of their Killing forms. These can be also be identified by their maximal compact subalgebras as follows:

Exceptional Lie Algebra	Maximal Compact Subalgebra
$F_4(52)$	F_4
$E_6(26)$	F_4
$E_7(25)$	$E_6 \oplus \mathfrak{so}(2)$
$E_8(24)$	$E_7 \oplus \mathfrak{so}(3)$
$F_4(-4)$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$E_6(-2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(3)$
$E_7(5)$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$
$E_8(24)$	$E_7 \oplus \mathfrak{so}(3)$

.

In section 5 we will give a general explanation of the close relationship between the maximal compact subalgebras of the algebras in one line of the split magic square $L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2)$ and the algebras in the preceding line of the compact magic square $L_3(\mathbb{K}_1, \mathbb{K}_2)$. We will use the same method to identify the maximal compact subalgebras of the doubly split magic square $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$.

4. Symmetrical Constructions of the $n = 3$ Magic Square

In this section we present two alternative constructions of Tits's magic square which are manifestly symmetric between the two composition algebras $\mathbb{K}_1, \mathbb{K}_2$. These are the constructions of Vinberg [12] and a construction using the triality algebra based on a suggestion of Ramond [14]. We begin by exploring the structure of the Lie algebras associated with the Jordan algebra $H_3(\mathbb{K})$.

4.1. The triality algebra $\operatorname{Tri}\mathbb{K}$ and $\operatorname{Der} H_3(\mathbb{K})$.

Definition 2. Let $\mathbb{K}$ be a composition algebra over $\mathbb{R}$. The *triality algebra* of $\mathbb{K}$ is defined to be

$$\operatorname{Tri}\mathbb{K} = \{(A, B, C) \in 3\mathfrak{so}(\mathbb{K}) : A(xy) = x(By) + (Cx)y,\ \forall x, y \in \mathbb{K}\}. \tag{4.1}$$

The structure of the triality algebras $\operatorname{Tri}\mathbb{K}$ can be analysed in a unified way as follows:

Lemma 4.1. *For any composition algebra* $\mathbb{K}$*,*

$$\operatorname{Tri}\mathbb{K} = \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}'$$

in which $\operatorname{Der}\mathbb{K}$ *is a Lie subalgebra and the other brackets are*

$$\begin{aligned}
[D, (a, b)] &= (Da, Db) \in 2\mathbb{K}' \\
[(a, 0), (b, 0)] &= \tfrac{2}{3}D_{a,b} + \left(\tfrac{1}{3}[a, b], -\tfrac{2}{3}[a, b]\right), \\
[(a, 0), (0, b)] &= \tfrac{1}{3}D_{a,b} - \left(\tfrac{1}{3}[a, b], \tfrac{1}{3}[a, b]\right), \\
[(0, a), (0, b)] &= \tfrac{2}{3}D_{a,b} + \left(-\tfrac{2}{3}[a, b], \tfrac{1}{3}[a, b]\right).
\end{aligned}$$

Proof. Define $T : \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K} \to \operatorname{Tri}\mathbb{K}$ by

$$T(D,a,b) = (D + L_a - R_b,\ D - L_{a+b} - R_b,\ D + L_a + R_{a+b}). \quad (4.2)$$

The alternative law guarantees that the right-hand side belongs to $\operatorname{Tri}\mathbb{K}$; the Lie algebra isomorphism property follows from the brackets

$$\begin{aligned}[L_x, L_y] &= \tfrac{2}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} + \tfrac{2}{3}R_{[x,y]}\\ [L_x, R_y] &= -\tfrac{1}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}\\ [R_x, R_y] &= \tfrac{2}{3}D_{x,y} - \tfrac{2}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}\end{aligned}$$

and the expression of the inverse map $T^{-1} : \operatorname{Tri}\mathbb{K} \to \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}$ as

$$T^{-1}(A,B,C) = (A - L_a + R_b, a, b)$$

where $a = \frac{1}{3}B(1) + \frac{2}{3}C(1)$ and $b = -\frac{2}{3}B(1) - \frac{1}{3}C(1)$. □

Note that $\operatorname{Tri}\mathbb{K}$ has a subalgebra $\operatorname{Der}\mathbb{K} \dot{+} \mathbb{K}'$ in which the second summand is the diagonal subspace of $2\mathbb{K}'$, containing the elements (a,a). Identifying $(a,a) \in 2\mathbb{K}'$ with $a \in \mathbb{K}'$, the brackets between elements of $\mathbb{K}'$ in this subalgebra are given by

$$[a,b] = 2D_{a,b} - [a,b].$$

The derivation and triality algebras of the four real division algebras, together with this intermediate algebra, are tabulated below.

$\mathbb{K}$	$\operatorname{Der}\mathbb{K}$		$\operatorname{Tri}\mathbb{K}$
$\mathbb{R}$	0	0	0
$\mathbb{C}$	0	$\mathfrak{so}(2)$	$\mathfrak{so}(2) \oplus \mathfrak{so}(2)$
$\mathbb{H}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$
$\mathbb{O}$	G_2	$\mathfrak{so}(7)$	$\mathfrak{so}(8)$

The identification $\operatorname{Tri}\mathbb{O} = \mathfrak{so}(8)$ is a form of the principle of triality [13].

We will now show that any two elements $x, y \in \mathbb{K}$ have an element of $\operatorname{Tri}\mathbb{K}$ assicated with them, of which the first component is the generator of rotations in the plane of x and y, defined as

$$S_{x,y}(z) = \langle x, z\rangle y - \langle y, z\rangle x. \quad (4.3)$$

Lemma 4.2. *For any $x, y \in \mathbb{K}$, let*

$$T_{x,y} = (4S_{x,y},\ R_y R_{\bar{x}} - R_x R_{\bar{y}},\ L_y L_{\bar{x}} - L_x L_{\bar{y}}).$$

Then $T_{x,y} \in \operatorname{Tri}\mathbb{K}$.

Proof. Write the action of $S_{x,y}$ as

$$\begin{aligned}2S_{x,y}(z) &= (x\bar{z} + z\bar{x})y - x(\bar{z}y + \bar{y}z) & (4.4)\\ &= -[x,y,z] + z(\bar{x}y) - (x\bar{y})z & (4.5)\end{aligned}$$

using the alternative law and the relation $[x, y, \bar{z}] = -[x, y, z]$. Since $\mathrm{Re}(\bar{x}y) = \mathrm{Re}(x\bar{y})$, we can write the last two terms as

$$z(\bar{x}y) - (x\bar{y})z = z\,\mathrm{Im}(\bar{x}y) - \mathrm{Im}(x\bar{y})z \tag{4.6}$$

$$= \tfrac{1}{2}z(\bar{x}y - \bar{y}x) - \tfrac{1}{2}(x\bar{y} - y\bar{x})z. \tag{4.7}$$

Now, by equation (2.10), we have

$$S_{x,y} = \tfrac{1}{6}D_{x,y} + L_a - R_b$$

with

$$a = -\tfrac{1}{6}[x, y] - \tfrac{1}{4}(x\bar{y} - y\bar{x}) \in \mathbb{K}'$$
$$b = -\tfrac{1}{6}[x, y] - \tfrac{1}{4}(\bar{x}y - \bar{y}x) \in \mathbb{K}'.$$

Hence by Lemma4.1 there is an element $(A, B, C) \in \mathrm{Tri}\,\mathbb{K}$ with $A = S_{x,y}$ and

$$B = \tfrac{1}{6}D_{x,y} - L_{a+b} - R_b = S_{x,y} - L_{2a+b},$$
$$C = \tfrac{1}{6}D_{x,y} + L_a + R_{a+b} = S_{x,y} + R_{a+2b}.$$

Writing $[x, y] = -\frac{1}{2}([\bar{x}, y] + [x, \bar{y}])$ gives

$$a + 2b = \tfrac{1}{4}(\bar{y}x - \bar{x}y),$$
$$2a + b = \tfrac{1}{4}(y\bar{x} - x\bar{y});$$

thus equations (4.4) and (4.6) imply that

$$S_{x,y} = -\tfrac{1}{2}A_{x,y} - R_{a+2b} + L_{2a+b}$$

where $A_{x,y}(z) = [x, y, z]$. Hence

$$Cz = -\tfrac{1}{2}[x, y, z] + \tfrac{1}{4}(y\bar{x} - x\bar{y})z$$
$$= -\tfrac{1}{4}[y, \overline{x}, z] + \tfrac{1}{4}[x, \overline{y}, z] + \tfrac{1}{4}(y\overline{x} - x\overline{y})z$$
$$= \tfrac{1}{4}y(\bar{x}z) - \tfrac{1}{2}x(\bar{y}z),$$

i.e.

$$C = \tfrac{1}{4}(L_y L_{\bar{x}} - L_x L_{\bar{y}}).$$

Similarly,

$$B = \tfrac{1}{4}(R_y R_{\bar{x}} - R_x R_{\bar{y}}).$$

Thus $(4S_{x,y}, 4C, 4B) = T_{x,y}$, which is therefore an element of $\mathrm{Tri}\,\mathbb{K}$. □

Define an automorphism of $\mathrm{Tri}\,\mathbb{K}$ as follows. For any linear map $A : \mathbb{K} \to \mathbb{K}$ let $\overline{A} = KAK$ where $K : \mathbb{K} \to \mathbb{K}$ is the conjugation $x \mapsto \bar{x}$, i.e.

$$\overline{A}(x) = \overline{A(\bar{x})}.$$

Lemma 4.3. *Given $T = (A, B, C) \in \mathrm{Tri}\,\mathbb{K}$, let*

$$\theta(T) = (\overline{B}, C, \overline{A}).$$

Then $\theta(T) \in \mathrm{Tri}\,\mathbb{K}$ and θ is a Lie algebra automorphism.

Proof. By Lemma 4.1, $T = T(D, a, b)$ for some $D \in \operatorname{Der}\mathbb{K}$ and $a, b \in \mathbb{K}'$. Then

$$\begin{aligned} A &= D + L_a - R_b \\ B &= D - L_{a+b} - R_b \\ C &= D + L_a + R_{a+b}. \end{aligned}$$

It follows that

$$\overline{B} = D + R_{a+b} + L_b = D + L_{a'} - R_{b'}$$

with $a' = b$, $b' = -a - b$. This is the first component of the triality $T' = (A', B', C')$, where

$$\begin{aligned} B' &= D - L_{a'+b'} - R_{b'} = C, \\ C' &= D + L_{a'} + R_{a'+b'} = \overline{A} \end{aligned}$$

i.e. $T' = (\overline{B}, C, \overline{A}) = \theta(T)$. It is clear that θ is a Lie algebra automorphism. □

Theorem 4.1. *For any composition algebra* $\mathbb{K}$*,*

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Tri}\mathbb{K} \dot{+} 3\mathbb{K}$$

in which $\operatorname{Tri}\mathbb{K}$ *is a Lie subalgebra; the brackets in* $[\operatorname{Tri}\mathbb{K}, 3\mathbb{K}]$ *are*

$$[T, F_i(x)] = F_i(T_i x) \in 3\mathbb{K}, \tag{4.8}$$

if $T = (T_1, \overline{T}_2, \overline{T}_3) \in \operatorname{Tri}\mathbb{K}$ *and* $F_1(x) + F_2(y) + F_3(z) = (x, y, z) \in 3\mathbb{K}$*; and the brackets in* $[\operatorname{Tri}\mathbb{K}, \operatorname{Tri}\mathbb{K}]$ *are given by*

$$[F_i(x), F_i(y)] = \theta^{1-i}(T_{x,y}) \in \operatorname{Tri}\mathbb{K}, \tag{4.9}$$

$$[F_i(x), F_j(y)] = F_k(\bar{y}\bar{x}) \in 3\mathbb{K}, \tag{4.10}$$

if $x, y \in \mathbb{K}$ *and* (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$*.*

Proof. Define elements $e_i, P_i(x)$ of $H_3(\mathbb{K})$ (where $i = 1, 2, 3$; $x \in \mathbb{K}$) by

$$\begin{pmatrix} \alpha & z & \bar{y} \\ \bar{z} & \beta & x \\ y & \bar{x} & \gamma \end{pmatrix} = \alpha e_1 + \beta e_2 + \gamma e_3 + P_1(x) + P_2(y) + P_3(z) \tag{4.11}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$ and $x, y, z \in \mathbb{K}$. The Jordan product in $H_3(\mathbb{K})$ is given by

$$e_i \cdot e_j = 2\delta_{ij} e_i \tag{4.12a}$$

$$e_i \cdot P_j(x) = (1 - \delta_{ij}) P_j(x) \tag{4.12b}$$

$$P_i(x) \cdot P_i(y) = 2\langle x, y\rangle (e_j + e_k) \tag{4.12c}$$

$$P_i(x) \cdot P_j(y) = P_k(\bar{y}\,\bar{x}) \tag{4.12d}$$

where in each of the last two equations (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Now let $D : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ be a derivation of this algebra. First suppose that

$$De_i = 0, \quad i = 1, 2, 3.$$

Then

$$\begin{aligned} e_i \cdot DP_i(x) &= 0, \\ e_i \cdot DP_j(x) &= DP_j(x) \quad \text{if } i \neq j. \end{aligned}$$

Thus $DP_j(x)$ is an eigenvector of each of the multiplication operators L_{e_i}, with eigenvalue 0 if $i = j$ and 1 if $i \neq j$. It follows that

$$DP_j(x) = P_j(T_j x) \tag{4.13}$$

for some $T_j : \mathbb{K} \to \mathbb{K}$. Now

$$DP_j(x) \cdot P_j(y) + P_j(x) \cdot DP_j(y) = 0$$

gives $T_j \in \mathfrak{so}(\mathbb{K})$; and the derivation property of D applied to (4.12d) gives

$$T_k(\bar{y}\bar{x}) = \bar{y}(\overline{T_i x}) + (\overline{T_j y})\bar{x}$$

i.e. $(T_k, \overline{T_i}, \overline{T_j}) \in \operatorname{Tri} \mathbb{K}$ and therefore $(T_1, \overline{T_2}, \overline{T_3}) \in \operatorname{Tri} \mathbb{K}$.

If $De_i \neq 0$, then from equation (4.12a) with $i = j$,

$$2e_i \cdot De_i = 2De_i$$

so De_i is an eigenvector of the multiplication L_{e_i} with eigenvalue 1, i.e. $De_i \in P_j(\mathbb{K}) + P_k(\mathbb{K})$ where (i, j, k) are distinct. Write

$$De_i = P_j(x_{ij}) + P_k(x_{ik});$$

then equation (4.12a) with $i \neq j$ gives

$$e_i \cdot P_k(x_{jk}) + e_i \cdot P_i(x_{ji}) + P_j(x_{ij}) \cdot e_j + P_k(x_{ik}) \cdot e_j = 0.$$

Hence

$$P_k(x_{jk} + x_{ik}) = 0.$$

It follows that the action of any derivation on the e_i must be of the form $De_i = F_1(x) + F_2(y) + F_3(z)$ where

$$\begin{aligned} F_i(x)e_i &= 0 \\ F_i(x)e_j = -F_i(x)e_k &= P_i(x), \end{aligned} \tag{4.14}$$

(i, j, k) being a cyclic permutation of $(1, 2, 3)$. Hence

$$\operatorname{Der} H_3(\mathbb{K}) \subseteq \operatorname{Tri} \mathbb{K} \oplus 3\mathbb{K}.$$

To show that such derivations $F_i(x)$ exist and therefore that the inclusion just obtained is an equality, consider the operation of commutation with the matrix

$$X = \begin{pmatrix} 0 & -z & \bar{y} \\ \bar{z} & 0 & -x \\ -y & \bar{x} & 0 \end{pmatrix}$$

$$= X_1(x) + X_2(y) + X_3(z),$$

i.e. define $F_i(x) = C_{X_i(x)}$ where $C_X : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ is the commutator map

$$C_X(H) = XH - HX. \tag{4.15}$$

This satisfies equation (4.14) and also

$$\begin{aligned} F_i(x)P_i(y) &= -2\langle x, y\rangle(e_j - e_k) \\ F_i(x)P_j(y) &= -P_k(\bar{y}\,\bar{x}) \\ F_i(x)P_k(y) &= P_j(\bar{x}\,\bar{y}). \end{aligned} \tag{4.16}$$

It is a derivation of $H_3(\mathbb{K})$ by virtue of the matrix identity (A.5).

The Lie brackets of these derivations follow from another matrix identity

$$[A, [B, H]] - [B, [A, H]] = [[A, B], H] + E(X, Y)H$$

(see (A.6)). If $A = X_i(x)$ and $B = X_j(y)$ we have $E(A, B) = 0$ and

$$[X_i(x), X_j(y)] = X_k(\bar{y}\bar{x})$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. This yields the Lie bracket (4.10). If $X = X_i(x)$ and $Y = X_i(y)$, the matrix commutator $Z = [X, Y]$ is diagonal with $z_{ii} = 0$, $z_{jj} = y\bar{x} - x\bar{y}$ and $z_{kk} = \bar{y}x - \bar{x}y$ $(i, j, k$ cyclic$)$. Hence the action of the commutator $[F_i(x), F_i(y)] = C_Z + E(X, Y)$ on $H_3(\mathbb{K})$ is

$$\begin{aligned} [F_i(x), F_i(y)]e_m &= 0 \qquad\qquad (m = i, j, k) \\ [F_i(x), F_i(y)]P_i(w) &= P_i\big(z_{jj}w - wz_{kk} - 2[x, y, w]\big) \\ &= 4P_i(S_{xy}w) \quad \text{by eq. (4.3).} \\ [F_i(x), F_i(y)]P_j(w) &= P_j\big(z_{kk}w - 2[x, y, w]\big) \\ &= P_j(\bar{y}(xw) - \bar{x}(yw)) \\ [F_i(x), F_i(y)]P_k(w) &= P_k\big(-wz_{jj} - 2[x, y, w]\big) \\ &= P_k\big((wx)\bar{y} - (wy)\bar{x}\big). \end{aligned}$$

Thus

$$\begin{aligned} [F_i(x), F_i(y)]P_i(w) &= P_i(Aw) = P_i(T_iw) \\ [F_i(x), F_i(y)]P_j(w) &= P_j(\overline{B}w) = P_j(T_jw) \\ [F_i(x), F_i(y)]P_k(w) &= P_k(\overline{C}w) = P_k(T_kw) \end{aligned}$$

where $T_{x,y} = (A, B, C) = (T_i, \overline{T_j}, \overline{T_k})$, so that $(T_1, \overline{T_2}, \overline{T_3}) = \theta^{1-i}(T_{x,y})$. This establishes the Lie bracket (4.9). □

Theorem 4.2. *For any composition algebra* $\mathbb{K}$*,*

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der}\mathbb{K} \dotplus A_3'(\mathbb{K}) \tag{4.17}$$

in which $\operatorname{Der}\mathbb{K}$ *is a Lie subalgebra, the Lie brackets between* $\operatorname{Der}\mathbb{K}$ *and* $A_3'(\mathbb{K})$ *are given by the elementwise action of* $\operatorname{Der}\mathbb{K}$ *on* 3×3 *matrices over* $\mathbb{K}$*, and for* $A, B \in A_3'(\mathbb{K})$

$$[A, B] = (AB - BA)' + \tfrac{1}{3}D(A, B)$$

where

$$D(A, B) = \sum_{ij} D_{a_{ij}, b_{ji}} \quad \in \operatorname{Der}\mathbb{K},$$

a_{ij} *and* b_{ij} *being the matrix elements of* A *and* B*.*

Proof. By Lemma 4.1 and Theorem 4.1

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der}\mathbb{K} \dotplus 2\mathbb{K}' \dotplus 3\mathbb{K}. \tag{4.18}$$

Identify $(a, b) + (x, y, z) \in 2\mathbb{K}' \dotplus 3\mathbb{K}$ with the traceless antihermitian matrix

$$A = \begin{pmatrix} -a-b & -z & \bar{y} \\ \bar{z} & a & -x \\ -y & \bar{x} & b \end{pmatrix} \quad \in A_3'(\mathbb{K});$$

then the actions of $2\mathbb{K}'$ and $3\mathbb{K}$ on $H_3(\mathbb{K})$ defined in Theorem 4.1 are together equivalent to the commutator action C_A defined by equation (4.15). By the identity (A.9),

$$[C_A, C_B] = C_{(AB-BA)'} + D(A, B),$$

so the bracket $[A, B]$ is as stated. □

It can be shown that this structure of $\operatorname{Der}\mathbb{H}_n(\mathbb{K})$ persists for all n if $\mathbb{K}$ is associative:

$$\operatorname{Der} H_n(\mathbb{K}) \cong A_n'(\mathbb{K}) \dotplus \operatorname{Der}\mathbb{K}.$$

If $\mathbb{K}$ is not associative, however (i.e. $\mathbb{K} = \mathbb{O}$ or $\widetilde{\mathbb{O}}$), the anticommutator algebra $H_n(\mathbb{K})$ is not a Jordan algebra for $n > 3$ and its derivation algebra collapses:

$$\operatorname{Der} H_n(\mathbb{O}) = A_n'(\mathbb{R}) \oplus \operatorname{Der}\mathbb{O} \cong \mathfrak{so}(n) \oplus \mathfrak{g}_2.$$

4.2. **The Vinberg Construction.** For this construction let $\mathbb{K}_1$ and $\mathbb{K}_2$ be composition algebras, and let $\mathbb{K}_1 \otimes \mathbb{K}_2$ be the tensor product algebra with multiplication

$$(u_1 \otimes v_1)(u_2 \otimes v_2) = u_1 v_1 \otimes u_2 v_2$$

and conjugation

$$\overline{u \otimes v} = \overline{u} \otimes \overline{v}.$$

Then the vector space

$$V_3(\mathbb{K}_1, \mathbb{K}_2) = A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2) \dotplus \operatorname{Der}\mathbb{K}_1 \dotplus \operatorname{Der}\mathbb{K}_2 \tag{4.19}$$

is clearly symmetric between $\mathbb{K}_1$ and $\mathbb{K}_2$. Vinberg showed that this is a Lie algebra when taken with the Lie brackets defined by the statements:

1. $\operatorname{Der}\mathbb{K}_1 \oplus \operatorname{Der}\mathbb{K}_2$ is a Lie subalgebra.
2. For $D \in \operatorname{Der}\mathbb{K}_1 \oplus \operatorname{Der}\mathbb{K}_2$ and $A \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$,
$$[D, A] = D(A) \tag{4.20}$$
where on the right-hand side D acts elementwise on the matrix A.
3. For $A = (a_{ij})$, $B = (b_{ij}) \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$,
$$[A, B] = (AB - BA)' + \tfrac{1}{3}\sum_{ij} D_{a_{ij}, b_{ji}} \tag{4.21}$$

where $D_{x,y}$ for $x, y \in \mathbb{K}_1 \otimes \mathbb{K}_2$ is defined by

$$D_{p\otimes q, u\otimes v} = \langle p, u\rangle D_{q,v} + \langle q, v\rangle D_{p,u}.$$

We will now use the results of Section 4.1 to show that the Lie algebra $V_3(\mathbb{K}_1, \mathbb{K}_2)$ is a Lie algebra isomorphic to the algebra $L_3(\mathbb{K}_1, \mathbb{K}_2)$ defined by the Tits-Freudenthal construction, since no proof of this is readily available.

Theorem 4.3. [12] *The Vinberg algebra defined above is a Lie algebra isomorphic to Tits's magic square algebra* $L_3(\mathbb{K}_1, \mathbb{K}_2)$.

Proof. The matrix part of the Vinberg vector space can be decomposed as

$$A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2) = A_3'(\mathbb{R} \otimes \mathbb{K}_2) \dotplus A_3'(\mathbb{K}_1' \otimes \mathbb{K}_2).$$

But $A_3'(\mathbb{K}_1' \otimes \mathbb{K}_2) \cong \mathbb{K}_1' \otimes H_3'(\mathbb{K}_2)$ (if H is a hermitian matrix over $\mathbb{K}_2$ and $a \in \mathbb{K}_1'$ is pure imaginary, then $a \otimes H$ is antihermitian over $\mathbb{K}_1 \otimes \mathbb{K}_2$), so

$$A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2) \cong A_3'(\mathbb{K}_2) \dotplus \mathbb{K}_1' \otimes H_3'(\mathbb{K}_2).$$

Using Theorem 4.2, we can write the Tits vector space as

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der}\mathbb{K}_1 \dotplus \operatorname{Der}\mathbb{K}_2 \dotplus A_3'(\mathbb{K}_2) \dotplus \mathbb{K}_1' \otimes H_3'(\mathbb{K}_2);$$

it is therefore isomorphic to $V_3(\mathbb{K}_1, \mathbb{K}_2)$. We will now prove that this is a Lie algebra isomorphism by showing that

$$[A_3'(\mathbb{R}\otimes\mathbb{K}_2), A_3'(\mathbb{R}\otimes\mathbb{K}_2)]_{\text{Vin}} = [A_3'(\mathbb{K}_2), A_3'(\mathbb{K}_2)]_{\text{Tits}},$$
$$[A_3'(\mathbb{R}\otimes\mathbb{K}_2), A_3'(\mathbb{K}_1'\otimes H_3'(\mathbb{K}_2)]_{\text{Vin}} = [A_3'(\mathbb{K}_2), \mathbb{K}_1'\otimes H_3'(\mathbb{K}_2)]_{\text{Tits}}$$
$$[A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2), A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2)]_{\text{Vin}} = [\mathbb{K}_1'\otimes H_3'(\mathbb{K}_2), \mathbb{K}_1'\otimes H_3'(\mathbb{K}_2)]_{\text{Tits}},$$

where $[,]_{\text{Vin}}$ denotes the Lie brackets in the Vinberg construction and $[,]_{\text{Tits}}$ denotes the Lie brackets in the Tits construction.

1. $[A_3'(\mathbb{R}\otimes\mathbb{K}_2), A_3'(\mathbb{R}\otimes\mathbb{K}_2)]_{\text{Vin}}$. In $V(\mathbb{K}_1, \mathbb{K}_2)$ the bracket is

$$[A, B]_{\text{Vin}} = (AB - BA)' + \tfrac{1}{3}\sum_{ij} D_{a_{ij}, b_{ji}}$$

where $A, B \in A_3'(\mathbb{K}_1\otimes\mathbb{R})$. In $L_3(\mathbb{K}_1, \mathbb{K}_2)$ the matrices A and B are identified with elements of $A_3'(\mathbb{K}_1) \subset \operatorname{Der} H_3(\mathbb{K}_1)$, where their Lie bracket $[A, B]_{\text{Tits}}$ is the same as the above by Theorem 4.2.

2. $[A_3'(\mathbb{R}\otimes\mathbb{K}_2), A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2)]_{\text{Vin}}$. Let $A \in A_3'(\mathbb{R}\otimes\mathbb{K}_2) = A_3'(\mathbb{K}_2)$ and $B \in A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2)$; we may take $B = b\otimes H$ with $b \in \mathbb{K}_1'$, $H \in H_3'(\mathbb{K}_2)$. Then

$$D_{a_{ij}, b_{ji}} = \langle 1, b\rangle D_{a_{ij}, h_{ji}} + \langle a_{ij}, h_{ji}\rangle D_{1,b} = 0$$

and by Lemma A.5, $\operatorname{tr}(AH - HA) = 0$. Hence the Vinberg bracket is

$$[A, b\otimes H]_{\text{Vin}} = b\otimes(AH - HA) = [A, b\otimes H]_{\text{Tits}}$$

since the action of A as an element of $\operatorname{Der} H_3(\mathbb{K}_2)$ is $H \mapsto AH - HA$.

3. $[A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2), A_3'(\mathbb{K}_1'\otimes\mathbb{K}_2)]_{\text{Vin}}$. Let $A = a\otimes H$, $B = b\otimes H$ with $a, b \in \mathbb{K}_1'$ and $H, K \in H_3'(\mathbb{K}_2)$. Then

$$\begin{aligned}[A, B]_{\text{Vin}} &= ab\otimes HK - ba\otimes KH - \tfrac{1}{3}(ab\otimes HK - ba\otimes KH)\\ &\quad + \tfrac{1}{3}\sum_{ij}\big(\langle a, b\rangle D_{h_{ij}, k_{ji}} + \langle h_{ij}, k_{ji}\rangle D_{a,b}\big)\\ &= -\langle a, b\rangle (HK - KH)'\\ &= \tfrac{1}{2}[a, b]\otimes(H * K) - \langle a, b\rangle D(H, K) + \tfrac{1}{3}\langle H, K\rangle D_{a,b}\end{aligned}$$

since $\operatorname{Re}(ab) = \operatorname{Re}(ba) = -\langle a, b\rangle$ and $\operatorname{Im}(ab) = -\operatorname{Im}(ba) = \tfrac{1}{2}[a, b]$; also

$$\sum D_{h_{ij}, k_{ij}} = -\sum D_{h_{ij}, \overline{k_{ij}}} = -\sum D_{h_{ij}, k_{ji}} = -D(H, K)$$

where $D(H, K)$ is defined in Theorem 4.2.

On the other hand,

$$[a\otimes H, b\otimes K]_{\text{Tits}} = \tfrac{1}{3}\langle H, K\rangle D_{a,b} - \langle a, b\rangle [L_H, L_K] + \tfrac{1}{2}[a, b]\otimes(H * K).$$

But the matrix identity (A.10) gives

$$[L_H, L_K] = (HK - KH)' + \tfrac{1}{3}D(H, K),$$

from which it follows that

$$[a \otimes H, b \otimes K]_{\mathrm{Tits}} = [a \otimes H, b \otimes K]_{\mathrm{Vin}}.$$

□

4.3. **The Triality Construction.** A second clearly symmetric formulation of the magic square can be given in terms of triality algebras.

Theorem 4.4. *For any two composition algebras* $\mathbb{K}_1$ *and* $\mathbb{K}_2$*,*

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Tri}\,\mathbb{K}_1 \oplus \mathrm{Tri}\,\mathbb{K}_2 \dotplus 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{4.22}$$

in which $\mathrm{Tri}\,\mathbb{K}_1 \oplus \mathrm{Tri}\,\mathbb{K}_2$ *is a Lie subalgebra and the other brackets are as follows. Define* $F_i(x \otimes y) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2$ *by*

$$F_1(x_1 \otimes x_2) + F_2(y_1 \otimes y_2) + F_3(z_1 \otimes z_2) = (x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2).$$

Then for $T_\alpha = (T_{\alpha 1}, \overline{T}_{\alpha 2}, \overline{T}_{\alpha 3}) \in \mathrm{Tri}\,\mathbb{K}_\alpha$ *and* $x_\alpha, y_\alpha, z_\alpha \in \mathbb{K}_\alpha (\alpha = 1, 2)$*,*

$$[T_1, F_i(x_1 \otimes x_2)] = F_i(T_{1i}x_1 \otimes x_2) \tag{4.23}$$

$$[T_2, F_i(x_1 \otimes x_2)] = F_i(x_1 \otimes T_{2i}x_2) \tag{4.24}$$

$$[F_i(x_1 \otimes x_2), F_j(y_1 \otimes y_2)] = F_k(\overline{y}_1\overline{x}_1 \otimes \overline{y}_2\overline{x}_2) \tag{4.25}$$

where (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$*; and*

$$[F_i(x_1 \otimes x_2), F_i(y_1 \otimes y_2)] = \langle x_2, y_2\rangle \theta^{1-i} T_{x_1 y_1} + \langle x_1, y_1\rangle \theta^{1-i} T_{x_2 y_2} \in \mathrm{Tri}\,\mathbb{K}_1 \oplus \mathrm{Tri}\,\mathbb{K}_2 \tag{4.26}$$

where θ *is the automorphism of Lemma 4.3.*

Proof. The vector space structure (3.2) of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ can be written, using Theorem 4.1 and Lemma 4.1, as

$$\begin{aligned} L_3(\mathbb{K}_1, \mathbb{K}_2) &= \mathrm{Der}\, H_3(\mathbb{K}_1) \dotplus H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dotplus \mathrm{Der}\,\mathbb{K}_2 \\ &= (\mathrm{Tri}\,\mathbb{K}_1 \dotplus 3\mathbb{K}_1) \dotplus (2\mathbb{K}_2' \dotplus 3\mathbb{K}_1 \otimes \mathbb{K}_2') \dotplus \mathrm{Der}\,\mathbb{K}_2 \\ &= \mathrm{Tri}\,\mathbb{K}_1 \dotplus (\mathrm{Der}\,\mathbb{K}_2 \dotplus 2\mathbb{K}_2') \dotplus (3\mathbb{K}_1 \otimes \mathbb{K}_2' \dotplus 3\mathbb{K}_1) \\ &\cong \mathrm{Tri}\,\mathbb{K}_1 \dotplus \mathrm{Tri}\,\mathbb{K}_2 \dotplus 3\mathbb{K}_1 \otimes \mathbb{K}_2. \end{aligned}$$

We need to consider the following five subspaces of $L_3(\mathbb{K}_1, \mathbb{K}_2)$:

1. $\mathrm{Tri}\,\mathbb{K} \subset \mathrm{Der}\, H_3(\mathbb{K}_1)$ contains elements $T = (T_1, \overline{T}_2, \overline{T}_3)$ acting on $H_3'(\mathbb{K}_1)$ as in Theorem 4.1:

   $$Te_i = 0, \quad TP_i(x) = P_i(T_i x) \quad (x \in \mathbb{K}; i = 1, 2, 3).$$

2. $3\mathbb{K}_1$ is the subspace of $\mathrm{Der}\, H_3(\mathbb{K}_1)$ containing the elements $F_i(x)$ defined in Theorem 4.1; these will be identified with the elements $F_i(x \otimes 1) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2$.

3. $2\mathbb{K}_2'$ is the subspace $\Delta\otimes\mathbb{K}_2'$ of $H_3(\mathbb{K}_1)\otimes\mathbb{K}_2'$, where $\Delta\subset H_3'(\mathbb{K}_1)$ is the subspace of real, diagonal, traceless matrices and is identified with a subspace of $\operatorname{Tri}\mathbb{K}_2$ as described in Lemma 4.1. We will regard $2\mathbb{K}_2'$ as a subspace of $3\mathbb{K}_2'$, namely
$$2\mathbb{K}_2' = \{(a_1,a_2,a_3)\in 3\mathbb{K}_2' : a_1+a_2+a_3=0\}$$
and identify $\mathbf{a}=(a_1,a_2,a_3)$ with the 3×3 matrix
$$\Delta(\mathbf{a}) = \begin{pmatrix} a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix} \in H_3'(\mathbb{K}_1)\otimes\mathbb{K}_2'$$
in the Tits description, and on the other hand with the triality $T(\mathbf{a}) = (T_1,\overline{T}_2,\overline{T}_3)$ where $T_i = L_{a_j} - R_{a_k}$.

4. $3\mathbb{K}_1\otimes\mathbb{K}_2'$ is the subspace of $H_3(\mathbb{K}_1\otimes\mathbb{K}_2')$ spanned by elements $P_i(x)\otimes a$ $(i=1,2,3 : x\in\mathbb{K}_1, a\in\mathbb{K}_2')$; in the triality description it is a subspace of $3\mathbb{K}_1\otimes\mathbb{K}_2$ in the obvious way.

5. $\operatorname{Der}\mathbb{K}_2$ is a subspace of $\operatorname{Tri}\mathbb{K}_2$, a derivation D being identified with $(D,D,D)\in\operatorname{Tri}\mathbb{K}_2$.

The proof is completed by verifying that the Lie brackets defined by Tits (eq. 3.3) coincide with those in the statement of the theorem. The above decomposition of $L_3(\mathbb{K}_1,\mathbb{K}_2)$ gives fifteen types of bracket to examine; for each of them the verification is straightforward [1]. □

The isomorphism between the Vinberg construction and the triality construction is easy to see directly at the vector space level: using Lemma 4.1,
$$\begin{aligned} L_3(\mathbb{K}_1,\mathbb{K}_2) &= \operatorname{Tri}\mathbb{K}_1 \dotplus \operatorname{Tri}\mathbb{K}_2 \dotplus 3\mathbb{K}_1\otimes\mathbb{K}_2\\ &= \operatorname{Der}\mathbb{K}_1 \dotplus 2\mathbb{K}_2' \dotplus \operatorname{Der}\mathbb{K}_2 \dotplus 2\mathbb{K}_2' \dotplus 3\mathbb{K}_1\otimes\mathbb{K}_2\\ &= \operatorname{Der}\mathbb{K}_1 \dotplus 2\mathbb{K}_1'\otimes\mathbb{R} \dotplus \operatorname{Der}\mathbb{K}_2 \dotplus 2\mathbb{R}\otimes\mathbb{K}_2' \dotplus 3\mathbb{K}_1\otimes\mathbb{K}_2\\ &= A_3(\mathbb{K}_1\otimes\mathbb{K}_2) \dotplus \operatorname{Der}\mathbb{K}_1 \dotplus \operatorname{Der}\mathbb{K}_2. \end{aligned}$$

Thus both ways of understanding the symmetry of the 3×3 magic square reduce to being different ways of looking at the same underlying vector space, which is an extension of the vector space of antisymmetric 3×3 matrices over $\mathbb{K}_1\otimes\mathbb{K}_2$. The Lie algebras of the square can therefore be understood as analogues of $\mathfrak{su}(3)$ with the complex numbers replaced by $\mathbb{K}_1\otimes\mathbb{K}_2$.

5. The rows of the magic square

In this section we will examine the non-symmetric magic square obtained by taking $\mathbb{K}_1$ to range over the split composition algebras $\mathbb{R}$, $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{O}}$. According to Theorem 3.2, the first three rows contain the derivation, structure and conformal algebras of the Jordan algebras $H_3(\mathbb{K})$, which we have defined to be the generalisations of the Lie algebras of antihermitian traceless 3×3 matrices, all traceless 3×3 matrices, and symplectic 6×6 matrices:

$$L_3(\mathbb{R}, \mathbb{K}) \cong \operatorname{Der} H_3(\mathbb{K}) = \mathfrak{sa}(3, \mathbb{K}),$$
$$L_3(\widetilde{\mathbb{C}}, \mathbb{K}) \cong \operatorname{Str}' H_3(\mathbb{K}) = \mathfrak{sl}(3, \mathbb{K}),$$
$$L_3(\widetilde{\mathbb{H}}, \mathbb{K}) \cong \operatorname{Con} H_3(\mathbb{K}) = \mathfrak{sp}(6, \mathbb{K}).$$

We will now determine the precise composition of these algebras in terms of matrices over $\mathbb{K}$.

Theorem 5.1. **(a)** $\mathfrak{sa}(3, \mathbb{K}) = A_3'(\mathbb{K}) \dotplus \operatorname{Der} \mathbb{K};$ (5.1)

(b) $\mathfrak{sl}(3, \mathbb{K}) = L_3'(\mathbb{K}) \dotplus \operatorname{Der} \mathbb{K};$ (5.2)

(c) $\mathfrak{sp}(6, \mathbb{K}) = Q_6'(\mathbb{K}) \dotplus \operatorname{Der} \mathbb{K}.$ (5.3)

In each case the Lie brackets are defined as follows:

1. $\operatorname{Der} \mathbb{K}$ *is a Lie subalgebra;*
2. *The brackets between* $\operatorname{Der} \mathbb{K}$ *and the other summand are given by the elementwise action of* $\operatorname{Der} \mathbb{K}$ *on matrices over* $\mathbb{K}$*;*
3. *The brackets between two matrices in the first summand are*

$$[X, Y] = (XY - YX)' + \frac{1}{n} D(X, Y) \tag{5.4}$$

where n (= 3 or 6) is the size of the matrix and $D(X, Y)$ is defined in (A.1).

Proof. **(a)** This is Theorem 4.2.

(b) The vector space of $\mathfrak{sl}(3, \mathbb{K})$ is

$$\begin{aligned} \mathfrak{sl}(3, \mathbb{K}) &= \operatorname{Str}' H_3(\mathbb{K}) = \operatorname{Der} H_3(\mathbb{K}) \dotplus H_3'(\mathbb{K}) \\ &= \operatorname{Der} \mathbb{K} \dotplus A_3'(\mathbb{K}) \dotplus H_3'(\mathbb{K}) \\ &= \operatorname{Der} \mathbb{K} \dotplus M_3'(\mathbb{K}) \end{aligned}$$

For $A, B \in A_3'(\mathbb{K})$ the Lie bracket is that of $\operatorname{Der} H_3(\mathbb{K})$, which is (5.4). For $A \in A_3'(\mathbb{K})$, $H \in H_3'(\mathbb{K})$ the bracket is given by the action of X as an element of $\operatorname{Der} H_3(\mathbb{K})$ on H, which according to Theorem 4.2 is

$$[A, H] = AH - HA. \tag{5.5}$$

Now by Lemma A.5, $\operatorname{tr}(AH - HA) = 0$ and $D(A, H) = 0$; hence (5.5) is the same as (5.4).

Finally, for $H, K \in H_3'(\mathbb{K})$ the $\operatorname{Str}' H_3(\mathbb{K})$ bracket is

$$[H, K] = L_H L_K - L_K L_H \in \operatorname{Der} H_3(\mathbb{K})$$

and in the Appendix it is shown that this commutator appears in the decomposition $\operatorname{Der} H_3(\mathbb{K})$ as

$$[L_H, L_K] = (HK - KH)' + \tfrac{1}{3} D(H, K).$$

Note that the action of $\mathrm{Str}H_3(\mathbb{K})$ on $H_3(\mathbb{K})$ is as follows. The subalgebra $\mathrm{Der}\,\mathbb{K}$ acts elementwise, while according to (4.2) the matrix part of $\mathrm{Der}\,H_3(\mathbb{K})$ acts by

$$H \mapsto AH - HA \qquad (A \in A_3'(\mathbb{K})).$$

The remaining matrix subspace $H_3'(\mathbb{K})$ acts by translations in the Jordan algebra $H_3(\mathbb{K})$:

$$H \mapsto KH + HK \qquad (K \in H_3'(\mathbb{K})).$$

Hence the action of the matrix part of $\mathrm{Str}'H_3(\mathbb{K})$ is

$$H \mapsto XH + HX^\dagger \qquad (X \in M_3'(\mathbb{K})). \tag{5.6}$$

(c) The vector space of $\mathfrak{sp}(6,\mathbb{K})$ is

$$\begin{aligned}\mathfrak{sp}(6,\mathbb{K}) &= \mathrm{Con}\,H_3(\mathbb{K}) = \mathrm{Str}\,H_3(\mathbb{K}) \dotplus 2H_3(\mathbb{K})\\ &= \mathfrak{so}(\mathbb{K}') \dotplus M_3'(\mathbb{K}) \dotplus \mathbb{R} \dotplus 2H_3(\mathbb{K}).\end{aligned} \tag{5.7}$$

On the other hand, a 6×6 matrix X belongs to $Q_6'(\mathbb{K})$ if and only if

$$\begin{aligned}&X^\dagger J + JX = 0 \quad \text{and} \quad \mathrm{tr}\,X = 0\\ \iff \quad & X = \begin{pmatrix} A & B \\ C & -A^\dagger \end{pmatrix} \quad \text{with } B, C \in H_3(\mathbb{K})\\ &\text{and} \quad A \in M_3(\mathbb{K}), \quad \mathrm{Im}(\mathrm{tr}\,A) = 0, \quad \text{so} \quad A \in M_3'(\mathbb{K}) \dotplus \mathbb{R}.\end{aligned}$$

Thus

$$Q_6'(\mathbb{K}) \cong M_3'(\mathbb{K}) \dotplus \mathbb{R} \dotplus 2H_3(\mathbb{K}),$$

the summand $\mathbb{R}$ representing $\mathrm{Re}(\mathrm{tr}\,A)$, so the vector space structure of $\mathfrak{sp}(6,\mathbb{K})$ is as stated in (c).

To examine the Lie brackets, we write (5.3) as

$$\mathfrak{sp}(4,\mathbb{K}) \cong \mathfrak{sl}(3,\mathbb{K}) \dotplus \mathbb{R} \dotplus 2H_3(\mathbb{K}).$$

An element A of the matrix part of $\mathfrak{sl}(3,\mathbb{K})$ corresponds in $\mathfrak{sp}(6,\mathbb{K})$ to the matrix $\widehat{A} = \begin{pmatrix} A & 0 \\ 0 & -A^\dagger \end{pmatrix}$. Since $\mathrm{Str}H_3(\mathbb{K})$ is a Lie subalgebra of $\mathrm{Con}H_3(\mathbb{K})$, the Lie bracket of two such elements in $\mathfrak{sp}(6,\mathbb{K})$ is given by

$$[\widehat{A}, \widehat{B}] = (AB - BA - \tfrac{1}{3}tI_3)\widehat{\ } + \tfrac{1}{3}D(A,B)$$

where $t = \mathrm{tr}(AB - BA)$, which is purely imaginary, being a sum of commutators in $\mathbb{K}$. Hence

$$\begin{aligned}(AB - BA - \tfrac{1}{3}tI_2)\widehat{\ } &= \widehat{A}\widehat{B} - \widehat{B}\widehat{A} - \tfrac{1}{3}tI_6\\ &= \widehat{A}\widehat{B} - \widehat{B}\widehat{A} - \tfrac{1}{6}\,\mathrm{tr}(\widehat{A}\widehat{B} - \widehat{B}\widehat{A})I_6\\ &= (\widehat{A}\widehat{B} - \widehat{B}\widehat{A})'.\end{aligned}$$

Also $D(A,B) = 2D(\widehat{A}, \widehat{B})$, so (5.4) holds in $\mathfrak{sp}(6,\mathbb{K})$ for elements X, Y of the form $\widehat{A}$.

For $X \in M_3'(\mathbb{K})$ and $Y \in \mathbb{R}$ or $X, Y \in \mathbb{R}$, both sides of (5.4) are zero.

For $X \in M_3'(\mathbb{K}) \dot{+} \mathbb{R}$ and $Y \in 2H_3(\mathbb{K})$, i.e.

$$X = \begin{pmatrix} A & 0 \\ 0 & -A^\dagger \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

with $A \in M_3(\mathbb{K})$, $B, C \in H_3(\mathbb{K})$, the Lie bracket in $\mathfrak{sp}(6, \mathbb{K})$ is given by the direct sum of the action of $\operatorname{Str} H_3(\mathbb{K})$ on $H_3(\mathbb{K})$ and its transform by the involution $*$ of Section 2. The action is given by (5.6), so $B \mapsto AB + BA^\dagger$, while the effect of the involution is to change the sign of the hermitian part of A, so $C \mapsto -A^\dagger C - CA$. Thus

$$[X, Y] = \begin{pmatrix} 0 & AB + BA^\dagger \\ -A^\dagger C - CA & 0 \end{pmatrix} = XY - YX.$$

Clearly $\operatorname{tr}(XY - YX) = 0$ and $D(X, Y) = 0$, so this is the same as (5.4).

Finally, for $X, Y \in 2H_3(\mathbb{K})$, say

$$X = \begin{pmatrix} 0 & H \\ K & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

the bracket is given by (2.17 - 2.18), i.e.

$$[X, Y] = \tfrac{1}{2}\left(L_{H \cdot C} + L_{K \cdot B} + [L_H, L_C] + [L_K, L_B]\right).$$

The first two terms on the right-hand side form an element of $\operatorname{Str} H_3(\mathbb{K})$ which corresponds in $\mathfrak{sp}(6, \mathbb{K})$ to the matrix

$$\frac{1}{2}\begin{pmatrix} HC + CH + KB + BK & 0 \\ 0 & -HC - CH - KB - BK \end{pmatrix}$$

while the second pair of terms forms an element of $\operatorname{Der} H_3(\mathbb{K})$ corresponding, according to (5.4), to the sum of the matrix

$$\frac{1}{2}\begin{pmatrix} (HC - CH)' + (KB - BK)' & 0 \\ 0 & (HC - CH)' + (KB - BK)' \end{pmatrix}$$

and the $\operatorname{Der} \mathbb{K}$ element

$$\tfrac{1}{6}D(H, C) + \tfrac{1}{6}D(H, C) + \tfrac{1}{6}D(K, B) = \tfrac{1}{6}D(X, Y).$$

Hence

$$\begin{aligned}
[X, Y] = &\begin{pmatrix} HC + KB & 0 \\ 0 & -CH - BK \end{pmatrix} \\
&\quad - \tfrac{1}{6}\operatorname{tr}(HC - CH + KB + BK)I_6 + \tfrac{1}{6}D(X, Y) \\
= &\ (XY - YX)' + \tfrac{1}{6}D(X, Y)
\end{aligned}$$

as asserted in (c). □

This description of the rows of the magic square was given a geometrical interpretation by Freudenthal [3]. A Lie algebra of 3×3 matrices corresponds to a Lie group of linear transformations of a 3-dimensional vector space, or projective transformations of a plane. The Lie group corresponding to $\mathfrak{sa}(3, \mathbb{K})$ preserves a hermitian form in the vector space or a *polarity* in the projective plane, i.e. a correspondence between points and lines. This defines the four (real, complex, quaternionic and octonionic) *elliptic* geometries, in which there is just one class of primitive geometric objects, the points, with a relation of polarity between them (inherited from orthogonality of lines in the vector space). The special linear Lie algebras $\mathfrak{sl}(3, \mathbb{K})$ correspond to the transformation groups of the four projective geometries, in which there are two primitive geometric objects, points and lines, with no relations between points and points or lines and lines but a relation of incidence between points and lines. The third row of the magic square, containing Lie algebras $\mathfrak{sp}(6, \mathbb{K})$, yields the transformation groups of five-dimensional symplectic geometries, whose primitive geometric objects are points, lines and planes. Freudenthal completed this geometrical schema to incorporate the last row of the magic square by defining *metasymplectic* geometries, which have a fourth type of primitive object, the *symplecta*. In metasymplectic geometry points can be *joined* (contained in a line, which is unique if the points are distinct), *interwoven* (contained in a plane, unique if the points are not joined) or *hinged* (contained in a symplecton, unique if the points are not interwoven).

6. Magic Squares of $n \times n$ Matrices

According to Theorem 3.1, Tits's construction (3.2–3.3) yields a Lie algebra for any Jordan algebra if the composition algebra $\mathbb{K}_2$ is associative. Hence for $\mathbb{K}_2 = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and their split versions we obtain a Lie algebra $L_n(\mathbb{K}_1, \mathbb{K}_2)$ for any $n > 3$ by taking $\mathbb{J} = H_n(\mathbb{K}_2)$ in $L(\mathbb{K}_1, \mathbb{J})$ (the case $n = 2$, which will be examined in section 8, lends itself naturally to a slightly different construction). The proof of Vinberg's model is valid for any size of matrix, so we have

Theorem 6.1. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be associative composition algebras over $\mathbb{R}$, and let $L_n(\mathbb{K}_1, \mathbb{K}_2)$ be the Lie algebra obtained by Tits's construction (3.2–3.3) with $\mathbb{K} = \mathbb{K}_1$ and $\mathbb{J} = H_n(\mathbb{K}_2)$. Then*

$$L_n(\mathbb{K}_1, \mathbb{K}_2) = A'_n(\mathbb{K}_1 \otimes \mathbb{K}_2) \dotplus \operatorname{Der} \mathbb{K}_1 \dotplus \operatorname{Der} \mathbb{K}_2$$

with brackets as in $V_3(\mathbb{K}_1, \mathbb{K}_2)$ (section 4.2).

By Theorem 3.2, the Lie algebras $\mathfrak{sa}(n,\mathbb{K})$, $\mathfrak{sl}(n,\mathbb{K})$ and $\mathfrak{sp}(2n,\mathbb{K})$ can now be identified for associative $\mathbb{K}$ as

$$\mathfrak{sa}(n,\mathbb{K}) = \operatorname{Der} H_n(\mathbb{K}) = L_n(\mathbb{R},\mathbb{K}) = A'_n(\mathbb{K}) \dotplus \operatorname{Der}\mathbb{K}, \tag{6.1}$$

$$\mathfrak{sl}(n,\mathbb{K}) = \operatorname{Str}' H_n(\mathbb{K}) = L_n(\widetilde{\mathbb{C}},\mathbb{K}) = M'_n(\mathbb{K}) \dotplus \operatorname{Der}\mathbb{K}, \tag{6.2}$$

$$\mathfrak{sp}(2n,\mathbb{K}) = \operatorname{Con} H_n(\mathbb{K}) = L_n(\widetilde{\mathbb{H}},\mathbb{K}) = Q'_n(\mathbb{K}) \dotplus \operatorname{Der}\mathbb{K}. \tag{6.3}$$

6.1. **The Santander-Herranz Construction.** Vinberg's approach to the magic square is extended to general dimensions n by Santander and Herranz in their construction of 'Cayley-Klein' (CK) algebras. This starts from a $(2N+2)\times(2N+1)$ matrix $I_\omega = \operatorname{diag}(1,\omega_{01},\omega_{02},\ldots,\omega_{0N})$ depending on $N+1$ fixed non-zero parameters ω_i, with $\omega_{0a} = \omega_0\omega_1\ldots\omega_a$. Let $\mathbb{I}_\omega = \left(\begin{smallmatrix}0 & I_\omega \\ -I_\omega & 0\end{smallmatrix}\right)$. A matrix X is defined to be G-antihermitian if $X^\dagger G + GX = 0$. Santander and Herranz define three series of classical CK-algebras:

1. The special antihermitian CK-algebra, $\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1,\mathbb{K})$. This is the Lie algebra of I_ω-antihermitian matrices, X, over $\mathbb{K}$ if $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$, or the subalgebra of traceless matrices if $\mathbb{K}=\mathbb{C}$.

2. The special linear CK-algebra, $\mathfrak{sl}_{\omega_1\ldots\omega_N}(N+1,\mathbb{K})$. This is the Lie algebra of all matrices $X \in \mathbb{K}^{(N+1)\times(N+1)}$ with $\operatorname{tr} X = 0$ if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $\operatorname{Re}(\operatorname{tr} X) = 0$ if $\mathbb{K}=\mathbb{H}$.

3. The special symplectic CK-algebra, $\mathfrak{sn}_{\omega_1\ldots\omega_N}(2(N+1),\mathbb{K})$. This is the Lie algebra of all $\mathbb{I}_\omega$-antihermitian matrices over $\mathbb{K}$ if $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$ and the subalgebra of matrices with zero trace if $\mathbb{K}=\mathbb{C}$.

For $N = 1,2$ these definitions can be extended to include $\mathbb{K}=\mathbb{O}$ by adding the derivations of $\mathbb{O}$ in each case. A fourth CK-algebra can also be added, the metasymplectic CK-algebra, $\mathfrak{mn}(N+1,\mathbb{K})$ based on the definition of the metasymplectic geometry given in [3].

Now define the set of matrices

$$J_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & -\omega_{ab} & \cdots \\ & \vdots & & \vdots & \\ \cdot & 1 & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{pmatrix}, \quad M_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \omega_{ab} & \cdots \\ & \vdots & & \vdots & \\ \cdot & 1 & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{pmatrix}$$

and

$$H_m = \begin{pmatrix} 1 & & & \cdot & \\ & & & \vdots & \\ \cdot & & \cdots & 1 & \cdots \\ & & & \vdots & \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & \cdots & \\ \vdots & & 0 \end{pmatrix},$$

where $a, b = 0, 1, \ldots, N$ with the condition that $a < b$; $m = 1, \ldots, N$; and matrix indices run over the range $0, \ldots, N$. Further if X is one of these matrices then define $X^i = e_i X$ and

$$\mathbb{X} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \; \mathbb{X}_1 = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \; \mathbb{X}_2 = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \; \mathbb{X}_3 = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}.$$

Note that there is an isomorphism $J \mapsto \mathbb{J}_{ab}$, $M_{ab} \mapsto \mathbb{M}_{ab;2}$, $M^1_{ab} \mapsto \mathbb{M}^1_{ab}$, $M^2_{ab} \mapsto \mathbb{M}^2_{ab}$.

The first three rows and columns of the Tits-Freudenthal magic square can now be generalised to the $(N+1)$-dimensional case using the three CK-algebra series as follows

Lie Algebra	Lie span of the generators		
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1,\mathbb{K})$	J_{ab}	$J_{ab}, M^1_{a,b}$	$J_{ab}, M^1_{a,b}, M^2_{ab}$
$\mathfrak{sl}_{\omega_1\ldots\omega_N}(N+1,\mathbb{K})$	J_{ab}, M_{ab}	$J_{ab}, M_{ab}, M^1_{a,b}$	$J_{ab}, M_{ab}, M^1_{a,b}, M^2_{ab}$
$\mathfrak{sn}_{\omega_1\ldots\omega_N}(N+1,\mathbb{K})$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}, \mathbb{M}^1_{ab}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}, \mathbb{M}^1_{ab}, \mathbb{M}^2_{ab}$

Then the symmetry of the $(N+1)$ dimensional magic square (and consequently of the 3×3 magic square) can be explained as follows.

Each algebra is a subalgebra of all the algebras to its right and below it: as we move from left to right and from top to bottom across the square, in each step the same new generators appear. Explicitly, moving from the top algebra ($\mathfrak{sa}$) to the bottom ($\mathfrak{sn}$), in each column M_{ab} appears in the first step ($\mathfrak{sa} \to \mathfrak{sl}$) and $\mathbb{M}_{ab;1}$ appears in the second ($\mathfrak{sl} \to \mathfrak{sn}$). Similarly, moving from left to right, M^1_{ab} is the additional generator after the first step and M^2_{ab} is the additional generator after the second.

In more recent work Santander [16] has gone on to define the tensor algebra $\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K}_1 \otimes \mathbb{K}_2)$, an extension of the Vinberg construction which includes all simple Lie algebras, i.e. *any* simple Lie algebra can be written in the form $\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K}_1 \otimes \mathbb{K}_2)$ for an appropriate choice of ω_i, N, $\mathbb{K}_1$ and $\mathbb{K}_2$. Explicitly this is the algebra of $(N+1) \times (N+1)$ matrices with entries in $\mathbb{K}_1 \otimes \mathbb{K}_2$ and the derivations of $\mathbb{K}_1$ and $\mathbb{K}_2$.

Thus we have a second way of approaching an explanation of the symmetry of the magic square and indeed a classification of all simple Lie algebras in terms of matrices with entries in the division algebras.

7. Maximal Compact Subalgebras

We now turn to the question of identifying the Lie algebras of Theorem 5.1 in the standard list of real forms of complex semisimple Lie algebras. The split magic square $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$ contains real forms of the complex Lie algebras $L_3(\mathbb{K}_1, \mathbb{K}_2)$ which are identified in Table 1; we will establish this identification by finding the maximal compact subalgebras.

Recall that a semi-simple Lie algebra over $\mathbb{R}$ is called *compact* if it has a negative-definite Killing form. A non-compact real form $\mathfrak{g}$ of a semi-simple complex Lie algebra L has a maximal compact subalgebra $\mathfrak{n}$ with an orthogonal complementary subspace $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{n} \dot{+} \mathfrak{p}$ and the brackets

$$\begin{aligned} [\mathfrak{n}, \mathfrak{n}] &\subseteq \mathfrak{n} \\ [\mathfrak{n}, \mathfrak{p}] &\subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq \mathfrak{n} \end{aligned} \tag{7.1}$$

(see, for example, [4]), from which it follows that $\langle \mathfrak{n}, \mathfrak{p} \rangle = 0$ where $\langle , \rangle$ is the Killing form of L. There exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{n}$ and $\mathfrak{p}$ are eigenspaces of σ with eigenvalues $+1$ and -1 respectively. A compact real form, $\mathfrak{g}'$, of L will also contain $\mathfrak{n}$ as a compact subalgebra of $\mathfrak{g}'$ but clearly in this case the maximal compact subalgebra will be $\mathfrak{g}'$ itself. We can obtain $\mathfrak{g}'$ from $\mathfrak{g}$ by keeping the same brackets in $[\mathfrak{n}, \mathfrak{n}]$ and $[\mathfrak{n}, \mathfrak{p}]$ but multiplying the brackets in $[\mathfrak{p}, \mathfrak{p}]$ by -1, i.e. by performing the *Weyl unitary trick* (putting $\mathfrak{g}' = \mathfrak{n} \dot{+} i\mathfrak{p}$).

We will use the following method to identify the maximal compact subalgebras in $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$. It is known that $L_3(\mathbb{K}_1, \mathbb{K}_2)$ gives a compact real form of each Lie algebra (from, for example [8]). Thus if $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$ shares a common subalgebra with $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$, say $\mathfrak{n}$, where

$$\begin{aligned} L_3(\mathbb{K}_1, \mathbb{K}_2) &= \mathfrak{n} \dot{+} \mathfrak{p}_1 \\ L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2) &= \mathfrak{n} \dot{+} \mathfrak{p}_2, \end{aligned}$$

and the brackets in $[\mathfrak{n}, \mathfrak{p}_1]$ are the same as those in $[\mathfrak{n}, \mathfrak{p}_2]$ but the brackets in $[\mathfrak{p}_1, \mathfrak{p}_1]$ are -1 times the equivalent brackets in $[\mathfrak{p}_2, \mathfrak{p}_2]$, then $\mathfrak{n}$ will be the maximal compact subalgebra of $L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2)$ and $\mathfrak{p}_2$ will be its orthogonal complementary subspace. We will see that this sign change in the brackets reflects precisely the change in sign in the Cayley-Dickson process [17] when moving from the division algebra to the corresponding split composition algebra.

First we consider the relation between $\operatorname{Der} \mathbb{K}$ and $\operatorname{Der} \widetilde{\mathbb{K}}$ where $\mathbb{K}$ is a division algebra. Both $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ can be obtained by the Cayley-Dickson process [17] from a positive-definite composition algebra $\mathbb{F}$; they are both of the form $F_\varepsilon^2 = \mathbb{F} \dot{+} l\mathbb{F}$, where l is the new imaginary unit and the multiplication is given by

$$\begin{aligned} x(ly) &= l(\overline{x}y) \\ (lx)y &= l(yx) \\ (lx)(ly) &= \varepsilon y\overline{x} \end{aligned} \tag{7.2}$$

where $\varepsilon = -1$ for $\mathbb{K}$ and $\varepsilon = 1$ for $\widetilde{\mathbb{K}}$.

A derivation D of $\mathbb{F}_\varepsilon^2$ can be specified by giving its action on $\mathbb{F}'$ (since $D(1) = 0$) and by specifying $D(l)$. Thus each derivation D of $\mathbb{F}$ can be extended to a derivation $\overline{D}$ of $\mathbb{F}_\varepsilon^2$ by defining $Dl = 0$. We also define derivations E_a, F_a for each $a \in \mathbb{F}'$, and G_S for each symmetric linear map $S : \mathbb{F}' \to \mathbb{F}'$, as follows: For $x \in \mathbb{F}$ and $a, b \in \mathbb{F}'$,

$$\begin{aligned} E_a x &= 0 \quad (x \in \mathbb{F}), & E_a l &= la; \\ F_a b &= l(ab - \langle a, b\rangle) \quad (b \in \mathbb{F}'), & F_a l &= -2\varepsilon a \\ & & \text{so that } F_a(lb) &= -\tfrac{1}{2}\varepsilon[a, b]; \\ G_S a &= l(Sa) \quad (a \in \mathbb{F}'), & G_S l &= 0. \end{aligned} \tag{7.3}$$

Theorem 7.1. $\mathrm{Der}(\mathbb{F}_\varepsilon^2)$ *is spanned by* $\overline{D}$ $(D \in \mathrm{Der}\,\mathbb{F})$, E_a, F_a $(a \in \mathbb{F}')$ *and* G_S *where* $S : \mathbb{F}' \to \mathbb{F}'$ *is symmetric and traceless if* $\mathbb{F} = \mathbb{H}$. *The Lie brackets are given by*

$$[D, E_a] = E_{Da}, \qquad [D, F_a] = F_{Da},$$

$$[D, G_S] = G_{[D,S]},$$

$$[E_a, E_b] = -E_{[a,b]}$$

$$[E_a, F_b] = \tfrac{1}{4}F_{[a,b]} - \tfrac{3}{2}G_{S(a,b)} + \left(1 - \tfrac{3}{m}\right)\langle a, b\rangle G_{\mathrm{id}}$$

(*where* $m = \dim \mathbb{F}'$, $S(a, b)$ *is the traceless symmetric map*

$$S(a, b)c = \langle a, c\rangle b + \langle b, c\rangle a - \frac{2}{m}\langle a, b\rangle c,$$

and G_{id} *is given by (7.3) when* S *is the identity map on* $\mathbb{F}'$);

$$[E_a, G_S] = \tfrac{1}{2}F_{Sa} - \tfrac{1}{4}G_{[D_a,S]}$$

where $D_a \in \mathrm{Der}\,\mathbb{F}$ *is the inner derivation* $D_a(x) = [a, x]$*;*

$$[F_a, F_b] = -\tfrac{1}{4}\varepsilon D_{[a,b]} - 2\varepsilon E_{[a,b]},$$

$$[F_a, G_S] = \tfrac{1}{2}\varepsilon D_{Sa} + 2\varepsilon E_{Sa},$$

$$[G_S, G_T] = \varepsilon[S, T].$$

Proof. First we note that any derivation of an algebra must annihilate the identity of the algebra. Let D be a derivation of $\mathbb{F}_\varepsilon^2$ satisfying $Dl = 0$. Then D is determined by its action on $a \in \mathbb{F}'$. Write

$$Da = Ta + l(Sa)$$

where T and S are maps from $\mathbb{F}'$ to $\mathbb{F}$. Then the derivation condition applied to the relations (7.2) requires T to be a derivation of $\mathbb{F}$ and S to be a map from $\mathbb{F}'$ to $\mathbb{F}'$ satisfying

$$\begin{aligned} S[a, b] &= -2a(Sb) + 2b(Sa) \\ &= 2(Sb)a - 2(Sa)b. \end{aligned}$$

Since $ab+ba = -2\langle a,b\rangle$ where $\langle , \rangle$ is the inner product on $\mathbb{F}$, this yields

$$\langle Sa, b\rangle = \langle a, Sb\rangle,$$

i.e. S is a symmetric operator on $\mathbb{F}'$, and

$$S[a,b] = -[Sa,b] - [a,Sb].$$

If $\mathbb{F} = \mathbb{H}$ there is an identity

$$S[a,b] + [Sa,b] + [a,Sb] = (\operatorname{tr} S)[a,b]$$

(a version of $\varepsilon_{mjk}\delta_{in} + \varepsilon_{imk}\delta_{jn} + \varepsilon_{ijm}\delta_{kn} = \varepsilon_{ijk}\delta_{mn}$), so that in this case $\operatorname{tr} S = 0$.

Thus

$$Dl = 0 \quad \Longrightarrow \quad D = T + G_S \ \text{ with } \ T \in \operatorname{Der}\mathbb{F}$$

where $S : \mathbb{F}' \to \mathbb{F}'$ is symmetric and traceless if $\mathbb{F} = \mathbb{H}$. To show that every such map is a derivation of $\mathbb{F}^2_\varepsilon$, it is sufficient to check the relations (7.2) for G_S where S is one of the elementary traceless symmetric maps of the form $S(a,b)$. This is straightforward.

It is also straightforward (though tedious) to check that the maps E_a, F_a are derivations of $\mathbb{F}^2_\varepsilon$ for any $a \in \mathbb{F}'$. Now let D be any derivation of $\mathbb{F}^2_\varepsilon$, and write

$$Dl = \alpha + a + l(\beta + b)$$

with $\alpha, \beta \in \mathbb{R}$ and $a, b \in \mathbb{F}'$. Since $l^2 = \varepsilon$ and $D\varepsilon = 0$, Dl must anticommute with l; hence $\alpha = \beta = 0$, so that

$$Dl = -\frac{1}{2\varepsilon}F_a(l) + E_b(l).$$

It follows, by the first part of the proof, that $D + (2\varepsilon)^{-1}F_a - E_b$ is the sum of a derivation of $\mathbb{F}$ and an element G_S. Thus the derivations $\overline{D}, E_a, F_a$ and G_S span $\operatorname{Der}\mathbb{F}^2_\varepsilon$.

The stated commutators can be verified by straightforward computation. □

Let $\operatorname{Der}_0 \mathbb{F}^2_\varepsilon$ be the subalgebra of $\operatorname{Der}\mathbb{F}^2_\varepsilon$ spanned by $\operatorname{Der}\mathbb{F}$ and E_a ($a \in \mathbb{F}'$), so that

$$D \in \operatorname{Der}_0 \mathbb{F}^2_\varepsilon \quad \Longleftrightarrow \quad D(\mathbb{F}) \subset \mathbb{F} \text{ and } D(l\mathbb{F}) \subset l\mathbb{F}; \tag{7.4}$$

and let $\operatorname{Der}_1 \mathbb{F}^2_\varepsilon$ be the subspace spanned by F_a and G_S, so that

$$D \in \operatorname{Der}_1 \mathbb{F}^2_\varepsilon \quad \Longleftrightarrow \quad D(\mathbb{F}) \subset l\mathbb{F} \text{ and } D(l\mathbb{F}) \subset \mathbb{F}. \tag{7.5}$$

Then $\operatorname{Der}\mathbb{F}^2_\varepsilon$ has the structure (7.1), with $\mathfrak{n} = \operatorname{Der}_0 \mathbb{F}^2_\varepsilon$ and $\mathfrak{p} = \operatorname{Der}_1 \mathbb{F}^2_\varepsilon$, and the brackets in $[\mathfrak{p},\mathfrak{p}]$, which include a factor ε, have opposite signs in $\mathbb{K}$ and $\widetilde{\mathbb{K}}$. Since $\operatorname{Der}\mathbb{K}$ is compact (being a subalgebra of $\mathfrak{so}(\mathbb{K})$), this identifies the maximal compact subalgebra of $\operatorname{Der}\widetilde{\mathbb{K}}$ as $\operatorname{Der}_0 \widetilde{\mathbb{K}} = \operatorname{Der}\mathbb{F} \dot{+} \mathbb{F}'$; explicitly,

$$\operatorname{Der}_0 \widetilde{\mathbb{C}} = 0, \qquad \operatorname{Der}_0 \widetilde{\mathbb{H}} = \mathfrak{so}(2), \qquad \operatorname{Der}_0 \widetilde{\mathbb{O}} = \mathfrak{so}(4).$$

Note that the algebra $\mathbb{F}_\varepsilon$ has a $\mathbb{Z}_2$-grading $\mathbb{F}_\varepsilon^2 = \mathbb{F} \dot{+} l\mathbb{F}$, and the above decomposition is the corresponding $\mathbb{Z}_2$-grading of the derivation algebra, i.e. $\operatorname{Der}_\delta \mathbb{F}_\varepsilon^2$ ($\delta = 0, 1$) is the subspace of derivations of degree δ. From the definition (2.10) it follows that the derivation $D_{x,y}$ has degree $\gamma + \delta$ (mod 2) if x has degree γ and y has degree δ.

Now consider the rows of the non-compact magic square $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$. Suppose $\widetilde{\mathbb{K}}_1 = (\mathbb{F}_2)_+^2 = \mathbb{F}_2 \dot{+} l\mathbb{F}_2$. Then Vinberg's construction gives

$$\begin{aligned} L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2) &= A_3'(\widetilde{\mathbb{K}}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der} \widetilde{\mathbb{K}}_1 \dot{+} \operatorname{Der} \mathbb{K}_2 \\ &= \mathfrak{n} \dot{+} \mathfrak{p} \end{aligned}$$

where

$$\begin{aligned} \mathfrak{n} &= A_3'(\mathbb{F}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der}_0 \widetilde{\mathbb{K}}_1 \dot{+} \operatorname{Der} \mathbb{K}_2, \\ \mathfrak{p} &= A_3(l\mathbb{F}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der}_1 \widetilde{\mathbb{K}}_1. \end{aligned}$$

The brackets (4.20–4.21), together with the $\mathbb{Z}_2$-grading of $\widetilde{\mathbb{K}}_1$ and $\operatorname{Der} \widetilde{\mathbb{K}}_1$, give the structure (7.1). The compact algebra $L_3(\mathbb{K}_1 \otimes \mathbb{K}_2)$ has the same structure, and the brackets are the same except for the sign in $[\mathfrak{p}, \mathfrak{p}]$, which contains a factor ε. Hence the maximal compact subalgebra of $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$ is

$$\begin{aligned} \mathfrak{n} &= A_3'(\mathbb{F}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der} \mathbb{K}_2 \dot{+} \operatorname{Der} \mathbb{F}_1 \dot{+} \mathbb{F}_1' \\ &= L_3(\mathbb{F}_1, K_2) \dot{+} \mathbb{F}_1'. \end{aligned}$$

Thus we have

Theorem 7.2. *The maximal compact subalgebra of the non-compact magic square algebra $L_3(\widetilde{\mathbb{K}}_1, K_2)$ is $L_3(\mathbb{F}_1, K_2) \dot{+} \mathbb{F}_1'$, where $\mathbb{F}_1$ is the division algebra preceding $\mathbb{K}_1$ in the Cayley-Dickson process.*

Applying this to the last row and column of the magic square gives the table at the end of Section 3.

For completeness, we identify the real Lie algebras occurring in the magic square $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$ when both composition algebras are split. Writing $\widetilde{\mathbb{K}}_i = (\mathbb{F}_i)_+^2$ (i=1,2) gives a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading

$$\begin{aligned} L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2) = & A_3'(\mathbb{F}_1 \otimes \mathbb{F}_2) + \operatorname{Der}_0 \widetilde{\mathbb{K}}_1 + \operatorname{Der}_0 \widetilde{\mathbb{K}}_2 \\ & + A_3'(l\mathbb{F}_1 \otimes \mathbb{F}_2) \dot{+} \operatorname{Der}_1 \widetilde{\mathbb{K}}_1 \\ & + A_3'(\mathbb{F}_1 \otimes l\mathbb{F}_2) \dot{+} \operatorname{Der}_1 \widetilde{\mathbb{K}}_2 \\ & + A_3'(l\mathbb{F}_1 \otimes l\mathbb{F}_2) \end{aligned}$$

in which the successive lines have gradings (0,0), (1,0), (0,1) and (1,1). By arguments similar to those used for $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$, the maximal compact subalgebra is the direct sum of the subspaces of degree (0,0) and

(1,1), namely

$$\begin{aligned}\mathfrak{n} &= A_3'(\mathbb{F}_1 \otimes \mathbb{F}_2) \dotplus \operatorname{Der}\mathbb{F}_1 \dotplus \mathbb{F}_1' \dotplus \operatorname{Der}\mathbb{F}_2 \dotplus \mathbb{F}_2' \dotplus A_3'(l_1\mathbb{F}_1 \otimes l_2\mathbb{F}_2)\\ &= L_3(\mathbb{F}_1, \mathbb{F}_2) \dotplus \mathbb{F}_1' \dotplus \mathbb{F}_2' \dotplus A_3'(l_1\mathbb{F}_1 \otimes l_2\mathbb{F}_2).\end{aligned}$$

Since the elements of $l_1\mathbb{F}_1 \otimes l_2\mathbb{F}_2$ are self-conjugate in $\mathbb{K}_1 \otimes \mathbb{K}_2$, the last summand contains antisymmetric 3×3 matrices which can be identified with the entries in the last row and column (excluding the diagonal element) of an antihermitian 4×4 matrix over $\mathbb{F}_1 \otimes \mathbb{F}_2$, while an element of $\mathbb{F}_1' \dotplus \mathbb{F}_2'$ can be identified with the last diagonal element of such a matrix. Thus we have a vector space isomorphism

$$\mathfrak{n} = L_4(\mathbb{F}_1, \mathbb{F}_2). \tag{7.6}$$

We will find that this is actually a Lie algebra isomorphism.

By inspection of the table of real forms of complex semi-simple Lie algebras [4, 12] we can now identify the non-compact Lie algebras of the doubly-split magic square $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$ as follows:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sp}(6,\mathbb{R})$	$F_4(-4)$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(6,\mathbb{R})$	$E_6(-6)$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{sl}(6,\mathbb{R})$	$\mathfrak{so}(6,6)$	$E_7(-7)$
$\mathbb{O}$	$F_4(-4)$	$E_6(-6)$	$E_7(-7)$	$E_8(-8)$

.

in which the real forms of the exceptional Lie algebras are identified by the signatures of their Killing forms. The maximal compact subalgebras are

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3)$	$\mathfrak{u}(3)$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$\mathbb{C}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(6)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{u}(3)$	$\mathfrak{so}(6)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

.

In this last table the 3×3 square labelled by $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ is isomorphic to

	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\mathfrak{so}(4)$	$\mathfrak{su}(4)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{su}(4)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(4)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

.

which has a non-compact form

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(4)$	$\mathfrak{su}(4)$	$\mathfrak{sq}(4)$
$\mathbb{C}$	$\mathfrak{sl}(4,\mathbb{R})$	$\mathfrak{sl}(4,\mathbb{C})$	$\mathfrak{sl}(4,\mathbb{H})$
$\mathbb{H}$	$\mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{sp}(8,\mathbb{C})$ $\cong \mathfrak{su}(4,4)$	$\mathfrak{sp}(8,\mathbb{H})$

.

in which we have changed the labels of the rows and columns from $\mathbb{K}$ to $\mathbb{F}$ where $\mathbb{K} = \mathbb{F}_\varepsilon^2$ with $\varepsilon = +1$ for the rows and $\varepsilon = -1$ for the columns. The rows of this table are $\mathfrak{sa}(4,\mathbb{F}_2)$, $\mathfrak{sl}(4,\mathbb{F}_2)$ and $\mathfrak{sp}(8,\mathbb{F}_2)$, and therefore by theorem 6.1 the Lie algebras in the table are $L_4(\widetilde{\mathbb{F}}_1,\mathbb{F}_2)$. The compact forms are therefore $L_4(\mathbb{F}_1,\mathbb{F}_2)$ as asserted in (7.6), and we have established that this is a Lie algebra isomorphism.

The involution of the compact Lie algebra $L_3(\mathbb{K}_1,\mathbb{K}_2)$ which defines the non-compact form $L_3(\widetilde{\mathbb{K}}_1,\widetilde{\mathbb{K}}_2)$ can be taken to be $X \mapsto -X^T$ for $X \in A_3'(\widetilde{\mathbb{K}}_1,\widetilde{\mathbb{K}}_2)$, together with the (essentially unique) non-trivial involution on both $\operatorname{Der}\mathbb{K}_1$ and $\operatorname{Der}\mathbb{K}_2$. The Cartan subalgebra of $L_3(\mathbb{K}_1,\mathbb{K}_2)$ can be chosen so that this involution takes each root element x_α to $x_{-\alpha}$ (and preserves the Cartan subalgebra). This explains why the rank of each of the Lie algebras $L_3(\widetilde{\mathbb{K}}_1,\widetilde{\mathbb{K}}_2)$ is equal in magnitude to the signature of its Killing form.

The magic squares $L_3(\mathbb{K}_1,\widetilde{\mathbb{K}}_2)$ and $L_3(\widetilde{\mathbb{K}}_1,\widetilde{\mathbb{K}}_2)$ contain all the real forms of the exceptional simple Lie algebras except the following two:

$F_4(20)$ with maximal compact subalgebra $\mathfrak{so}(9)$;

$E_6(14)$ with maximal compact subalgebra $\mathfrak{so}(10) \oplus \mathfrak{so}(2)$.

These can presumably be explained by a construction in which the antihermitian matrices $A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$ are replaced by matrices which are antihermitian with respect to a non-positive definite metric matrix $G = \operatorname{diag}(1,1,-1)$, i.e. by matrices X satisfying $\overline{X}^T G = -GX$, as in the Santander-Herranz construction.

8. The $n = 2$ Magic Square

It would be surprising, particularly in view of Freudenthal's geometrical interpretation (see Section 5), if $n = 3$ were the only case in which there were Lie algebras $L_n(\mathbb{K}_1,\mathbb{K}_2)$ for non-associative $\mathbb{K}_1$ and $\mathbb{K}_2$; we would expect Lie algebras corresponding to $n = 2$ to arise as subalgebras of the $n = 3$ algebras. Indeed, the algebra of 2×2 hermitian matrices $H_2(\mathbb{K})$ is a Jordan algebra if $H_3(\mathbb{K})$ is, and therefore the Tits construction of Theorem 3.1 yields a Lie algebra $L_2(\mathbb{K}_1,\mathbb{K}_2) = L(\mathbb{K}_1, H_2(\mathbb{K}_2))$ for associative $\mathbb{K}_1$ and for any composition algebra $\mathbb{K}_2$. We will now show how to extend this construction to allow $\mathbb{K}_1$ to be any composition algebra.

For $n = 2$ the hermitian Jordan algebra $H_2(\mathbb{K})$ takes a particularly simple form. The usual identification of $\mathbb{R}$ with the subspace of scalar multiples of the identity gives $H_2(\mathbb{K}) = \mathbb{R} \dotplus H_2'(\mathbb{K})$, and the Jordan product in the traceless subspace $H_2'(\mathbb{K})$ is given by

$$A \cdot B = \langle A, B\rangle \mathbf{1} \tag{8.1}$$

where the inner product is defined by

$$\langle A, B\rangle = \tfrac{1}{2}\operatorname{tr}(A \cdot B)$$

so that

$$A = \begin{pmatrix} \lambda & x \\ \overline{x} & -\lambda \end{pmatrix}, B = \begin{pmatrix} \mu & y \\ \overline{y} & -\mu \end{pmatrix} \quad \Longrightarrow \quad \langle A, B\rangle = 2\big(\lambda\mu + \langle x, y\rangle\big), \tag{8.2}$$

i.e.

$$H_2'(\mathbb{K}) = \mathbb{R} \oplus \mathbb{K} \tag{8.3}$$

(recall that we use $\oplus$ to denote that the summands are orthogonal subspaces). The anticommutator algebra of $H_2(\mathbb{K})$ is therefore a subalgebra of that of the Clifford algebra of the vector space $\mathbb{R} \oplus \mathbb{K}$; since the Clifford algebra is associative, its anticommutator algebra is a (special) Jordan algebra. It is immediate from (8.1) that the derivations of this Jordan algebra are precisely the antisymmetric linear endomorphisms of $H_2'(\mathbb{K})$. To summarise,

Theorem 8.1. *If $\mathbb{K}$ is any composition algebra, the anticommutator algebra $H_2(\mathbb{K})$ is a Jordan algebra with product given by* (8.1)*, and its derivation algebra is*

$$\operatorname{Der} H_2(\mathbb{K}) \cong \mathfrak{so}(\mathbb{R} \oplus \mathbb{K}). \tag{8.4}$$

There is also a description of $\operatorname{Der} H_2(\mathbb{K})$ in terms of 2×2 matrices like, but interestingly different from, the description of $\operatorname{Der} H_3(\mathbb{K})$ in Theorem 4.2:

Theorem 8.2. *For any composition algebra $\mathbb{K}$,*

$$\operatorname{Der} H_2(\mathbb{K}) = \mathfrak{so}(\mathbb{K}') \dotplus A_2'(\mathbb{K})$$

in which $\mathfrak{so}(\mathbb{K}')$ is a Lie subalgebra, the Lie brackets between $\mathfrak{so}(\mathbb{K}')$ and $A_2'(\mathbb{K})$ are given by the elementwise action of $\mathfrak{so}(\mathbb{K}')$ on 2×2 matrices over $\mathbb{K}$, and

$$[A, B] = (AB - BA)' + 2S(A, B) \tag{8.5}$$

where $A, B \in A_2'(\mathbb{K})$, the prime denotes the traceless part, and

$$S(A, B) = \sum_{ij} S_{a_{ij}, b_{ij}} \in \mathfrak{so}(\mathbb{K}').$$

Proof. Write $H_2'(\mathbb{K}) = \sigma_1(\mathbb{K}') \dotplus \mathbb{R}\sigma_2 \dotplus \mathbb{R}\sigma_3$ where $\sigma_1 : \mathbb{K}' \to H_2'(\mathbb{K})$ and $\sigma_2, \sigma_3 \in H_2'(\mathbb{K})$ are defined by

$$\sigma_1(a) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\operatorname{Der} H_2(\mathbb{K}) = \mathfrak{so}(H_2'(\mathbb{K})) = \mathfrak{so}(\mathbb{K}') \dotplus \mathbb{R}\theta_1 \dotplus \theta_2(\mathbb{K}') \dotplus \theta_3(\mathbb{K}')$$

where the actions of the derivations θ_1, $\theta_2(a)$ and $\theta_3(b)$ $(a, b \in \mathbb{K}')$ are given by

$$\begin{aligned}
&\theta_1(a\sigma_1) = 0, && \theta_1(\sigma_2) = \sigma_3, && \theta_1(\sigma_3) = -\sigma_2,\\
&\theta_2(a)(b\sigma_1) = -\langle a, b\rangle\sigma_3, && \theta_2(a)\sigma_2 = 0, && \theta_2(a)\sigma_3 = a\sigma_1,\\
&\theta_3(a)(b\sigma_1) = \langle a, b\rangle\sigma_2, && \theta_3(a)\sigma_2 = -a\sigma_1, && \theta_3(a)\sigma_3 = 0.
\end{aligned}$$

These actions are reproduced by

$$\begin{aligned}
\theta_1(H) &= [\sigma_1, H]\\
\theta_2(a)(H) &= [a\sigma_2, H]\\
\theta_3(a)(H) &= [a\sigma_3, H]
\end{aligned} \tag{8.6}$$

where the brackets denote matrix commutators, so

$$\mathbb{R}\theta_1 \dotplus \theta_2(\mathbb{K}') \dotplus \theta_3(\mathbb{K}') = \mathbb{R}\sigma_1 \dotplus \mathbb{K}'\sigma_2 \dotplus \mathbb{K}'\sigma_3 = A_3'(\mathbb{K}),$$

and the action of $A \in A_3(\mathbb{K})$ as a derivation of $H_2(\mathbb{K})$ is the commutator map C_A. By the matrix identity (A.14), the Lie bracket in $\operatorname{Der} H_2(\mathbb{K})$ is given by

$$[A, B] = (AB - BA)' + \tfrac{1}{2}F(A, B).$$

where $F(A, B)$ is defined in (A.4). Now for $A, B \in A_2(\mathbb{H})$ and $z \in \mathbb{K}'$ we have

$$\begin{aligned}
4S(A, B)z &= \sum_{ij} \big\{\langle a_{ij}, z\rangle b_{ji} - \langle b_{ji}, z\rangle a_{ij}\big\}\\
&= \sum_{ij} \big\{(a_{ij}\overline{z} + z\overline{a_{ij}})b_{ji} + b_{ji}(\overline{a_{ij}}z + \overline{z}a_{ij})\\
&\qquad\qquad - (b_{ji}\overline{z} - z\overline{b_{ji}})a_{ij} - a_{ij}(\overline{b_{ji}}z + \overline{z}b_{ji})\big\}\\
&= \sum_{ij} \big\{z(\overline{a_{ij}}b_{ji} - \overline{b_{ji}}a_{ij}) - (b_{ji}\overline{a_{ij}} - a_{ij}\overline{b_{ji}})z\big\}\\
&= F(A, B)z.
\end{aligned}$$

The Lie bracket can therefore be written as (8.5). □

Comparison with Theorem 4.2 suggests that in passing from $n = 3$ to $n = 2$, $\operatorname{Der}\mathbb{K}$ should be replaced by $\mathfrak{so}(\mathbb{K}')$. This has no effect if $\mathbb{K}$ is associative, since then these two Lie algebras coincide[1] (see Section 2). Thus we will define $L_2(\mathbb{K}_1, \mathbb{K}_2)$ by making this replacement in the definition of $L_n(\mathbb{K}_1, \mathbb{K}_2)$ for $n \geq 3$, which gives the vector space

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\mathbb{K}_1') \dotplus \operatorname{Der} H_2(\mathbb{K}_2) \dotplus \mathbb{K}_1' \otimes H_2'(\mathbb{K}_2) \tag{8.7}$$

[1]This seems to be a genuine coincidence since it does not survive at the group level: $\operatorname{Aut}\mathbb{C} = \mathrm{O}(\mathbb{C}')$ but $\operatorname{Aut}\mathbb{H} = \mathrm{SO}(\mathbb{H}')$.

To obtain the replacement for the Lie bracket (3.3), note that it follows from (8.1) that the traceless Jordan product $A * B$ is identically zero in $H_2(\mathbb{K})$. Moreover, if $\mathbb{K}_1$ is associative the derivation $D_{a,b} \in \operatorname{Der}\mathbb{K}_1$ is the generator $4S_{a,b}$ of rotations in the plane of a and b, for

$$\begin{aligned} 4S_{a,b}(c) &= 2\langle a,c\rangle b + 2b\langle a,c\rangle - 2\langle b,c\rangle a - 2a\langle b,c\rangle \\ &= -(ac+ca)b - b(ac+ca) + (bc+cb)a + a(bc+cb) \\ &= [[a,b],c]. \end{aligned}$$

Thus the following definition of $L_2(\mathbb{K}_1,\mathbb{K}_2)$ is the case $n = 2$ of the general definition of L_n if $\mathbb{K}_1$ is associative:

Definition 3. The algebra $L_2(\mathbb{K}_1,\mathbb{K}_2)$ consists of the vector space (8.7) with brackets in the first two summands given by the Lie algebra $\mathfrak{so}(\mathbb{K}_1) \oplus \operatorname{Der} H_2(\mathbb{K}_2)$, brackets between these and the third summand given by the usual action on $\mathbb{K}_1' \otimes H_2'(\mathbb{K}_2)$, and further brackets

$$[a\otimes A, b\otimes B] = 2\langle A,B\rangle S_{a,b} - 4\langle a,b\rangle [L_A, L_B] \tag{8.8}$$

$(A, B \in H_2'(\mathbb{K}_2))$.

This is readily identified as a Lie algebra:

Theorem 8.3. *If $\mathbb{K}_1$ and $\mathbb{K}_2$ are composition algebras, $L_2(\mathbb{K}_1,\mathbb{K}_2)$ as defined above is a Lie algebra isomorphic to $\mathfrak{so}(\mathbb{K}_1\otimes\mathbb{K}_2)$.*

Proof. The orthogonal Lie algebra $\mathfrak{so}(V)$ of a real inner-product vector space V is spanned by the elementary generators $S_{x,y}$, defined as in (4.3) with $x, y \in V$. Hence

$$\mathfrak{so}(V\oplus W) \cong \mathfrak{so}(V) \dot{+} \mathfrak{so}(W) \dot{+} S_{V,W}$$

where $S_{V,W}$, spanned by $S_{v,w} = -S_{w,v}$ with $v \in V$, $w \in W$, is isomorphic to $V \otimes W$. Taking $V = \mathbb{K}_1'$, $W = \mathbb{R}\oplus\mathbb{K}_2 \cong H_2'(\mathbb{K}_2)$, we have a vector space isomorphism θ between $\mathfrak{so}(V\oplus W) \cong \mathfrak{so}(\mathbb{K}_1\oplus\mathbb{K}_2)$ and $L_2(\mathbb{K}_1,\mathbb{K}_2)$ such that $\theta|\mathfrak{so}(\mathbb{K}_1')$ is the identity, $\theta|\mathfrak{so}(\mathbb{R}\oplus\mathbb{K}_2)$ is the isomorphism of Theorem 8.1, and $\theta|S_{V,W}$ is given by

$$\theta(S_{a,A}) = \frac{1}{\sqrt{2}} a\otimes A \qquad (a\in\mathbb{K}_1',\ A\in H_2'(\mathbb{K}_2')).$$

Then θ is an algebra isomorphism: for $X \in \mathfrak{so}(\mathbb{K}_1')$,

$$\theta([X, S_{a,A}]) = \theta(S_{Xa,A}) = 2(Xa\otimes A) = [\theta(X), \theta(Sa, A)],$$

and similarly for $Y \in \mathfrak{so}(\mathbb{R}\oplus\mathbb{K}_2)$. Finally,

$$\theta\big([S_{a,A}, S_{b,B}]\big) = \theta(\langle a,b\rangle S_{A,B} + \langle A,B\rangle S_{a,b}\big)$$

while

$$[\theta(S_{a,A}), \theta(S_{b,B})] = \langle A,B\rangle S_{a,b} - 2\langle a,b\rangle [L_A, L_B]$$

and (8.1) gives $[L_A, L_B] = -\frac{1}{2}S_{A,B}$. Hence $L_2(\mathbb{K}_1,\mathbb{K}_2)$ is a Lie algebra and θ is a Lie algebra isomorphism. □

This theorem shows that the compact and doubly split magic squares $L_2(\mathbb{K}_1, \mathbb{K}_2)$ and $L_2(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$ ($\mathbb{K}_1, \mathbb{K}_2 = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$), like the $n = 3$ squares, are symmetric. The complex types of these Lie algebras are identified below.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K}, \mathbb{R})$	D_1	$A_1 \cong B_1 \cong C_1$	$C_2 \cong B_2$	B_4
$L_2(\mathbb{K}, \mathbb{C})$	$A_1 \cong B_1 \cong C_1$	$A_1 \oplus A_1$	$A_3 \cong D_3$	D_5
$L_2(\mathbb{K}, \mathbb{H})$	$C_2 \cong B_2$	$A_3 \cong D_3$	D_4	D_6
$L_2(\mathbb{K}, \mathbb{O})$	B_4	D_5	D_6	D_8

.

The following table shows the mixed square $L_2(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K}, \mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$L_2(\widetilde{\mathbb{C}}, \mathbb{K})$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$L_2(\widetilde{\mathbb{H}}, \mathbb{K})$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$L_2(\widetilde{\mathbb{O}}, \mathbb{K})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

Note that the maximal compact subalgebras in each row of this square are related to the previous row as in the $n = 3$ magic square (Theorem 7). Because the definition of $L_2(\mathbb{K}_1, \mathbb{K}_2)$ coincides with the Tits construction $T(\mathbb{K}_1, H_2(\mathbb{K}_2))$ if $\mathbb{K}_1$ is associative, Theorem 3.2 gives the same relation between the rows of the non-compact $n = 2$ square $L_2(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$ and the matrix Lie algebras $\mathfrak{sa}(2, \mathbb{K})$, $\mathfrak{sl}(2, \mathbb{K})$ and $\mathfrak{sp}(4, \mathbb{K})$ as for $n = 3$ as shown below.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\widetilde{\mathbb{O}}$
Der $H_2(\mathbb{K}) \cong L_2(\mathbb{R}, \mathbb{K})$	$\mathfrak{so}(2)$	$\mathfrak{su}(2)$	$\mathfrak{sq}(2)$	$\mathfrak{so}(9)$
Str $H_2(\mathbb{K}) \cong L_2(\widetilde{\mathbb{C}}, \mathbb{K})$	$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{H})$	$\mathfrak{sl}(2, \mathbb{O})$
Con $H_2(\mathbb{K}) \cong L_2(\widetilde{\mathbb{H}}, \mathbb{K})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{su}(2,2)$	$\mathfrak{sp}(4, \mathbb{H})$	$\mathfrak{sp}(4, \mathbb{O})$
$L_2(\widetilde{\mathbb{O}}, \mathbb{K})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

The differences between the definitions of L_2 and L_3, however, affect the description of these matrix Lie algebras as follows:

Theorem 8.4. [18] *For any composition algebra* $\mathbb{K}$*,*

(a) $$\mathfrak{sl}(2,\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}');$$

(b) $$\mathfrak{sl}(2,\mathbb{K}) = M_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}');$$

(c) $$\mathfrak{sp}(4,\mathbb{K}) = Q_4'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}').$$

In each case the Lie brackets are defined as follows:

1. $\mathfrak{so}(\mathbb{K}')$ *is a Lie subalgebra;*
2. *The brackets between* $\mathfrak{so}(\mathbb{K}')$ *and the other summand are given by the elementwise action of* $\mathfrak{so}(\mathbb{K}')$ *on matrices over* $\mathbb{K}$*;*
3. *The brackets between two matrices in the first summand are*

$$[X,Y] = (XY - YX)' + \frac{1}{n}F(X,Y) \tag{8.9}$$

where n *(= 2 or 4) is the size of the matrix and* $F(X,Y) \in \mathfrak{so}(\mathbb{K}')$ *is defined by*

$$F(X,Y)a = \sum_{ij}\big([[x_{ij},y_{ji}],a] + 2[x_{ij},y_{ji},a]\big) \qquad (a \in \mathbb{K}').$$

Proof. **(a)** The first isomorphism comes from Theorem 8.2, while the brackets are given by (8.5). This establishes (8.9) for $\mathfrak{sa}(2,\mathbb{K})$.

(b) For $\mathfrak{sl}(2,\mathbb{K})$ the vector space is

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{K}) = \mathrm{Str}' H_2(\mathbb{K}) &= \mathrm{Der}\, H_2(\mathbb{K}) \dot{+} H_2'(\mathbb{K}) \\ &= \mathfrak{so}(\mathbb{K}') \dot{+} A_2'(\mathbb{K}) \dot{+} H_2'(\mathbb{K}) \\ &= \mathfrak{so}(\mathbb{K}') \dot{+} L_2'(\mathbb{K}).\end{aligned}$$

For $A,B \in H_2'(\mathbb{K})$ the Lie bracket is that of $\mathrm{Der}\, H_2(\mathbb{K})$, which we have just seen to be given by (8.9). For $A \in A_2'(\mathbb{K})$, $H \in H_2'(\mathbb{K})$, the bracket is given by the action of A as an element of $\mathrm{Der}\, H_2(\mathbb{K})$ on H, which according to (8.6) is

$$[A,H] = AH - HA. \tag{8.10}$$

Now by Lemma A.5, $\mathrm{tr}(XH - HX) = 0$ and $F(A,H) = 0$. Hence (8.10) is the same as (8.9).

Finally, for $H,K \in H_2'(\mathbb{K})$ the $\mathrm{Str}'\, H_3(\mathbb{K})$ bracket is

$$[H,K] = L_H L_K - L_K L_H \in \mathrm{Der}\, H_2(\mathbb{K})$$

and by Lemma A.15 this commutator appears in the decomposition $\mathrm{Der}\, H_2(\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}')$ as

$$[L_H, L_K] = (HK - KH)' + \tfrac{1}{2}F(H,K).$$

As in Theorem 5.1, the action of the matrix part of $\mathrm{Str}'H_2(\mathbb{K})$ on $H_2(\mathbb{K})$ is

$$H \mapsto XH + HX^\dagger \qquad (X \in M_2'(\mathbb{K})). \tag{8.11}$$

(c) The proof of (c) is the same as that of Theorem 5.1(c) with the derivation $D(X,Y)$ replaced by the orthogonal map $F(X,Y)$. □

Appendix A. Matrix identities

In this appendix we prove various identities for matrices with entries in a composition algebra $\mathbb{K}$. For associative algebras these are familiar Jacobi-like identities and trace identities; in general they hold only for certain classes of matrix, and some need to be modified by terms containing the following elements of $\mathrm{Der}\,\mathbb{K}$ and $\mathfrak{so}(\mathbb{K}')$ defined for pairs of 3×3 matrices X, Y:

$$D(X,Y) = \sum_{ij} D_{x_{ij},y_{ji}} \tag{A.1}$$

where $D_{x,y}$ is the derivation of $\mathbb{K}$ defined in (2.10);

$$S(X,Y) = \sum_{ij} S_{x_{ij},y_{ji}} \tag{A.2}$$

where $S_{x,y}$ is the generator of rotations in the plane of x and y, defined in (4.3);

$$E(X,Y)z = \sum_{ij} [x_{ij}, y_{ji}, z] \qquad (z \in \mathbb{K}), \tag{A.3}$$

and

$$F(X,Y)z = \sum_{ij} [[x_{ij}, y_{ji}], z] + 2[x_{ij}, y_{ji}, z]. \tag{A.4}$$

In all of the following identities $\mathbb{K}$ is a composition algebra. The square brackets denote matrix commutators and the chain brackets denote matrix anticommutators:

$$[X,Y] = XY - YX, \qquad \{X,Y\} = XY + YX.$$

Lemma A.1. *The following matrix identities hold for $A, B \in A_3'(\mathbb{K})$ and $H, K, L \in H_3(\mathbb{K})$:*

(a) $$[A, \{H,K\}] = \{[A,H],K\} + \{H,[A,K]\}, \tag{A.5}$$

(b) $$[A,[B,H]] - [B,[A,H]] = [[A,B],H] + E(A,B)H, \tag{A.6}$$

(c) $$\{H,\{K,L\} - \{K,\{H,L\}\} = [[H,K],L] + E(H,K)L. \tag{A.7}$$

Proof. **(a)** The difference between the two sides of (A.5) can be written in terms of matrix associators, whose (i,j)th element is

$$\sum_{mn} \big([a_{im}, h_{mn}, k_{nj}] + [a_{im}, k_{mn}, h_{nj}] + [h_{im}, k_{mn}, a_{nj}] + [k_{im}, h_{mn}, a_{nj}] - [h_{im}, a_{mn}, k_{nj}] - [k_{im}, a_{mn}, h_{nj}]\big). \quad \text{(A.8)}$$

Suppose $i \neq j$ and let k be the third index. Since the diagonal elements of H and K are real, any associator containing them vanishes. Hence the terms containing a_{ij} or a_{ji} are

$$\sum_n \big([a_{ij}, h_{jn}, k_{nj}] + [a_{ij}, k_{jn}, h_{nj}]\big) + \sum_m \big([h_{im}, k_{mi}, a_{ij}] + [k_{im}, h_{mi}, a_{ij}]\big) - [h_{ij}, a_{ji}, k_{ij}] + [k_{ij}, a_{ji}, h_{ij}] = 0$$

by the alternative law, the hermiticity of H and K, and the fact that an associator changes sign when one of its elements is conjugated. The terms containing a_{ik} or a_{ki} are

$$[a_{ik}, h_{ki}, k_{ij}] + [a_{ik}, k_{ki}, h_{ij}] - [h_{ik}, a_{ki}, k_{ij}] - [k_{ik}, a_{ki}, h_{ij}] = 0$$

using also $a_{ki} = -\overline{a}_{ik}$. Similarly, the terms containing a_{jk} or a_{kj} vanish. Finally, the terms containing a_{ii}, a_{jj} and a_{kk} are

$$[a_{ii}, h_{ik}, k_{kj}] + [a_{ii}, k_{ik}, h_{kj}] + [h_{ik}, k_{kj}, a_{jj}] + [k_{ik}, h_{kj}, a_{jj}] - [h_{ik}, a_{kk}, k_{kj}] - [k_{ik}, a_{kk}, h_{kj}] = 0$$

since $a_{ii} + a_{jj} + a_{kk} = 0$.

Now consider the (i,i)th element. The last two terms of equation (A.8) become

$$-\sum_{mn} ([h_{im}, a_{mn}, k_{ni}] + [k_{in}, a_{nm}, h_{mi}]) = 0.$$

Let j be one of the other two indices. The terms containing a_{ij} or a_{ji} are

$$[a_{ij}, h_{jk}, k_{ki}] + [a_{ij}, k_{jk}, h_{ki}] + [h_{ik}, k_{kj}, a_{ji}] + [k_{ik}, h_{kj}, a_{ji}] = 0,$$

where k is the third index. There are no terms containing a_{jk} or a_{kj}. The terms containing a_{ii}, a_{jj} or a_{kk} are

$$\sum_n ([a_{ii}, h_{in}, k_{ni}] + [a_{ii}, k_{in}, h_{ni}]) + \sum_m ([h_{im}, k_{mi}, a_{ii}] + [k_{im}, h_{mi}, a_{ii}]) = 0.$$

Thus in all cases the expression (A.8) vanishes, proving (a).

(b), (c) Similar arguments establish equations (A.6) and (A.7). □

Lemma A.2. *The identities of Lemma A.1 hold for $A, B \in A_2(\mathbb{K})$ and $H, K, L \in H_2(\mathbb{K})$.*

Proof. The 2×2 case can be deduced from Lemma A.1 by applying it to the matrices

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -\operatorname{tr} B \end{pmatrix},$$

$$\widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}. \quad \widetilde{X} = \begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that A and B do not need to be traceless. □

In the next identities we use the notation L_H for the multiplication by H in the Jordan algebras $H_n(\mathbb{K})$ ($n = 2, 3$) and C_X for the commutator with any matrix X:

$$L_H(K) = \{H, K\}, \qquad C_X(Y) = [X, Y]$$

Lemma A.3. **(a)** *For $A, B \in A'_3(\mathbb{K})$ and $H \in H_3(\mathbb{K})$,*

$$[C_A, C_B]H = C_{(AB-BA)'}H + \tfrac{1}{3}D(A, B)H. \tag{A.9}$$

(b) *For $H, K, M \in H'_3(\mathbb{K})$,*

$$[L_H, L_K]M = C_{(HK-KH)'}M + \tfrac{1}{3}D(H, K)M. \tag{A.10}$$

Proof. By Lemma A.1,

$$[C_A, C_B]H = C_{(AB-BA)'}H + [t_1, H] + E(X, Y)H$$

and

$$[L_H, L_K]M = C_{(HK-KH)'}M + [t_2, M] + E(H, K)M$$

where

$$t_1 = \tfrac{1}{3}\operatorname{tr}(AB - BA) \quad \text{and} \quad t_2 = \tfrac{1}{3}\operatorname{tr}(HK - KH). \tag{A.11}$$

But for any matrices X, Y,

$$[\tfrac{1}{3}\operatorname{tr}(XY - YX), z] + E(X, Y)z = \sum_{ij}\left(\tfrac{1}{3}[[x_{ij}, y_{ji}], z] + [x_{ij}, y_{ji}, z]\right) \tag{A.12}$$

$$= \tfrac{1}{3}D(X, Y)z, \tag{A.13}$$

so the stated identities follow. □

Lemma A.4. **(a)** *For $A, B \in A'_2(\mathbb{K})$ and $H \in H_2(\mathbb{K})$,*

$$[C_A, C_B]H = C_{(AB-BA)'}H + \tfrac{1}{2}F(A, B)H. \tag{A.14}$$

(b) *For $H, K, M \in H'_2(\mathbb{K})$,*

$$[L_H, L_K]M = C_{(HK-KH)'}M + \tfrac{1}{2}F(H, K)M. \tag{A.15}$$

Proof. The proof is the same as that of Lemma A.3 except that in (A.11) the fraction occurring is $\frac{1}{2}$ rather than $\frac{1}{3}$, which means that in (A.12) $D(X,Y)$ must be replaced by $F(X,Y)$. □

Lemma A.5. *For $A \in A_n(\mathbb{K})$ and $H \in H_n(\mathbb{K})$,*

$$\operatorname{tr}[A,H] = 0 \qquad \text{and} \qquad D(A,H) = F(A,H) = 0.$$

Proof.

$$\operatorname{tr}[A,H] = \sum_{ij}[a_{ij},h_{ji}] = \sum_{ij}[\overline{a_{ij}},\overline{h_{ji}}] = -\sum_{ij}[a_{ji},h_{ij}] = -\operatorname{tr}[A,H],$$

so $\operatorname{tr}[A,H]=0$; and similarly,

$$D(A,H) = \sum_{ij} D_{a_{ij},h_{ji}} = \sum_{ij} D_{\overline{a_{ij}},\overline{h_{ji}}} = \sum_{ij} D_{-a_{ji},h_{ij}} = -D(A,H),$$

so $D(A,H)=0$. A similar argument shows that $F(A,H)=0$. □

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